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APPLICATION OF MULTIBODY METHODOLOGY TO ROTATING SHAFT PROBLEMS

M. A. Brown

*Abbott Laboratories Diagnostic Division D*307 *AP*31, 200 *Abbott Park Road*, *Abbott Park*, *IL* 60064-3537, *U.S.A.*

AND

A. A. Shabana

Department of Mechanical Engineering, *University of Illinois at Chicago*, 842 *West Taylor Street*, *Chicago*, *IL* 60607-7022, *U.S.A.*

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The equations of motion presented in the literature for a circular shaft rotating about its centroidal axis assume a constant angular velocity and include a single coupling effect resulting from the Coriolis inertia force component of the gyroscopic moment. The objective of this investigation is to address the limitations of the classical equations of motion and provide a comprehensive model for a shaft rotating about its axis at an arbitrary angular velocity. The general equations of motion for a flexible body are derived through the application of the principle of virtual work in dynamics and are tailored to the specific case of the rotating shaft problem. The equations are shown to include both the Coriolis and centrifugal inertia forces, and the effect of the inertia terms on the system dynamic stability is demonstrated. The effect of the rotary inertia on the axial and transverse deformations is formulated and the coupling terms are obtained. The case of a shaft rotating with a non-constant angular velocity is also examined and the effect of the angular acceleration on the stability of the shaft is addressed. The generality of the approach presented in this study is further demonstrated by considering the dynamics of a rotating shaft subject to a base excitation. The coupling between the base motion and the deformation of the shaft is examined numerically and the effect of the support motion on the dynamics of the shaft is discussed for both a low level and high level disturbance. The results presented in this investigation demonstrate that the general flexible body formulation can be used to study rotating shafts. As a consequence, general purpose flexible multibody computer algorithms can be used to systematically solve more general rotating shaft problems.

1. INTRODUCTION

Systems such as internal combustion engines, power generators and turbines, flexible antennas, rotating airfoils, and high speed machining are driven by rotating shafts which are subject to dynamic inertia forces. In many applications, the rotational speed is low enough that the effects of the dynamic inertia forces can be ignored [1]. However, in other cases such as in high speed machining, the inertia effects must be taken into account. It was reported in reference [2], that high-speed spindles operating at up to 100 000 r.p.m. are used in the fabrication of precision tools for the plastic injection molding industry. These spindles drive grinders for machining operations requiring a surface finish with an accuracy of 50×10^{-6} in and details located at $\pm 0{\cdot}0001$ in tolerance. With the

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advent of high-speed machining, not only has an improvement been realized in grinding operations, but the accuracy in milling applications has also improved by an order of magnitude. A conventional horizontal mill when machining 24 in aluminium cooling fins can hold a thickness tolerance of $\pm 0{\cdot}01$ inch. When fitted with a high-speed spindle, a thickness tolerance of less than $\pm 0{\cdot}003$ inch is possible, and it is becoming common practice to machine highly vibration sensitive materials [3] which were not possible with conventional systems. In addition, it was demonstrated in references [4] and [5] that spindle and bearing vibration can be reduced, and the overall system stability improved by regulating the rotational speed of the spindle system.

As the shaft/rotor configuration limits the speed and stability of the spindle, a mathematical model must be developed that comprehends the complete dynamics of the rotating shaft. This model could be used to predict the system vibrational behavior for any given shaft/rotor configuration and rotational speed, and should also be capable of predicting the vibrational behavior when subjected to external disturbances such as base excitation.

2. BACKGROUND

The dynamics of shafts rotating about their axis has been the subject of many studies concerned with rotating machinery. A comprehensive treatment of early state-of-the-art rotor dynamics and self excited vibrations was provided in reference [6], including the effects of the centrifugal forces, with numerous solutions to practical vibration problems. A detailed analysis was presented explaining how the angular velocity of the rotating shaft influences the critical speeds of the system, with reference to the stability limitations resulting from the centrifugal force. The method of modal analysis was introduced in reference [7] for a rotating beam supported by flexible bearings subjected to a mass imbalance, initial curvation, and gravity. This paper followed the classical formulation in developing the equations of motion for the ideal shaft but did not include any inertia effects. The dynamics of a continuous shaft subjected to an externally applied torque was investigated in reference [8]. In this investigation, the classical differential equations of motion describing the Timoshenko beam were developed with specific attention to the effects of transverse shear, gyroscopic moments, rotary inertia, externally applied torque, and their interactions with the critical speeds of high speed rotors. The effect of the centrifugal force component of the gyroscopic moment was not considered, and only the effect of the rotary inertia on the shaft's mass was identified. In each of these studies, the primary concern of the analysis was to investigate the rotor critical speeds, and to investigate stability between critical speed ranges. In these articles, the rotor is treated as a distributed parameter system describable by partial differential equations of motion.

Modal analysis and assumed shape functions were used in reference [9] to treat an undamped Rayleigh beam supported on isotropic bearings and rotating at a constant angular velocity. A second solution was developed utilizing a set of approximate shape functions based on the closed form solution to a stationary shaft. It was demonstrated that both the modal analysis formulation and the shape function approximation yield correct results. The shape function approximation was shown to be in excellent agreement with closed form solutions for simple cases.

The above study was extended in reference [10] to include a rotating, Rayleigh beam with rotor, supported on anisotropic bearings. The effects of rotary inertia and the gyroscopic moment were included. In reference [11], the Euler–Bernoulli, Rayleigh and Timoshenko beam theories were used to model a simply supported shaft rotating at a constant angular velocity. The equations of motion were derived utilizing the

Newton–Euler equations for dynamic equilibrium and the gyroscopic inertia was treated as an external moment. The results show good agreement with available solutions for the stationary beam subject to moving loads. A study in reference [12] extended the Timoshenko beam rotating at a constant angular velocity to include an accelerating, distributed transverse load having an exponential velocity.

The Lagrangian development of the equations of motion was presented in reference [13, 14] for the rotating Euler beam. The instability resulting from the rotation of the beam was recognized, but the effect of the Coriolis coupling on the stability was not addressed. The dynamic equations of motion for the rotating shaft were presented in reference [16] through the application of Hamilton's principle. The resulting Euler–Lagrange equations fully contain both the Coriolis and centrifugal inertia forces and a number of examples were studied assuming constant angular velocities. The focus of this investigation was the determination of intermediate support locations to correct for the instability resulting from the shaft rotation. The mathematical development used the assumed mode method in the specification of shape function approximations and the results were obtained using numerical techniques.

In this paper, the response of an isotropic shaft rotating about its isotropic axis at an arbitrary angular velocity is investigated. The cross-sectional dimensions and the shaft deflections are assumed small in comparison to its length and the effect of shear deformation is neglected. Both the classical formulation using the Newton–Euler equations of dynamic equilibrium and the principle of virtual work in dynamics are used to develop the equations of motion. In the classical formulation, the equations of motion are presented for the specific case of a shaft rotating at a constant angular velocity about its axis. The equations of motion for the general formulation are then developed using the multibody methodology. This formulation includes the effects of the Coriolis inertia force, as well as two additional inertia forces which influence the effective stiffness of the shaft. The gyroscopic moments are developed directly through the virtual work concept.

A more general formulation will also be developed to examine the dynamics of a rotating shaft when the effects of rotary inertia are included. The constant matrices which describe the effects of the rotary inertia on the effective mass will also be obtained. The generality of the approach presented is further demonstrated by considering the dynamics of a rotating shaft subject to a base excitation. The coupling between the base motion and the deformation of the shaft is examined numerically and the effect of the support motion on the vibrational behavior of the shaft is discussed for both a low level and high level distubance.

3. CLASSICAL FORMULATION

In this section, the classical formulations of the equations of motion of a shaft rotating about its axis at a constant angular velocity are reviewed. It is assumed that the cross-sectional dimensions of the shaft are small in comparison to its length and the effect of shear deformation is neglected. The equations of motion are derived by applying Newton–Euler equations to a cross-section of an infinitesimal length of the rotating shaft shown in Figure 1. The effect of the induced gyroscopic moment is obtained using the definition of the angular momentum as it applies to the rotation of the cross-section of the shaft. The purpose of the analysis presented in this section is to demonstrate the limitations of, and the approximations used in deriving the classical formulations of rotating shaft problems, as compared to the more general multibody formulation presented in the following sections.

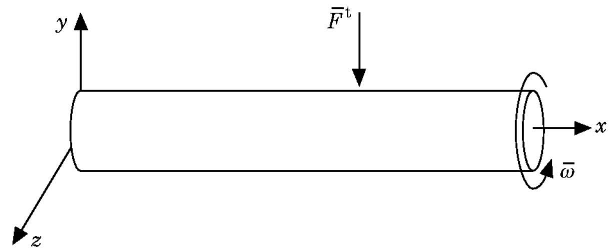

Figure 1. Rotating shaft.

The relationship between the angular accelerations and the applied moments are defined by Euler's equation

$$\mathbf{T} = \rho \mathbf{I} \partial^2 \Psi / \partial t^2, \tag{1}$$

where $\mathbf{T}$ is the vector of moments, $\mathbf{I}$ is the inertia tensor of an infinitesimal cross-section of the shaft, such that

$$\mathbf{I} = \begin{bmatrix} 2I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{2}$$

I is the moment of inertia of the cross-section about the y- and z-axes, ρ is the shaft density, and the angular displacement of the shaft is given by

$$\Psi = [0 \quad \psi_2 \quad \psi_3]^{\mathrm{T}},$$

where ψ_2 and ψ_3 are the rotations about the y- and z-axes, respectively. In equation (1), it is assumed that the shaft rotates with a constant angular velocity ω_1 about the x-axis, and as a consequence, its angular acceleration about the x-axis is identically equal to zero. The angular displacements can be written as

$$\psi_2 = -\partial u_3 / \partial x, \qquad \psi_3 = \partial u_2 / \partial x,$$

where u_2 and u_3 are the displacements in the y and z directions.

The angular momentum of the shaft is given by

$$\bar{\mathbf{H}} = \rho \mathbf{I} \boldsymbol{\omega}, \tag{3}$$

where

$$\boldsymbol{\omega} = [\omega_1 \quad 0 \quad 0]^{\mathrm{T}}. \tag{4}$$

Using the rate of change of the angular momentum and following the procedure described in reference [17], the classical equations of motion for the rotating shaft are defined as

$$\rho A \ddot{u}_2 + EIu_2'''' - \rho I \ddot{u}_2'' - 2\rho I \omega_1 \dot{u}_3'' = F^t(x, t), \qquad \rho A \ddot{u}_3 + EIu_3'''' - \rho I \ddot{u}_3'' + 2\rho I \omega_1 \dot{u}_2'' = 0, \tag{5a, b}$$

where A is the cross-sectional area, E is the modulus of elasticity, F^t is a transverse force acting in the y direction, $(\cdot)$ implies differentiation with respect to time, and $(')$ implies differentiation with respect to the spatial co-ordinate.

In obtaining the classical equations of motion presented in this section, several simplifying assumptions are made. These assumptions limit the use of this formulation in the analysis of more general cases. The limitations and approximations can be summarized as follows:

1. The angular velocity of the shaft is assumed constant, and as such, the classical formulation cannot be used in the analysis of shafts that have non-zero angular accelerations.
2. In the classical formulation, the effect of the longitudinal displacement resulting from the transverse applied load is neglected. As a result of the rotary inertia effect, bending deformations affect the longitudinal displacement of the rotating shaft. Such an effect cannot be examined using the classical approach.
3. The classical formulation does not take into consideration the support vibration resulting from support movements, or bearing looseness and joint clearances. Therefore, the use of this formulation is limited to special rotating shaft problems where base excitations and joint and bearing flexibility are ignored.
4. In the classical formulation, the complete effect of the deformation on the expression of the angular momentum is not taken into consideration. At high speed rotations, such as those encountered in modern rotating spindle applications, a more precise definition for the inertia forces must be used.

4. FLEXIBLE MULTIBODY EQUATIONS

Using the principle of virtual work in dynamics, it can be shown that the equations of motion of a deformable body that undergoes an arbitrary displacement can be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}_v + \mathbf{Q}_e, \tag{6}$$

where $\mathbf{M}$ is the body mass matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{Q}_v$ is the vector of Coriolis and centrifugal forces, and $\mathbf{Q}_e$ is the vector of generalized applied forces [18]. The vector of generalized co-ordinates $\mathbf{q}$ can be written in a partitioned form as

$$\mathbf{q} = [\mathbf{R}^{\mathrm{T}} \quad \boldsymbol{\theta}^{\mathrm{T}} \quad \mathbf{q}_f^T]^{\mathrm{T}} \tag{7}$$

in which $\mathbf{R}$ is the reference displacement, $\boldsymbol{\theta}$ is the set of parameters that define the body orientation, and $\mathbf{q}_f$ is the vector of elastic co-ordinates. According to this co-ordinate partitioning, the equation of motion can be written as

$$\begin{bmatrix} \mathbf{m}_{RR} & \mathbf{m}_{R\theta} & \mathbf{m}_{Rf} \\ \mathbf{m}_{\theta R} & \mathbf{m}_{\theta\theta} & \mathbf{m}_{\theta f} \\ \mathbf{m}_{fR} & \mathbf{m}_{f\theta} & \mathbf{m}_{ff} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{R}} \\ \ddot{\boldsymbol{\theta}} \\ \ddot{\mathbf{q}}_f \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{k}_{ff} \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \boldsymbol{\theta} \\ \mathbf{q}_f \end{bmatrix} = \begin{bmatrix} (\mathbf{Q}_v)_R \\ (\mathbf{Q}_v)_\theta \\ (\mathbf{Q}_v)_f \end{bmatrix} + \begin{bmatrix} (\mathbf{Q}_e)_R \\ (\mathbf{Q}_e)_\theta \\ (\mathbf{Q}_e)_f \end{bmatrix}, \tag{8}$$

where the elements of the mass matrix, stiffness matrix, quadratic velocity vector $\mathbf{Q}_v$, and the vector of generalized external forces, $\mathbf{Q}_e$, are defined in explicit form in references [17]. The elements of the mass matrix are defined as

$$\mathbf{M} = \begin{bmatrix} \mathbf{m}_{RR} & \mathbf{m}_{R\theta} & \mathbf{m}_{Rf} \\ \mathbf{m}_{\theta R} & \mathbf{m}_{\theta\theta} & \mathbf{m}_{\theta f} \\ \mathbf{m}_{fR} & \mathbf{m}_{f\theta} & \mathbf{m}_{ff} \end{bmatrix} = \int \rho \begin{bmatrix} \mathbf{I} & \mathbf{A}\tilde{\mathbf{u}}\bar{\mathbf{G}} & \mathbf{A}\mathbf{S} \\ & \bar{\mathbf{G}}^{\mathrm{T}}\tilde{\mathbf{u}}^{\mathrm{T}}\tilde{\mathbf{u}}\bar{\mathbf{G}} & \bar{\mathbf{G}}^{\mathrm{T}}\tilde{\mathbf{u}}\mathbf{S} \\ \textbf{Symmetric} & & \mathbf{S}^{\mathrm{T}}\mathbf{S} \end{bmatrix} \mathrm{d}V, \tag{9}$$

where the integration is taken over the volume of the body V, $\mathbf{A}$ is the transformation matrix that defines the orientation of the body co-ordinate system, $\tilde{\mathbf{u}}$ is the skew symmetric matrix associated with the vector $\bar{\mathbf{u}}$ which defines the position of an arbitrary point on the body with respect to the body co-ordinate system, $\mathbf{S}$ is the shape function matrix, and $\bar{\mathbf{G}}$

is the matrix relating the angular velocity vector to the vector of time derivatives of the orientation co-ordinates. That is

$$\bar{\mathbf{u}} = \bar{\mathbf{u}}_0 + \bar{\mathbf{u}}_f = \bar{\mathbf{u}}_0 + \mathbf{S}\mathbf{q}_f, \qquad \bar{\boldsymbol{\omega}} = \bar{\mathbf{G}}\dot{\boldsymbol{\theta}}, \tag{10}$$

where $\bar{\mathbf{u}}_0$ is the position of an arbitrary point on the body in the undeformed state, $\bar{\mathbf{u}}_f$ is the time and space dependent deformation vector and $\bar{\boldsymbol{\omega}}$ is the angular velocity vector defined in the body co-ordinate system. The vector of the Coriolis and centrifugal forces is defined as

$$(\mathbf{Q}_v) = \begin{bmatrix} (\mathbf{Q}_v)_R \\ (\mathbf{Q}_v)_\theta \\ (\mathbf{Q}_v)_f \end{bmatrix} = \begin{bmatrix} -\mathbf{A}\{\tilde{\bar{\boldsymbol{\omega}}}^2\bar{\mathbf{S}}_f - \int \rho\tilde{\bar{\mathbf{u}}}\,\mathrm{d}V\dot{\bar{\mathbf{G}}}\dot{\theta} + 2\tilde{\bar{\boldsymbol{\omega}}}\bar{\mathbf{S}}\dot{\mathbf{q}}_f\} \\ -\dot{\bar{\mathbf{G}}}^T\bar{\mathbf{I}}_{\theta\theta}\bar{\boldsymbol{\omega}} - 2\dot{\bar{\mathbf{G}}}^T\bar{\mathbf{I}}_{\theta f}\dot{\mathbf{q}}_f - \bar{\mathbf{G}}^T\dot{\bar{\mathbf{I}}}_{\theta\theta}\bar{\boldsymbol{\omega}} \\ -\int \rho\{\mathbf{S}^T[\tilde{\bar{\boldsymbol{\omega}}}^2\bar{\mathbf{u}} + 2\tilde{\bar{\boldsymbol{\omega}}}\dot{\bar{\mathbf{u}}}_f]\}\,\mathrm{d}V \end{bmatrix}, \tag{11}$$

where

$$\bar{\mathbf{S}} = \int \rho\mathbf{S}\,\mathrm{d}V, \qquad \bar{\mathbf{S}}_f = \int \rho\bar{\mathbf{u}}\,\mathrm{d}V, \qquad \bar{\mathbf{I}}_{\theta\theta} = \int \rho\tilde{\bar{\mathbf{u}}}^T\tilde{\bar{\mathbf{u}}}\,\mathrm{d}V, \qquad \bar{\mathbf{I}}_{\theta f} = \int \rho\tilde{\bar{\mathbf{u}}}^T\mathbf{S}\,\mathrm{d}V, \tag{12}$$

and $\tilde{\bar{\boldsymbol{\omega}}}$ is the skew symmetric matrix associated with the angular velocity vector $\bar{\boldsymbol{\omega}}$ defined in reference [18].

5. APPLICATION TO THE ROTATING SHAFT PROBLEM

For the constrained shaft as pictured in Figure 1, the origin of the shaft co-ordinate system is assumed to coincide with the origin of the global reference system, such that $\mathbf{R}$ is the null vector, and, as a consequence of the angular velocity being specified, the equations of motion associated with the elastic co-ordinates need only be used to completely describe the system deformations. The equations of motion associated with the elastic co-ordinates of equation (8) can be written, in the case of zero reference translations, as

$$\mathbf{m}_{f\theta}\ddot{\boldsymbol{\theta}} + \mathbf{m}_{ff}\ddot{\mathbf{q}}_f + \mathbf{k}_{ff}\mathbf{q}_f = (\mathbf{Q}_v)_f + (\mathbf{Q}_e)_f. \tag{13}$$

Using the equations presented in the preceding section, equation (13) can be expressed as

$$\mathbf{m}_{ff}\ddot{\mathbf{q}}_f + 2\mathbf{S}_\omega\dot{\mathbf{q}}_f + (\mathbf{k}_{ff} + \mathbf{S}_{\omega\omega})\mathbf{q}_f = \bar{\mathbf{I}}_{\theta f}^T(\bar{\mathbf{G}}\ddot{\boldsymbol{\theta}} + \dot{\bar{\mathbf{G}}}\dot{\boldsymbol{\theta}}) - \mathbf{S}_{\omega 0} + (\mathbf{Q}_e)_f, \tag{14}$$

where

$$\mathbf{S}_\omega = \int \rho\mathbf{S}^T\tilde{\bar{\boldsymbol{\omega}}}\mathbf{S}\,\mathrm{d}V, \qquad \mathbf{S}_{\omega\omega} = \int \rho\mathbf{S}^T(\tilde{\bar{\boldsymbol{\omega}}})^2\mathbf{S}\,\mathrm{d}V, \qquad \mathbf{S}_{\omega 0} = \int \rho\mathbf{S}^T(\tilde{\bar{\boldsymbol{\omega}}})^2\bar{\mathbf{u}}_0\,\mathrm{d}V, \tag{15}$$

and $\bar{\mathbf{I}}_{\theta f}$ is previously defined.

5.1. SIMPLE ROTATION

For a uniform shaft rotating about its x-axis, the undeformed position and angular velocity vectors are given by

$$\bar{\mathbf{u}}_0 = [x \quad 0 \quad 0]^T, \qquad \bar{\boldsymbol{\omega}} = [\omega_1 \quad 0 \quad 0]^T. \tag{16}$$

By letting the shape function be defined in a general matrix form as

$$\mathbf{S} = [\mathbf{S}_1^T \quad \mathbf{S}_2^T \quad \mathbf{S}_3^T]^T, \tag{17}$$

where $\mathbf{S}_k$ is the kth row of the shaft shape function, the quadratic velocity vector terms appearing in equation (15) reduce to

$$\mathbf{S}_{\omega\omega} = -\omega_1^2\hat{\mathbf{S}}, \qquad \mathbf{S}_{\omega} = -\omega_1\tilde{\mathbf{S}}, \qquad \mathbf{S}_{\omega 0} = \mathbf{0}, \tag{18}$$

where

$$\hat{\mathbf{S}} = \int \rho\mathbf{S}^{\mathrm{T}}\bar{\mathbf{I}}_{\mathrm{c}}\mathbf{S}\,\mathrm{d}V, \qquad \tilde{\mathbf{S}} = \int \rho\mathbf{S}^{\mathrm{T}}\tilde{\mathbf{I}}\mathbf{S}\,\mathrm{d}V, \tag{19}$$

$$\bar{\mathbf{I}}_{\mathrm{c}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \tilde{\mathbf{I}} = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{20}$$

Using these identities and definitions, equation (14) reduces to

$$\mathbf{m}_{\mathrm{ff}}\ddot{\mathbf{q}}_{\mathrm{f}} - 2\omega_1\tilde{\mathbf{S}}\dot{\mathbf{q}}_{\mathrm{f}} - (\mathbf{k}_{\mathrm{ff}} + \omega_1^2\hat{\mathbf{S}})\mathbf{q}_{\mathrm{f}} = \bar{\mathbf{I}}_{\theta\mathrm{f}}^{\mathrm{T}}(\bar{\mathbf{G}}\ddot{\boldsymbol{\theta}} + \dot{\bar{\mathbf{G}}}\dot{\boldsymbol{\theta}}) + (\mathbf{Q}_{\mathrm{e}})_{\mathrm{f}}. \tag{21}$$

Note that the differences between equation (21) and the more generalized form presented by equation (14) is due to the special forms of the vectors $\bar{\mathbf{u}}_0$ and $\bar{\boldsymbol{\omega}}$ as defined by equation (16). Since $\bar{\mathbf{u}}_0$ and $\bar{\boldsymbol{\omega}}$ are assumed to be parallel vectors, their cross product is equal to zero, and as a consequence, the centrifugal force vector $\mathbf{S}_{\omega 0}$ is identically equal to zero. If the dimensions of the shaft are such that the y and z components of the vector $\bar{\mathbf{u}}_0$ cannot be neglected, the vector $\mathbf{S}_{\omega 0}$ is not equal to zero. For instance, if

$$\bar{\mathbf{u}}_0 = [x \quad y \quad z]^{\mathrm{T}}, \tag{22}$$

one has

$$\mathbf{S}_{\omega 0} = \int \rho\mathbf{S}^{\mathrm{T}}(\tilde{\boldsymbol{\omega}})^2\bar{\mathbf{u}}_0\,\mathrm{d}V = \int \rho\mathbf{S}^{\mathrm{T}}(\tilde{\boldsymbol{\omega}})^2\begin{bmatrix} x \\ y \\ z \end{bmatrix}\mathrm{d}V. \tag{23}$$

It follows that

$$\mathbf{S}_{\omega 0} = -\omega_1^2\int \rho[y\mathbf{S}_2^{\mathrm{T}} + z\mathbf{S}_3^{\mathrm{T}}]\,\mathrm{d}V. \tag{24}$$

In applications such as high speed machining, the centrifugal force vector can be very significant even in the cases where the dimensions of the cross-section are small compared to the length of the shaft. One also notes that in the case of a simple rotation,

$$\dot{\bar{\mathbf{G}}}\dot{\boldsymbol{\theta}} = 0, \qquad \bar{\mathbf{G}}\ddot{\boldsymbol{\theta}} = \boldsymbol{\alpha}, \tag{25}$$

where $\boldsymbol{\alpha}$ is the angular acceleration of the shaft. Using these identities and the definitions of equation (16), the generalized equation of motion for a rotating shaft given by equation (21) can further reduce to

$$\mathbf{m}_{\mathrm{ff}}\ddot{\mathbf{q}}_{\mathrm{f}} - 2\omega_1\tilde{\mathbf{S}}\dot{\mathbf{q}}_{\mathrm{f}} + (\mathbf{k}_{\mathrm{ff}} - \omega_1^2\hat{\mathbf{S}})\mathbf{q}_{\mathrm{f}} - \bar{\mathbf{I}}_{\theta\mathrm{f}}^{\mathrm{T}}\alpha = (\mathbf{Q}_{\mathrm{e}})_{\mathrm{f}}. \tag{26}$$

Note that $\bar{\mathbf{I}}_{\theta\mathrm{f}}^{\mathrm{T}}$ can be written as the sum of two matrices:

$$\bar{\mathbf{I}}_{\theta\mathrm{f}}^{\mathrm{T}} = \bar{\mathbf{I}}_{\theta 1}^{\mathrm{T}} + \bar{\mathbf{I}}_{\theta 2}^{\mathrm{T}}, \tag{27}$$

where

$$\bar{\mathbf{I}}_{\theta 1}^{\mathrm{T}} = \int \rho \mathbf{S}^{\mathrm{T}} \tilde{\mathbf{u}}_0 \, \mathrm{d}V, \qquad \bar{\mathbf{I}}_{\theta 2}^{\mathrm{T}} = \int \rho \mathbf{S}^{\mathrm{T}} \tilde{\mathbf{u}}_{\mathbf{f}} \, \mathrm{d}V, \tag{28}$$

and $\tilde{\mathbf{u}}_0$ and $\tilde{\mathbf{u}}_{\mathbf{f}}$ are previously defined. Note that if the cross-section dimensions of the shaft are neglected in the definition of the vector $\tilde{\mathbf{u}}_0$, the vectors $\bar{\mathbf{u}}_0$ and α are parallel, and as a consequence,

$$\bar{\mathbf{u}}_0 \times \boldsymbol{\alpha} = \tilde{\mathbf{u}}_0 \boldsymbol{\alpha} = \mathbf{0}.$$

It follows that

$$\bar{\mathbf{I}}_{\theta 1}^{\mathrm{T}} \boldsymbol{\alpha} = 0.$$

Using simple vector and matrix operations, it can be shown that

$$\bar{\mathbf{I}}_{\theta 2}^{\mathrm{T}} \boldsymbol{\alpha} = \alpha_1 \int \rho \mathbf{S}^{\mathrm{T}} \tilde{\mathbf{I}} \mathbf{S} \, \mathrm{d}V \mathbf{q}_{\mathbf{f}} = \alpha_1 \tilde{\mathbf{S}} \mathbf{q}_{\mathbf{f}},$$

where $\tilde{\mathbf{S}}$ is defined in equation (19), and α_1 is the amplitude of the angular acceleration vector. Hence,

$$\bar{\mathbf{I}}_{\theta \mathbf{f}}^{\mathrm{T}} \boldsymbol{\alpha} = \alpha_1 \tilde{\mathbf{S}} \mathbf{q}_{\mathbf{f}}$$

and the equations of motion of the rotating shaft reduce to

$$\mathbf{m}_{\mathbf{ff}} \ddot{\mathbf{q}}_{\mathbf{f}} - 2\omega_1 \tilde{\mathbf{S}} \dot{\mathbf{q}}_{\mathbf{f}} + (\mathbf{k}_{\mathbf{ff}} - \omega_1^2 \hat{\mathbf{S}} - \alpha_1 \tilde{\mathbf{S}}) \mathbf{q}_{\mathbf{f}} = (\mathbf{Q}_{\mathbf{e}})_{\mathbf{f}}. \tag{29}$$

This equation describes the dynamics of an accelerating shaft and therefore its use is not limited only to the case of constant angular velocity.

5.2. GENERALIZATION

If the dimensions of the cross-section of the shaft are neglected in defining the vector $\bar{\mathbf{u}}_0$, and the angular acceleration is prescribed, equation (29) in the absence of external forces, defines a homogeneous system of ordinary differential equations. This system has zero response if the initial conditions are equal to zero since the inertia forces resulting from the rotation of the shaft are of the passive type [19].

If the transverse dimensions of the cross-section of the shaft are considered in defining the vector $\bar{\mathbf{u}}_0$ as defined by equation (22), $\bar{\mathbf{I}}_{\theta 1}^{\mathrm{T}}$ can be written as

$$\bar{\mathbf{I}}_{\theta 1}^{\mathrm{T}} \boldsymbol{\alpha} = \int \rho \mathbf{S}^{\mathrm{T}} \tilde{\mathbf{u}}_0 \, \mathrm{d}V \boldsymbol{\alpha} = \alpha_1 \int \rho (z \mathbf{S}_2^{\mathrm{T}} - y \mathbf{S}_3^{\mathrm{T}}) \, \mathrm{d}V \tag{30}$$

and the equations of motion of the accelerating shaft can be written as

$$\mathbf{m}_{\mathbf{ff}} \ddot{\mathbf{q}}_{\mathbf{f}} - 2\omega_1 \tilde{\mathbf{S}} \dot{\mathbf{q}}_{\mathbf{f}} + (\mathbf{k}_{\mathbf{ff}} - \omega_1^2 \hat{\mathbf{S}} - \alpha_1 \tilde{\mathbf{S}}) \mathbf{q}_{\mathbf{f}} = \bar{\mathbf{I}}_{\theta 1}^{\mathrm{T}} \boldsymbol{\alpha} - \mathbf{S}_{\omega 0} + (\mathbf{Q}_{\mathbf{e}})_{\mathbf{f}}, \tag{31}$$

where $\mathbf{S}_{\omega 0}$ is defined by equation (24).

In the absence of external forces, the preceding system defines a set of non-homogeneous differential equations, and as such, the response of this system is significantly different from that of the system defined by equation (29), in which the cross-section dimensions of the shaft are neglected.

6. EFFECT OF ROTARY INERTIA

In the previous sections, the effect of rotary inertia due to the angular deflection of the cross-section was not considered. In this section, the effect of the rotary inertia on the dynamics of the rotating shaft is examined. It will be demonstrated that these effects can be systematically captured in the flexible multibody formulation.

Let $\mathbf{r}$, as shown in Figure 2, represent the global position of an arbitrary point on the deformable body such that

$$\mathbf{r} = \mathbf{R} + \mathbf{A}(\bar{\mathbf{u}}_0 + \bar{\mathbf{u}}_f + \mathbf{A}_r\bar{\mathbf{u}}_L),$$

where the undeformed position vector $\bar{\mathbf{u}}_0$, the time and space dependent deformation vector $\bar{\mathbf{u}}_f = \mathbf{S}\mathbf{q}_f$, the shape function $\mathbf{S}$, the global transformation matrix $\mathbf{A}$, and the vector of generalized time dependent elastic co-ordinates $\mathbf{q}_f$ are as defined in the preceding sections. The transformation matrix $\mathbf{A}_r$ describes the infinitesimal rotation of the cross-section with respect to the shaft co-ordinate system, and the position vector $\bar{\mathbf{u}}_L$ describes the position of an arbitrary point in the rotated plane of the deformed shaft. The transformation matrices can be written as

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \qquad \mathbf{A}_r = \begin{bmatrix} 1 & -\psi_3 & \psi_2 \\ \psi_3 & 1 & 0 \\ -\psi_2 & 0 & 1 \end{bmatrix}, \tag{32}$$

where the vectors $\bar{\mathbf{u}}_0$ and $\bar{\mathbf{u}}_L$ are defined as

$$\bar{\mathbf{u}}_0 = [x \quad 0 \quad 0]^T, \qquad \bar{\mathbf{u}}_L = [0 \quad y \quad z]^T,$$

and ψ_2 and ψ_3 are the rotations of the cross section about the y and z axes of the shaft coordinate system. It can be shown that the global position vector $\mathbf{r}$ can be restated as

$$\mathbf{r} = \mathbf{R} + \mathbf{A}(\bar{\mathbf{u}}_{0t} + \mathbf{S}_t\mathbf{q}_f) = \mathbf{R} + \mathbf{A}\bar{\mathbf{u}}, \tag{33}$$

where

$$\bar{\mathbf{u}}_{0t} = [x \quad y \quad z]^T, \qquad \mathbf{S}_t = \begin{bmatrix} S_1 & -y\partial S_2/\partial x & -z\partial S_3/\partial x \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}. \tag{34}$$

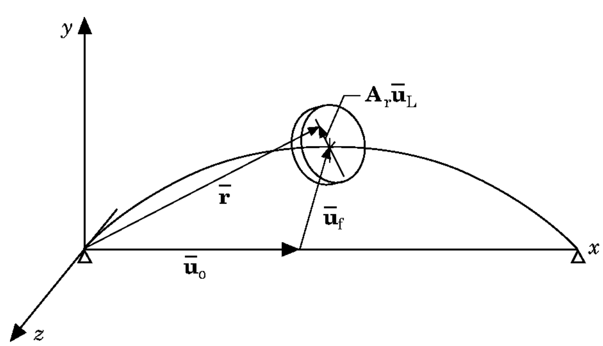

Figure 2. Co-ordinates of an arbitrary point on the cross-section.

Using equation (33), and applying the virtual work principle, one can show that the equation of motion of the rotating shaft which includes the effect of the rotary inertia can be written as

$$\mathbf{m}_{\mathrm{ff}}\ddot{\mathbf{q}}_{\mathrm{f}} - 2\omega_1\tilde{\mathbf{S}}_{\mathrm{t}}\dot{\mathbf{q}}_{\mathrm{f}} + (\mathbf{k}_{\mathrm{ff}} - \omega_1^2\hat{\mathbf{S}}_{\mathrm{t}} - \alpha_1\tilde{\mathbf{S}}_{\mathrm{t}})\mathbf{q}_{\mathrm{f}} = \alpha_1\bar{\mathbf{I}}_{0\mathrm{t}} + \omega_1^2\hat{\mathbf{S}}_{0\mathrm{t}} + (\mathbf{Q}_{\mathrm{e}})_{\mathrm{f}}, \tag{35}$$

where

$$\mathbf{m}_{\mathrm{ff}} = \int \rho \mathbf{S}_{\mathrm{t}}^{\mathrm{T}}\mathbf{S}_{\mathrm{t}}\,\mathrm{d}V, \qquad \tilde{\mathbf{S}}_{\mathrm{t}} = \int \rho \mathbf{S}_{\mathrm{t}}^{\mathrm{T}}\tilde{\mathbf{I}}\mathbf{S}_{\mathrm{t}}\,\mathrm{d}V, \qquad \hat{\mathbf{S}}_{\mathrm{t}} = \int \rho \mathbf{S}_{\mathrm{t}}^{\mathrm{T}}\bar{\mathbf{I}}_{\mathrm{c}}\mathbf{S}_{\mathrm{t}}\,\mathrm{d}V,$$

$$\bar{\mathbf{I}}_{0\mathrm{t}} = \int \rho \mathbf{S}_{\mathrm{t}}^{\mathrm{T}}\tilde{\mathbf{I}}\bar{\mathbf{u}}_{0\mathrm{t}}\,\mathrm{d}V, \qquad \hat{\mathbf{S}}_{0\mathrm{t}} = \int \rho \mathbf{S}_{\mathrm{t}}^{\mathrm{T}}\bar{\mathbf{I}}_{\mathrm{c}}\bar{\mathbf{u}}_{0\mathrm{t}}\,\mathrm{d}V. \tag{36}$$

A significant consequence of including the effect of the rotary inertia in the dynamic equations is the coupling of the axial displacement to the in-plane and out-of-plane bending modes [17].

7. APPLICATION

In this section, an example that demonstrates the use of the generalized equations of motion developed in the preceding sections is considered. This application will be used to examine the effect of the coupling between the modes of deformation of the rotating shaft. The closed form solution to a stationary shaft with simply supported end conditions will be used to approximate the transverse modes of vibration and the effect of the centrifugal and Coriolis forces will be examined.

In this section, the deformation of the rotating shaft is described using three mode shapes. These mode shapes correspond to the closed form solution for a stationary shaft with simply supported end conditions and can be written as

$$\mathbf{S} = \begin{bmatrix}\mathbf{S}_1\\ \mathbf{S}_2\\ \mathbf{S}_3\end{bmatrix} = \begin{bmatrix}\sin((\pi/\ell)x) & 0 & 0\\ 0 & \sin((\pi/\ell)x) & 0\\ 0 & 0 & \sin((\pi/\ell)x)\end{bmatrix} = \sin((\pi/\ell)x)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{37}$$

where only the fundamental modes of vibration are used.

The shape function matrix can be used to evaluate the constant matrices that appear in the equations of motion of the rotating shaft. We will consider the special case of the constrained shaft of section 5, where the effect of the rotary inertia is neglected. In this case, the dynamics of the shaft is governed by equation (29). The constant matrices that appear in this equation can be evaluated in the case of simply supported shaft as

$$\mathbf{m}_{\mathrm{ff}} = \int \rho \mathbf{S}^{\mathrm{T}}\mathbf{S}\,\mathrm{d}V = \begin{bmatrix}m/2 & 0 & 0\\ 0 & m/2 & 0\\ 0 & 0 & m/2\end{bmatrix}, \qquad \tilde{\mathbf{S}} = \int \rho \mathbf{S}^{\mathrm{T}}\tilde{\mathbf{I}}\mathbf{S}\,\mathrm{d}V = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & m/2\\ 0 & -m/2 & 0\end{bmatrix},$$

$$\hat{\mathbf{S}} = \int \rho \mathbf{S}^{\mathrm{T}}\bar{\mathbf{I}}_{\mathrm{c}}\mathbf{S}\,\mathrm{d}V = \begin{bmatrix}0 & 0 & 0\\ 0 & \dfrac{m}{2} & 0\\ 0 & 0 & \dfrac{m}{2}\end{bmatrix},$$

$$\mathbf{k}_{\mathrm{ff}} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} = \begin{bmatrix} EA\dfrac{\pi^2}{2\ell^2} & 0 & 0 \\ 0 & EI\dfrac{\pi^4}{2\ell^3} & 0 \\ 0 & 0 & EI\dfrac{\pi^4}{2\ell^3} \end{bmatrix}.$$

The vector of elastic co-ordinates is a three-dimensional vector which can be written as

$$\mathbf{q}_{\mathrm{f}} = [q_1 \quad q_2 \quad q_3]^{\mathrm{T}}, \tag{38}$$

where q_1 is the time dependent elastic co-ordinate associated with the axial mode of deformation, and q_2 and q_3 are the time dependent elastic co-ordinates associated with the in-plane and out-of-plane bending deformations, respectively.

Substituting the constant matrices and vectors into equation (29), the matrix equation of motion of the rotating shaft can be written as

$$\frac{m}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ddot{\mathbf{q}}_{\mathrm{f}} + \omega_1 m \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\dot{\mathbf{q}}_{\mathrm{f}}$$

$$+ \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 - \omega_1^2\dfrac{m}{2} & -\alpha_1\dfrac{m}{2} \\ 0 & \alpha_1\dfrac{m}{2} & k_3 - \omega_1^2\dfrac{m}{2} \end{bmatrix}\mathbf{q}_f = (\mathbf{Q}_e)_f \tag{39}$$

8. EFFECT OF THE CORIOLIS COUPLING

To illustrate the effect of the coupling between the in-plane and out-of-plane deformations in the absence of any external forces for a rotating shaft, the equations of motion will be examined for the case of a shaft rotating at a constant angular velocity. With the assumption that the angular acceleration is zero, the equations of motion for the in-plane and out-of-plane bending can be obtained from equation (39) as

$$\begin{bmatrix} m_e & 0 \\ 0 & m_e \end{bmatrix}\ddot{\mathbf{q}}_{\mathrm{f}} + \begin{bmatrix} 0 & -2\omega_1 m_e \\ 2\omega_1 m_e & 0 \end{bmatrix}\dot{\mathbf{q}}_{\mathrm{f}} + \begin{bmatrix} K_e & 0 \\ 0 & K_e \end{bmatrix}\mathbf{q}_{\mathrm{f}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{40}$$

where m_e and K_e are the effective mass and stiffness coefficients, defined as

$$m_e = m/2, \qquad K_e = EI\pi^4/2\ell^3 - \omega_1^2 m_e. \tag{41}$$

In the absence of an applied force, initial conditions must be specified in order to obtain a non-trivial solution for the preceding set of homogeneous second order differential equations.

8.1. CASE WITH NO CORIOLIS COUPLING

If the effect of the out-of-plane bending is neglected in the preceding equation, one obtains a single-degree-of-freedom system. Considering the shaft rotation without coupling, the in-plane equation of motion is given by $m_e\ddot{\mathbf{q}}_2 + K_e q_2 = 0$ which can be shown to have the general solution $q_2(t) = X\sin(\omega_f t + \phi)$, where X is the amplitude, ϕ is the phase angle and ω_f is the natural frequency given by

$$\omega_f = \sqrt{K_e/m_e}.$$

As long as K_e remains positive, the response is stable. Since K_e is a function of the square of the angular velocity, as given by equation (41), the response will have a limited stability range. That is, for the case of simply supported end conditions, the angular velocity of the shaft must be chosen such that $\omega_1 < \sqrt{EI\pi^4/2m_e\ell^3}$ for the response to be stable.

8.2. COUPLING DUE TO THE CORIOLIS FORCE

It was demonstrated that if only one transverse mode of vibration is considered, the first order velocity dependent term in equation (40) vanishes. As a consequence, the system reaches a stability limit as the angular velocity of the shaft increases. In the following development, the effect of the mode coupling due to the Coriolis force on the dynamics and stability of the rotating shaft is examined. A solution of equation (40) can be assumed in the form

$$\mathbf{q_f} = \mathbf{X}\mathbf{e}^{st}.$$

Substituting into equation (40), the characteristic equation can be obtained as

$$m_e^2 s^4 + (2m_e K_e + 4\omega_1^2 m_e^2)s^2 + K_e^2 = 0.$$

The roots of this equation are

$$s_1 = \mathrm{j}\Omega_1, \qquad s_2 = -\mathrm{j}\Omega_1, \qquad s_3 = \mathrm{j}\Omega_2, \qquad s_4 = -\mathrm{j}\Omega_2,$$

with

$$\Omega_1 = \sqrt{[(\xi + \omega_1^2 m_e) - 2\omega_1\sqrt{\xi m_e}]/m_e}, \qquad \Omega_2 = \sqrt{[(\xi + \omega_1^2 m_e) + 2\omega_1\sqrt{\xi m_e}]/m_e},$$

where $\xi = EI\pi^4/2\ell^3$ and $\mathrm{j} = \sqrt{-1}$.

The solution of the equation of motion of the rotating shaft can then be written as

$$\mathbf{q_f} = \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} C_{11}\sin(\Omega_1 t + \phi_{11}) + C_{12}\sin(\Omega_2 t + \phi_{12}) \\ C_{21}\sin(\Omega_1 t + \phi_{21}) + C_{22}\sin(\Omega_2 t + \phi_{22}) \end{bmatrix}, \tag{42}$$

where C_{11}, C_{12}, C_{21}, C_{22}, ϕ_{11}, ϕ_{12}, ϕ_{21}, ϕ_{22} are arbitrary constants which can be expressed in terms of four independent constants using the amplitude ratios (mode shapes), and are determined using the initial conditions specified for the rotating shaft problem. The preceding equation, however, demonstrates that regardless of the value of the angular velocity of the rotating shaft, the solution is stable, and the stability limitation observed when one mode of vibration is considered is eliminated.

With the mathematical expressions demonstrating the coupling effect developed above, numerical integration techniques can be used to obtain the time dependent elastic deflections for a shaft of 0·020 m diameter, modulus of elasticity $E = 2{\cdot}07 \times 10^{11}$ N/m^2, density $\rho = 7700$ kg/m^3, and shaft length $\ell = 1$ m. The equations of motion are generalized to allow the inclusion of more than one mode in each direction with an initial condition of $q_{20} = 0{\cdot}25 \times 10^{-3}$ m deflection assumed for both the in-plane fundamental and second harmonic, in-plane velocity $\dot{q}_{20} = 0$, and out-of-plane initial deflection and velocity of $q_{30} = \dot{q}_{30} = 0$. It is readily seen that for the case of zero coupling between planes and in the absence of any applied force, the out-of-plane response is zero. Figures 3(a–f) show the in-plane deformations, with respect to the time dependent co-ordinate q_2, for angular velocities of $\omega_1 = 2250$ r.p.m. (235·6 rad/s), 10 000 r.p.m. (1047·2 rad/s), and 25 000 r.p.m. (2618 rad/s), respectively. The out-of-plane deformations are 90° out of phase with the in-plane deformations and follow the identical development.

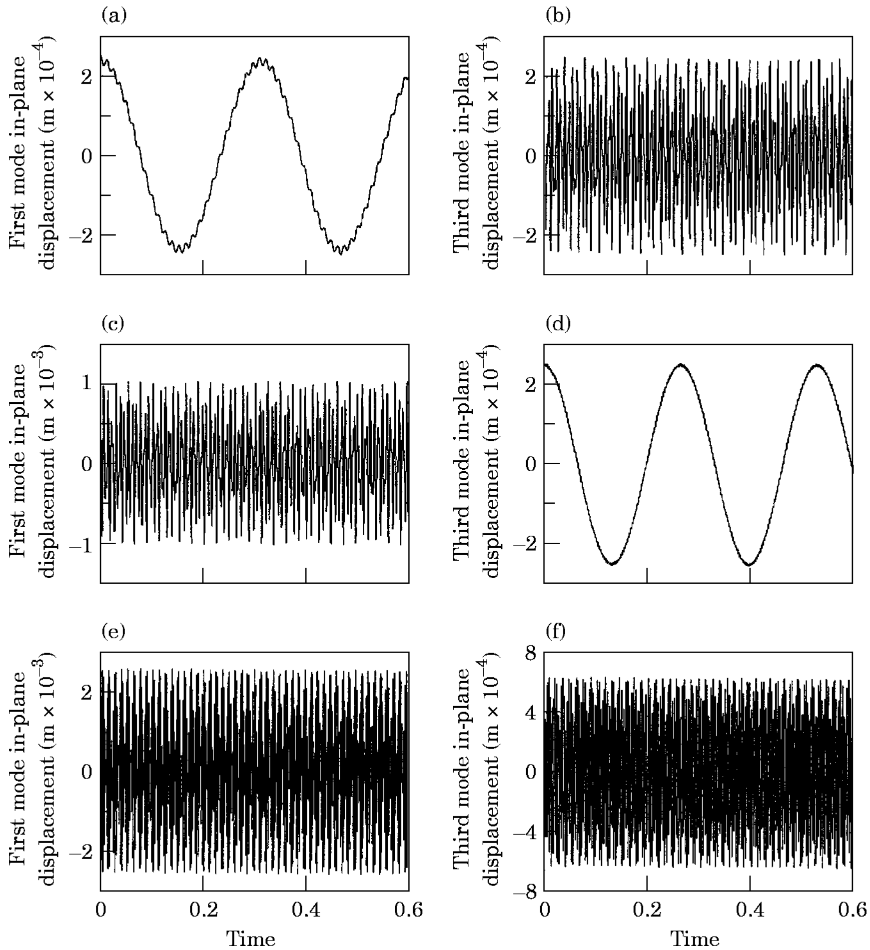

Figure 3. In-plane deflection of a shaft rotating at a constant angular velocity of ω_1 (r.p.m.) (a) 2250, fundamental mode; (b) 2250, second harmonic; (c) 10 000, fundamental mode; (d) 10 000, second harmonic; (e) 25 000, fundamental mode; (f) 25 000, second harmonic.

9. NON-CONSTANT ANGULAR VELOCITY

For the constrained shaft rotating with a specified non-constant angular velocity, the equations of motion, which do not include the effect of the rotary inertia, can be obtained from equation (39) as

$$\begin{bmatrix} m_e & 0 \\ 0 & m_e \end{bmatrix}\ddot{\mathbf{q}}_\mathbf{f} + \begin{bmatrix} 0 & -2\omega_1(t)m_e \\ 2\omega_1(t)m_e & 0 \end{bmatrix}\dot{\mathbf{q}}_\mathbf{f} + \begin{bmatrix} K_e(t) & -\alpha_1(t)m_e \\ \alpha_1(t)m_e & K_e(t) \end{bmatrix}\mathbf{q}_\mathbf{f} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (43)$$

The characteristic equation of the homogeneous system of equation (43) can be expressed as

$$m_e^2 s^4 + [2m_e K_e(t) + 4m_e^2\omega_1^2(t)]s^2 + 4m_e^2\omega_1(t)\alpha_1(t)s + K_e^2(t) + m_e^2\alpha_1^2(t) = 0, \qquad (44)$$

where the stiffness coefficient, angular velocity and angular acceleration are functions of time. It is clear for any specific point in time that the roots of this equation are dependent on the direction of the angular acceleration, and could have a significant effect on the stability of the rotating shaft.

As an example, consider a shaft rotating at an angular velocity given by

$$\omega_1(t) = V_s(1 - \mathrm{e}^{-\tau t}), \tag{45}$$

with angular acceleration

$$\alpha_1(t) = V_s \tau\, \mathrm{e}^{-\tau t}, \tag{46}$$

which describes a velocity function starting at $x = 0$ with $\omega_1(0) = 0$, then exponentially approaches a steady value, V_s, and remains at that value. The acceleration function begins at its maximum value and exponentially decreases to zero over time. The parameter τ controls how fast the shaft reaches the steady state.

Given identical physical parameters and initial conditions as specified for the case of the shaft rotating at a constant angular velocity, the time dependent elastic deflections can be obtained by applying numerical techniques to equation (43). For steady state angular velocity amplitudes of $\omega_1 = 2250$ r.p.m., 10 000 r.p.m., and 25 000 r.p.m., respectively, Figures 4(a–f) represent the magnitude and frequency of the vibration resulting from the given initial conditions. For the case shown in Figures 4(a) and 4(b), where $\omega_1 < \omega_n (\omega_n$ is the natural frequency of the stationary shaft), it is apparent that the ramping function alters the vibration response in the 0·2 second time period but does not reduce the magnitude or the frequency of the vibration. As the angular velocity is increased, such that $\omega_1 > \omega_n$, as seen in Figures 4(c) and 4(d), to a steady state value of $\omega_1 = 10\,000$ r.p.m., there is a reduction in vibration amplitude in the in-plane modes by an order of magnitude, as compared to the constant angular velocity of Figures 3(c) and 3(d), though the frequency remains the same. The second harmonic response is altered, but the amplitude is equal to the applied initial displacement. As the angular velocity is increased further, as shown in Figures 4(e) and 4(f), for the steady state angular velocity of $\omega_1 = 25\,000$ r.p.m., a similar decrease in the amplitude in the fundamental mode is observed. In each case, the maximum amplitude of vibration is equal to the specified initial deflection.

10. SUPPORT EXCITATION

The effect of the base excitation can be examined using the flexible multibody formulation for the rotating shaft problem developed in section 4. The dynamic equations of motion for the case of a specified rigid body translation can be obtained from the general flexible multibody equations as

$$\bar{\mathbf{S}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\ddot{\mathbf{R}} + \mathbf{m}_{\mathrm{ff}}\ddot{\mathbf{q}}_{\mathrm{f}} - 2\omega_1\tilde{\mathbf{S}}\mathbf{q}_{\mathrm{f}} + (\mathbf{k}_{\mathrm{ff}} - \omega_1^2\hat{\mathbf{S}} - \alpha_1\tilde{\mathbf{S}})\mathbf{q}_{\mathrm{f}} = (\mathbf{Q}_{\mathrm{e}})_{\mathrm{f}}, \tag{47}$$

where, for the case of the rotating shaft, each matrix is as previously defined. A similar equation can also be defined if the effect of the rotary inertia is considered. In this case, the shape function $\mathbf{S}$ is replaced by the matrix $\mathbf{S}_t$ defined in the preceding section.

Consider a base motion to be harmonic in the y direction, such that

$$\mathbf{R} = V_b \sin\omega_b t[0 \quad 1 \quad 0]^{\mathrm{T}}, \tag{48}$$

where V_b is the vibration amplitude and ω_b is the frequency of excitation. In the case of a simply supported shaft, one has

$$\bar{\mathbf{S}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\ddot{\mathbf{R}} = \frac{4m_e}{\pi}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\omega_1 t & \sin\omega_1 t \\ 0 & -\sin\omega_1 t & \cos\omega_1 t \end{bmatrix}(-V_b\omega_b^2\sin\omega_b t)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

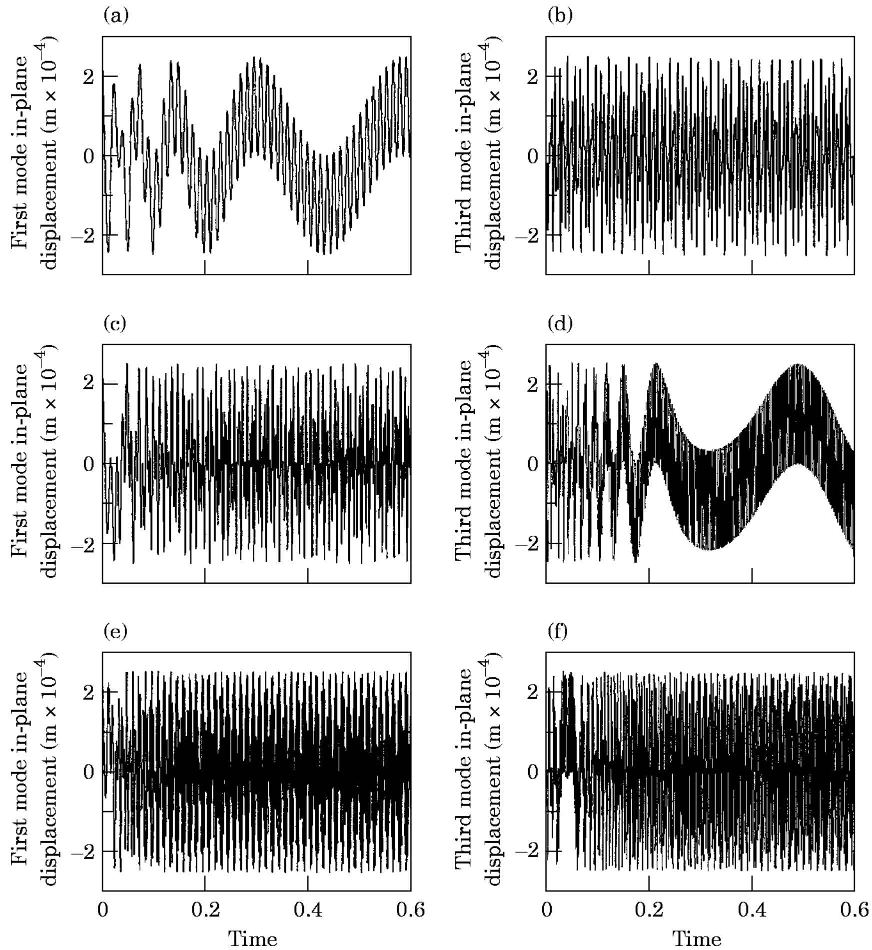

Figure 4. In-plane deflection of a shaft rotating at a non-constant angular velocity $\omega_1 = V_s(1 - e^{-\tau t})$ with $\tau = 12$ and V_s (r.p.m.): (a) 2250, fundamental mode; (b) 2250, second harmonic; (c) 10 000, fundamental mode; (d) 10 000, second harmonic; (e) 25 000, fundamental mode; (f) 25 000, second harmonic.

Using this equation, equation (47) leads to

$$\begin{bmatrix} m_e & 0 \\ 0 & m_e \end{bmatrix}\ddot{\mathbf{q}}_{\mathbf{f}} + 2\omega_1 m_e \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\dot{\mathbf{q}}_{\mathbf{f}} + \begin{bmatrix} K_e & 0 \\ 0 & K_e \end{bmatrix}\mathbf{q}_{\mathbf{f}} = \frac{4m_e}{\pi} V_b \omega_b^2 \sin \omega_b t \begin{bmatrix} -\cos \omega_1 t \\ \sin \omega_1 t \end{bmatrix}, \quad (49)$$

where m_e and K_e are the effective mass and stiffness coefficients defined by equation (41). A similar procedure, as previously applied, can be used to formulate the rotating shaft problem in terms of more fundamental modes and degrees of freedom.

For the case of a low level base excitation of 60 Hz (377 rad/s) and amplitude of 0·005 in ($1{\cdot}27 \times 10^{-5}$ m), this disturbance super-imposes an additional vibration on a shaft rotating at an angular velocity of $\omega_1 = 2250$ r.p.m. For angular velocities of 5000 r.p.m. and greater, the numerical analysis indicates that the low level excitation does not affect the deformations of the rotating shaft.

The effect of a high level base excitation of 1000 Hz (6283 rad/s) and amplitude of 0·005 in ($1{\cdot}27 \times 10^{-5}$ m), is presented in Figures 5(a–f). Figures 5(a) and 5(b) indicate that the high level disturbance has a significant effect on the shaft rotating at 2250 r.p.m. The deformation has a super-imposed excitation increasing the amplitude of vibration by a factor of 4 relative to the non-translating case shown in Figures 3(a) and 3(b). As the angular velocity of the shaft increases, the high level disturbance has less of an impact on the shaft's vibrational amplitude. For a shaft rotating at 5000 r.p.m. [17], the amplitude of the vibration increases by a factor of 2. Figures 5(c) and 5(d) represents the response of a shaft rotating at 10 000 r.p.m. to the 1000 Hz excitation. The amplitude of vibration is observed to increase by a factor of 1·5. For a shaft rotating at an angular velocity of 25 000 r.p.m., shown in Figures 5(e) and 5(f), the amplitude is approximately the same as the case with no base excitation. It is apparent, as the rotational speed of the shaft increases, the effect of the base excitation diminishes. The out-of-plane deformations exhibit an identical vibration response, though displaced 90°, as demonstrated in the in-plane deformation discussed in this section.

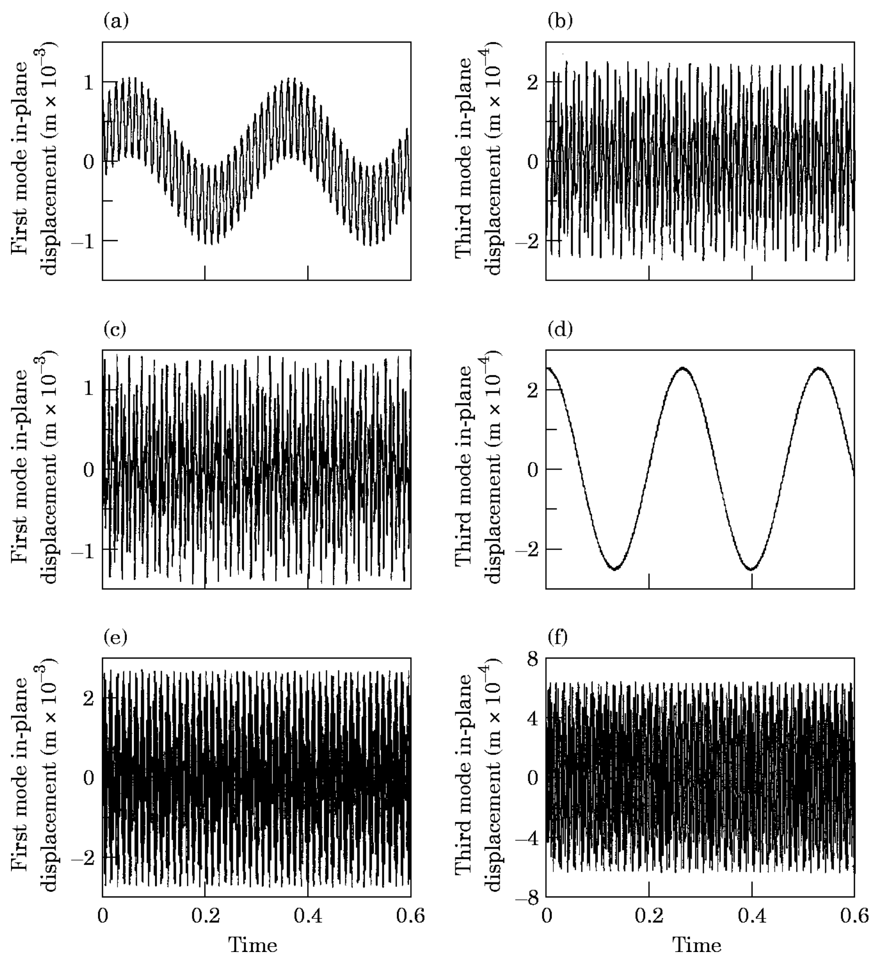

Figure 5. In-plane deflection of a rotating shaft subject to a high level base excitation of 1000 Hz, rotating at a constant angular velocity of ω_1 (r.p.m.): (a) 2250, fundamental mode; (b) 2250, second harmonic; (c) 10 000, fundamental mode; (d) 10 000, second harmonic; (e) 25 000, fundamental mode; (f) 25 000, second harmonic.

11. SUMMARY AND CONCLUSIONS

In this investigation, the dynamics of rotating shafts was examined. The classical equations of rotating shafts were presented, and the limitations of the classical theory were outlined. The general flexible body equations of motion were presented and also tailored to the rotating shaft problem. It was shown that the general flexible body formulation can be used to examine more general cases of rotating shaft applications and can be used to systematically include the effects of the change in the angular velocity, the effect of the rotary inertia, and the effect of support vibrations which cannot be examined using the classical approach. It was demonstrated that the centrifugal inertia forces, which are proportional to the square of the angular velocity, tend to reduce the system stiffness coefficients. Such a system becomes unstable if the effect of the coupling between the in-plane and out-of-plane bending is neglected. It was demonstrated in this investigation that the coupling between the in-plane and out-of-plane bending modes is the result of the Coriolis inertia forces and has a significant effect on the stability of the rotating shaft. In order to illustrate the generality of the flexible multibody formulation, a case study was presented. The closed form solution for a stationary shaft with simply supported end conditions is used to describe the modes of deformation of the rotating shaft. It was shown that these equations also allow the use of a non-constant angular velocity.

It was also shown that the effect of the rotary inertia can be systematically incorporated in the flexible multibody formulations. All that is required is a change in the form of the shape function matrix in order to include the effect of the rotation of the cross-section.

The effects of a base excitation, common to machining shafts with loose or misaligned supports, on the dynamics of the rotating shaft was also examined. Numerical results were presented for different conditions of the base excitation in order to demonstrate the generality of the flexible body formulation.

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NOMENCLATURE

A	cross-sectional area (m^2)
$\mathbf{A}$	three-dimensional rotation matrix which defines the orientation of the body in the global system
$\mathbf{A}_r$	matrix which describes the infinitesimal rotation of the cross-section of the shaft with respect to the shaft co-ordinate system
$\boldsymbol{\alpha}$	vector of angular accelerations
α_1	angular acceleration of the shaft in the x direction (rad/s^2)
β	mode shape
$\mathbf{D}$	damping matrix
ρ	density (kg/m^3)
$\delta(x - vt)$	Dirac delta function; v is the linear velocity of the moving force
$\mathbf{E}$	matrix of elastic co-ordinates
E	Young's modulus of elasticity (N/m^2)
$\boldsymbol{\epsilon}$	vector of strains
F	magnitude of the external force (N)
$\bar{\mathbf{G}}$	matrix which relates the angular velocity vector to the time derivatives of the orientation co-ordinates
$\bar{\mathbf{H}}$	vector of angular momentum
$\mathbf{I}_{\theta\theta}$	inertia tensor
$\dot{\bar{\mathbf{I}}}_{\theta\theta}$	deformation dependent submatrix associated with the Coriolis inertia force in the orientation co-ordinate
$\bar{\mathbf{I}}_{\theta f}$	deformation dependent submatrix associated with coupling between the rotation and the elastic deformation
$\bar{\mathbf{I}}_{0t}$	time invariant vector associated with the undeformed co-ordinates when the effect of rotary inertia is considered
$\tilde{\bar{\mathbf{I}}}_0$	time invariant submatrix associated with the coupling between the translation and rotation of the co-ordinate system
$\bar{\mathbf{I}}_0$	time invariant vector associated with the undeformed co-ordinates
$\mathbf{K}$	stiffness matrix
K_e	effective stiffness coefficient
$\mathbf{k}_{ff}$	stiffness matrix associated with the elastic co-ordinates of the deformable body
ℓ	length (m)
$\mathbf{M}$	mass matrix
m	mass (kg)
m_e	effective mass coefficient
$\mathbf{m}_{RR}$	submatrix associated with the translation of the body reference
$\mathbf{m}_{R\theta}$	submatrix which represents the coupling between the translation and rotation of the co-ordinate system
$\mathbf{m}_{\theta\theta}$	submatrix associated with the orientation of the body; a function of the inertia tensor $\bar{\mathbf{I}}_{\theta\theta}$
$\mathbf{m}_{Rf}$	submatrix which describes the coupling between the translation and the elastic deformation
$\mathbf{m}_{\theta f}$	submatrix which describes the coupling between the rotation and the elastic deformation
$\mathbf{m}_{ff}$	submatrix associated with the deformation co-ordinates of the body
Ω	vector of rate of precession
$\mathbf{Q}$	matrix of generalized applied forces
$\mathbf{Q}_e$	vector of generalized forces associated with the body generalized co-ordinates
$\mathbf{Q}_v$	vector consisting of the centrifugal and Coriolis inertia forces
$\mathbf{q}$	vector of time dependent co-ordinates
$\mathbf{q}_f$	vector of time dependent elastic co-ordinates of the deformable body
$\mathbf{r}$	global position vector
$\mathbf{R}$	vector which defines the global position of the body co-ordinate system
$\mathbf{S}$	matrix of position dependent shape functions
σ	vector of stresses
$\mathbf{S}_t$	shape function matrix including the effects of the rotary inertia
$\boldsymbol{\theta}$	vector of orientation co-ordinates
θ	angle of rotation of the shaft (radians)
u_1	displacement in the x direction
u_2	displacement in the y direction
u_3	displacement in the z direction
$\bar{\mathbf{u}}$	vector which defines the co-ordinates of the arbitrary point with

	respect to the body co-ordinate system
$\bar{\mathbf{u}}_0$	vector describing the position of the arbitrary point in the undeformed state
$\bar{\mathbf{u}}_{\mathbf{f}}$	time and space dependent deformation vector
δW_i	virtual work of the inertial force
δW_e	virtual work of the externally applied force
δW_s	virtual work of the elastic force
$\boldsymbol{\omega}$	vector of angular velocities
$\tilde{\bar{\omega}}$	skew symmetric matrix associated with the angular velocity vector $\bar{\omega}$
ω_1	angular velocity of the shaft in the x direction (r.p.m., rad/s)
ω_n	natural frequency of the non-rotating shaft (r.p.m., rad/s)
ψ	vector of angular displacements of the rotating shaft