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DAMAGE DETECTION USING A MODIFIED LAPLACIAN OPERATOR ON MODE SHAPE DATA

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Localized damage to a structure affects its dynamic properties, and much work has been undertaken investigating the variation of natural frequencies with damage. However, use of mode shape data has seen much less effort. This paper develops and presents a technique for identifying the location of structural damage in a beam. The procedure operates solely on the mode shape from the damaged structure, and does not require *a priori* knowledge of the undamaged structure. The procedure is developed using a one-dimensional finite element model of a beam, and demonstrated by experiment. When damage is severe (a localized thickness reduction of more than 10%), applying a finite difference approximation of Laplace's differential operator to the mode shape successfully identifies the location of the damage. However, when damage is less severe, further processing of the Laplacian output is required before the location can be determined. This post-processing enables the location of thickness reductions of less than 0·5% to be identified. The procedure is best suited to the mode shape obtained from the fundamental natural frequency. The mode shapes from higher natural frequencies can be used to verify the location of damage, but they are not as sensitive as the lower modes.

1. INTRODUCTION

When a structure suffers localized damage, its dynamic properties can change. Specifically, crack damage can cause a stiffness reduction, with an inherent reduction in natural frequencies, an increase in modal damping, and a change to the mode shapes. There has been a significant effort to detect the location of cracks using one or more of these characteristics. The most easily observable change is the reduction in natural frequencies, and most of the reported effort uses this feature in one way or another. Varying success is reported using the change in modal damping, and little work is reported on using the change in mode shape to detect the location of damage.

Much of the reported work using modal analysis for damage detection concerns trusses and frameworks and is generally targeted at offshore oil platforms and space structures such as satellites and the space station. On many steel platforms the changes in natural frequencies caused by the failure of a single, structurally important member can be large enough to be detected, and can be used to identify damage [1–10]. For trusses, the change in pole/zero information caused when an "at risk" member fails can be analyzed using a neural network; see Manning [11]. Alternatively, Liu [12] derived an optimization program in which the error norm of the eigenequation is minimized. Some authors employ both natural frequency changes and mode shape information. Chen and Garba [13] investigated the variation of modal amplitude caused by damage and used kinetic and potential energy distribution to identify damage. Li and Smith [14] adjusted the physical properties of a truss in a finite element model, and Adams *et al*. [15] used natural frequency shifts and anomalies in mode shapes as damage indicators. Lim and Kashangaki [16] identified

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damage with a multiple mode analysis. Finite element and experimental data for a scale modal of an offshore platform were presented by Shahrivar and Bouwkamp [17], who concluded that the fundamental mode shape was more sensitive to damage than the fundamental vibration frequency.

Damage detection using changes in damping has been investigated. The effect of debonding on the modal damping of a sandwich panel was investigated by Peroni *et al*. [18], who showed that for some modes, damage caused a slight increase in damping coefficient. Lai and Young [19] reported extensive work on a composite structure, including the effect of high temperatures and prolonged exposure to humidity. They showed that while delamination decreases the natural frequency of the fundamental mode, it increases the damping. However, they concluded that change in damping coefficient was an unreliable parameter on which to base a damage detection algorithm. Vantomme [20] correlated modal parameters and the accumulated damage of composite joints of a stiffened plate structure and concluded that modal frequency measurements were more suitable than modal damping measurements.

Some researchers have considered using a spatial stiffness matrix to locate damage; however, Lin [21] observed that higher modal frequencies contribute to the spatial stiffness matrix values to a greater extent than lower ones—to obtain a good estimate of the spatial stiffness matrix one needs to measure all the modes of the structure, especially the high frequency modes. This presents a problem for stiffness-based damage detection methods relying on experimental modal data, and various techniques and degrees of success are reported [22–25]. A variation is to perturb the stiffness matrix of a finite element model and compare the resulting changes in natural frequencies with observations, Hearn and Testa [26]. An alternative to the stiffness matrix, considered by Pandey and Biswas [27], is to consider the flexibility matrix, which converges better on increasing frequency.

Compared to trusses and space frames, there is very little reported work on beams and plates, and most is designed for concrete structures, such as bridges, and composites. When a bridge is damaged, a change in natural frequencies is one of the most observable effects, and a committee report by Javor [28] gives international technical guidelines that recommend monitoring the fundamental frequency for the long-term observation of such structures. Miller *et al*. [29] reported on a reinforced concrete bridge, and compared experimental and finite element results to identify damage to the shoulders of the bridge using differences in natural frequencies, and anomalies in mode shapes. Casas and Aparicio [30] minimized a scalar performance error, which included the sum-square difference between footprint and measured mode shapes and natural frequencies. They concluded that the measurement of only one mode shape was not sufficient to distinguish structural damage. The use of a neutral network to find the location and size of delamination in a composite panel from changes in natural frequencies was considered by Okafor *et al*. [31].

Other approaches to damage detection include that of Rizos *et al*. [32] who developed a method based on the amplitudes at two points in a structure vibrating at one of its natural frequencies and an analytical solution of the dynamic response, and Springer *et al*. [33] who used variations in natural frequency to identify damage in members that can be modelled as longitudinally vibrating beams. Doyle [34] considered a structure to be a collection of multiply connected waveguides, and investigated an iterative procedure to detect cracks in beams.

Most of the techniques use changes in natural frequencies. Some use mode shape information, although often this is only to verify the same natural frequency is being considered. Yuen [35] investigated the systematic change in the fundamental mode shape for a cantilever with respect to the location of the damage. The use of strain measured mode shapes has been considered by several authors. Yao *et al*. [36] concluded that strain measured mode shapes were more effective at identifying the location of damage than displacement mode shapes for damage in a steel truss. This was also found by Chang *et al*. [37], who investigated the sensitivity of modal parameters to damage and compared strain measured mode shapes with displacement mode shapes. This finding is consistent with a study by Pandey *et al*. [38], who investigated the change in curvature of mode shape for a damaged beam. Curvature is proportional to the surface strain, and Pandey showed that the difference between curvature mode shapes for an intact and damaged beam could find a localized change in elastic modulus of about 30%.

Cumulatively, the published work suggests that a change in natural frequency is the single most effective dynamic indicator of structural damage. However, locating the damage is not a simple matter. Nearly all the published work requires reference to an undamaged dynamic model of some form. Most often this is a validated finite element model, although some work refers to a model obtained from an undamaged structure. Requiring this footprint can limit the application of these techniques. This paper presents a technique that solely uses the mode shape obtained from a damaged beam; no footprint or natural frequency information are required. The first part of the identification procedure is to apply a Laplacian operator to the discretely measured mode shape. When damage is severe, this single step is sufficient to identify the location of the damage. However, when damage is less severe, further processing of the Laplacian yields a damage detection procedure that can identify a localized thickness change as small as 0·5%. This procedure is developed with finite element models of a cantilever and a free–free beam, and demonstrated with the results from an experimental modal analysis of a beam.

2. THE LAPLACE OPERATOR

Cracks and other forms of localized damage in a beam can introduce a reduction in the flexural stiffness (EI), but minimal change to the mass. For the uniform, rectangular cross-section beam considered here, localized stiffness damage can be introduced by reducing the thickness for one element of the finite element model, but leaving the mass matrix unchanged. This approach is consistent with previously published work, including references [9] and [38]. The percentage reduction in thickness is called the percentage damage applied to the beam. For the numerical examples, a model with a reduced matrix size of 20 was used, and although this damage detection procedure does not require natural frequency information, for interest Figures 1 and 2 show the variation of natural frequencies with damage.

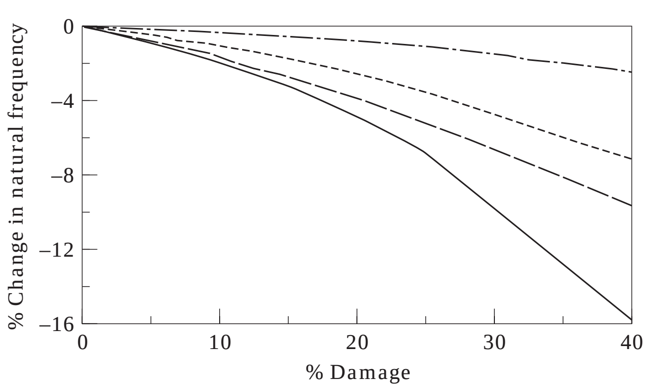

Figure 1. Effect of damage on natural frequencies for a free–free beam; ——, mode 1; -----, mode 2; —-—, mode 3; ———, mode 4.

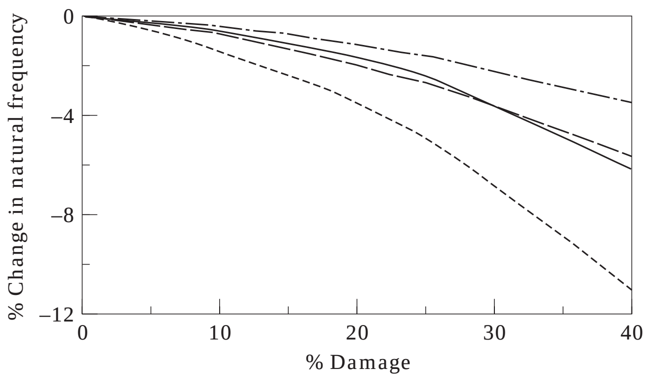

Figure 2. Effect of damage on natural frequencies for a cantilever: key as for Figure 1.

Localized changes to EI result in a mode shape that has a localized change in slope. Experimental mode shape data are discrete in space, and therefore the change in slope can be estimated using a finite difference approximation. The Laplacian difference equation [39] is a common method used to calculate an estimate of the second difference of a discrete function, but it is normally applied to problems involving two dimensions. A beam can be analyzed as a one-dimensional structure, and in this case the one-dimensional Laplacian, $\mathscr{L}_i$, of the discrete mode shape, y_i, is given by

$$\mathscr{L}_i = (y_{i+1} + y_{i-1}) - 2y_i. \tag{1}$$

A Laplacian calculated using equation (1) and the shape for the first bending mode of a finite element free–free beam with 50% damage between nodes # 7 and # 8 is shown in Figure 3. This level of damage is severe, and also causes a noticeable anomaly in the mode shape. When the damage is less severe, the Laplacian retains its characteristic shape, but the effect is less pronounced. This is shown, for 5% damage to the same beam, in Figure 4. The Laplacian has a similar shape and identifies damage in a similar fashion to the curvature shapes in reference [38]. The main difference is that reference [38] considers the difference in curvature between undamaged and damaged beams, whereas this study only considers the damaged model.

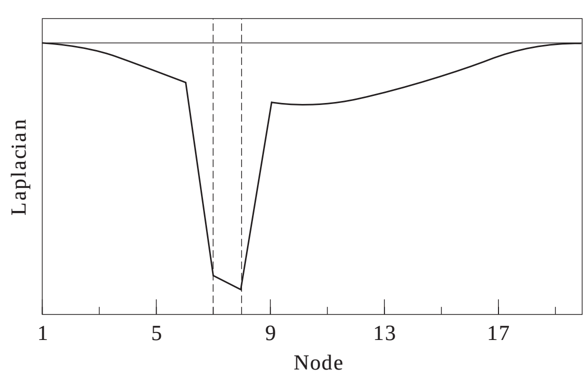

Figure 3. Laplacian for 50% damage.

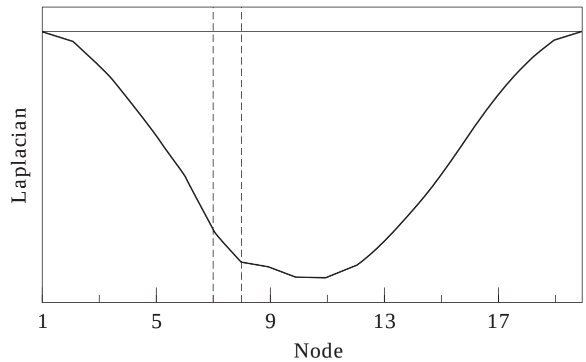

Figure 4. Laplacian for 5% damage.

3. MODIFYING THE LAPLACIAN

As the level of damage further reduces, the distinctive shape of the Laplacian continues to become less pronounced. It still holds the information required to locate the damage, but the location is not immediately apparent. Various methods of enhancing the discontinuity in the graph, such as piecewise linearization and cubic spline, were tried. The method that was most effective was to fit a cubic polynomial to the Laplacian, and calculate a difference function between the cubic and Laplacian. A separate cubic was determined for each element of the Laplacian in turn, with the coefficients being determined from the data on either side of the element, but excluding the actual element. For example, the cubic calculated for the ith element of the Laplacian, $\mathscr{L}_i$, at position x_i along the beam, was defined as

$$a_0 + a_1 x_i + a_2 x_i^2 + a_3 x_i^3. \tag{2}$$

The coefficients a_0, a_1, a_2 and a_3 were determined using Laplacian elements:

$$\mathscr{L}_{i-2}, \qquad \mathscr{L}_{i-1}, \qquad \mathscr{L}_{i+1}, \qquad \mathscr{L}_{i+2}. \tag{3}$$

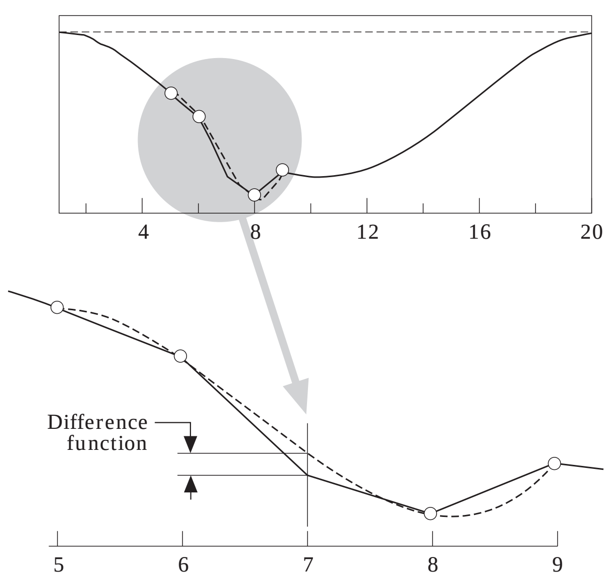

Figure 5. Calculation of the difference function for Node # 7: —, Laplacian; ○, data for cubic; ----; cubic.

The difference function, δ_i, was calculated from the cubic and the Laplacian:

$$\delta_i = (a_0 + a_1 x_i + a_2 x_i^2 + a_3 x_i^3) - \mathscr{L}_i. \tag{4}$$

This calculation is shown graphically for $i = 7$ in Figure 5. The difference function for 5% damage between nodes # 7 and # 8 is shown in Figure 6. While the difference function does not identify the location of damage as accurately as the Laplacian, it is still sufficiently detailed to suggest where damage may be present. For example, Figure 6 suggests damage somewhere between nodes # 5 and # 10. This "smearing" of location is a result of the difference function calculation.

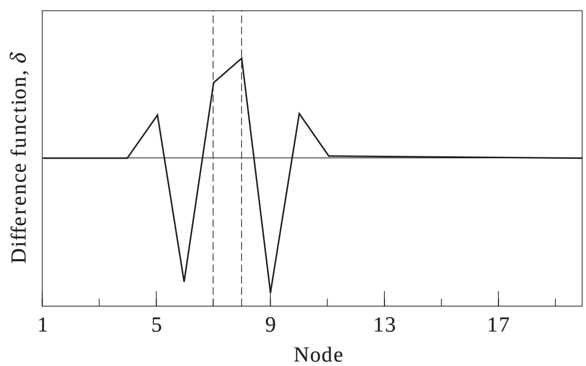

Figure 6. Difference function δ for 5% damage (free–free).

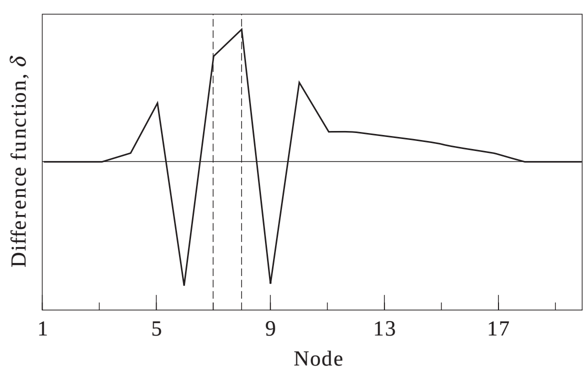

Figure 7. Difference function δ for 0·5% damage (free–free).

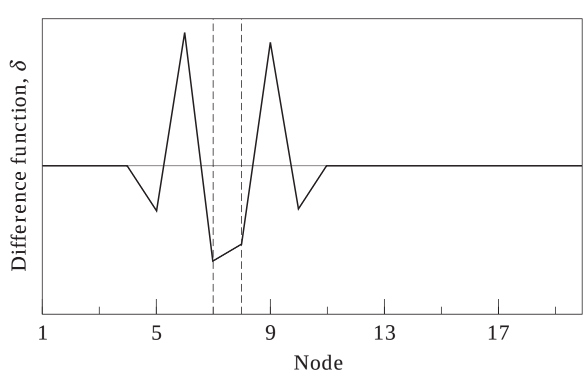

Figure 8. Difference function δ for 0·5% damage (cantilever).

4. SENSITIVITY

4.1. SEVERITY OF DAMAGE

As an indication of the potential for identifying small amounts of localized damage, Figure 7 shows the difference function for 0·5% damage between node # 7 and # 8 of a free–free beam. This damage is equivalent to locally reducing the thickness of the 5 mm beam to 4·975 mm. Figure 8 shows the difference function for a cantilever with the same damage. Note that the difference functions are scaled to the full size of the graph—absolute values are not considered here. The unprocessed Laplacian can identify thickness variations of about 10%. However, the difference function can identify the location of less than 0·5% damage.

4.2. MODE NUMBER

An important consideration for experimental mode shape data is the effect of increasing the mode number. Usually, the higher the mode number, the more difficult it is to measure and accurately identify a mode shape. Figures 9–12 show the difference function for several natural frequencies for a cantilever with 10% damage. The figures show that the difference function for the fundamental gives the strongest indication of damage. Mode shapes for higher natural frequencies are not as effective, but can still be used to verify the location of damage. The findings are similar for a free–free beam.

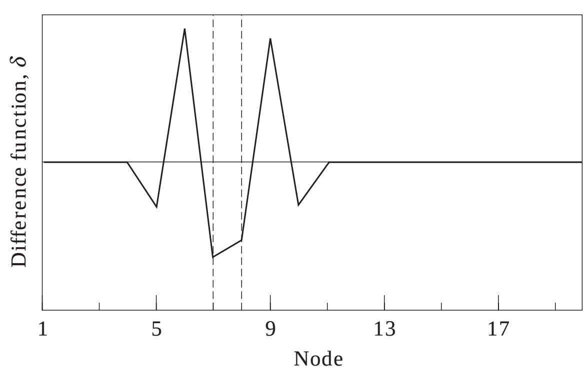

Figure 9. Cantilever mode 1 difference function δ for 10% damage.

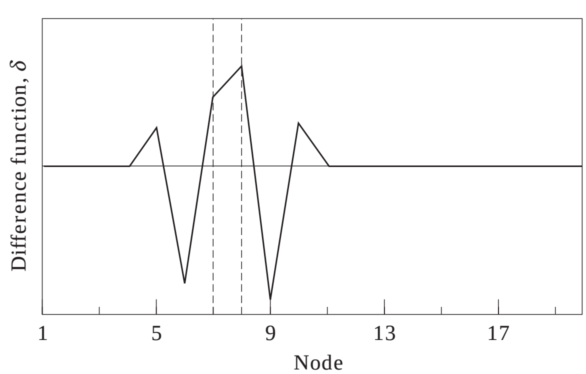

Figure 10. Cantilever mode 2 difference function δ for 10% damage.

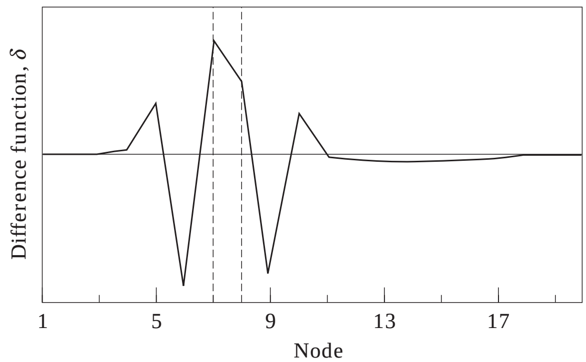

Figure 11. Cantilever mode 3 different function δ for 10% damage.

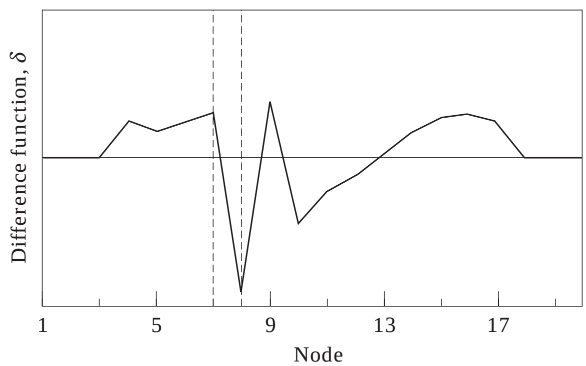

Figure 12. Cantilever mode 4 difference function δ for 10% damage.

4.3. DAMAGE LOCATION

Calculation of the difference function is most effective when there are at least two spatial co-ordinates on each side of the damage location. When damage is very close to either end of the beam the "end effects" caused by the numerical operations can partly obscure the characteristic shape of the difference function. When damage is not near an end, the sensitivity of the procedure is almost independent of the damage location. This is shown in Figures 13 and 14 for a cantilever with 10% damage applied toward the root and tip, respectively.

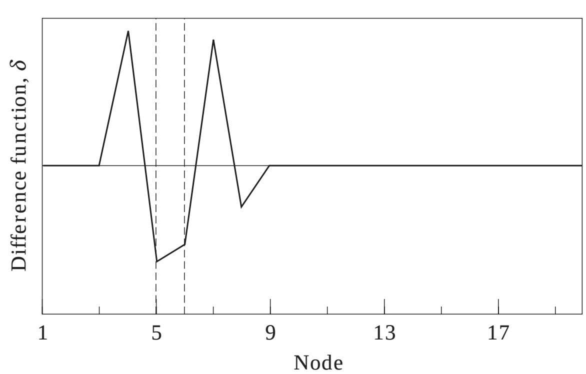

Figure 13. Cantilever difference function δ for 10% damage near root.

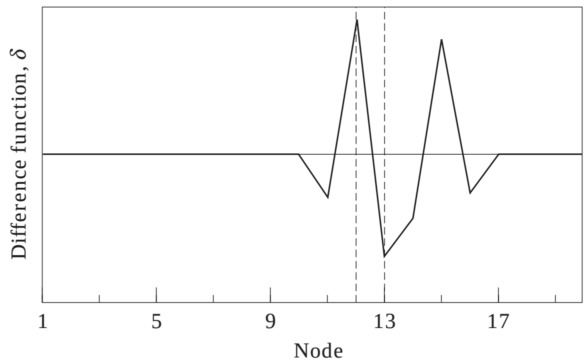

Figure 14. Cantilever difference function δ for 10% damage near the tip.

5. STRAIN MODAL DATA

The analysis conducted so far has considered displacement mode shapes. There is a rising interest in using strain gauges to measure mode shapes [40–42], and the surface strain can be related to the radius of curvature, R, of the beam. For pure bending:

$$\varepsilon = t/2R. \tag{5}$$

When displacements are small, the radius of curvature can be approximated by

$$R = |(1 + (\mathrm{d}y/\mathrm{d}x)^2)^{3/2}/(\mathrm{d}^2y/\mathrm{d}x^2)| \approx |1/(\mathrm{d}^2y/\mathrm{d}x^2)| = |1/\mathscr{L}|. \tag{6}$$

Hence the Laplacian is proportional to the surface strain:

$$\mathscr{L} = \mathrm{d}^2y/\mathrm{d}x^2 \approx 1/R = (2/t)\,\varepsilon. \tag{7}$$

The experimental significance is that the Laplacian can be obtained directly from a measured strain mode shape, which eliminates the need to calculate it from displacement data. This potentially offers a significant improvement in the sensitivity of the procedure presented in this paper, and helps explain the findings in references [36] and [37] that strain mode shapes are more effective as damage location indicators.

6. EXPERIMENTAL DEMONSTRATION

A flat steel beam, approximately 0·6 m × 0·25 m × 4 mm thick (24 in × 10 in × 1/6 in) was suspended through two small holes with S-hooks and rubber cords. The beam had very light damping, and therefore, to make it more representative of an engineering

TABLE 1

Effect of damage on the natural frequencies and viscous damping ratios

	Natural frequency (Hz)		Viscous damping ratio (%)	
Mode	Undamaged	Damaged	Undamaged	Damaged
1	60·82	54·82	2·21	2·45
2	167·65	151·88	0·81	0·90
3	328·97	324·86	0·42	0·43
4	544·02	509·65	0·26	0·29
5	811·01	755·61	0·18	0·20

structure, free layer damping was applied to one side. This gave the beam a damping ratio in the range 0·5–2%. A uniform 20 × 7 mesh of co-ordinates was marked on the beam, which was damaged by cutting a through-thickness slot (the thickness of a saw blade) in the middle, across approximately half the width, and 0·2 m from one end. This introduced a stiffness change, with minimal effect on the mass of the plate. The test method was impact excitation, referenced to a fixed accelerometer, and the 140 frequency response functions were recorded and subject to a modal analysis of the first five natural frequencies. For comparison only, the natural frequencies and damping ratios before and after damaging the beam are shown in Table 1. For all modes there is a small increase in damping, although the increase is within the bounds of experimental error. All the natural frequencies are reduced, and a comparison with the equivalent data for the finite element beam, Figure 1, suggests the slot caused stiffness damage comparable to that produced by about 30% damage.

The mode shapes for the first two natural frequencies, and their Laplacian and difference functions, are shown in Figures 15–20. These results are consistent with the finite element

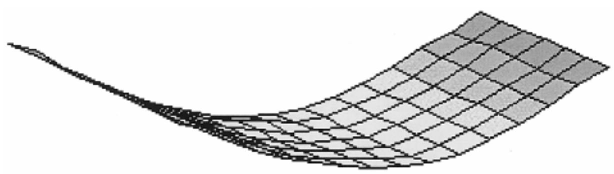

Figure 15. Experimental mode 1.

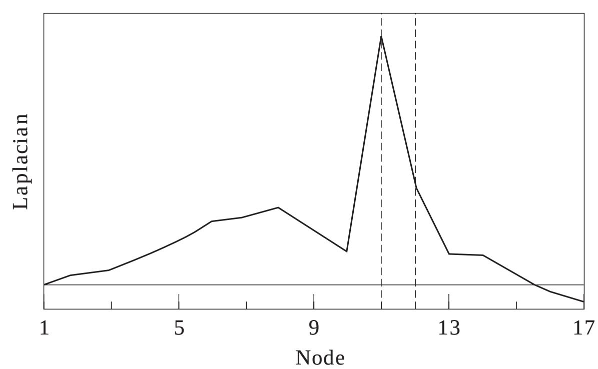

Figure 16. Laplacian for mode 1.

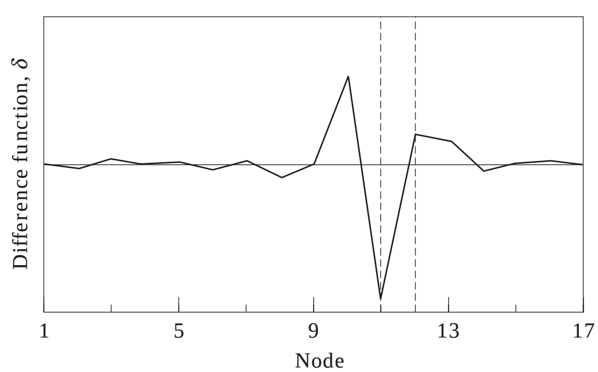

Figure 17. Difference function δ for mode 1.

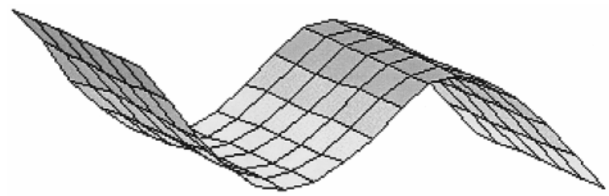

Figure 18. Experimental mode 2.

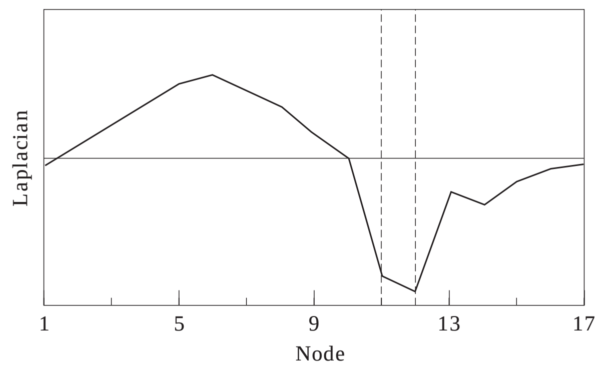

Figure 19. Laplacian for mode 2.

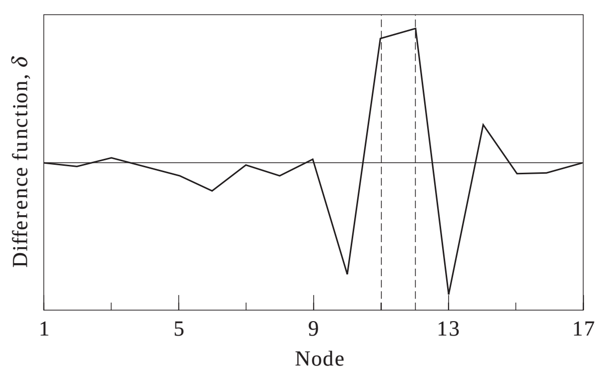

Figure 20. Difference function δ for mode 2.

findings, and the Laplacian has identified the location of the damage. However, the difference function gives a clearer indication of the damage location.

7. DISCUSSION AND CONCLUSIONS

A finite difference Laplacian function can successfully be used to identify the location of stiffness damage of as little as about 10% in an otherwise uniform beam. When damage is less severe, further processing of the Laplacian can locate damage of less than 0·5%. The post-processing consists of determining a cubic polynomial to fit the Laplacian locally at each spatial co-ordinate. A difference function between the cubic and Laplacian provides the information necessary to identify the location of damage. Mode shape data from the fundamental mode are most suited to the technique. However, data from the next three or four natural frequencies can be of use, particularly for verification of the results from the fundamental. The findings were supported by experiment, where a slot cut into a steel beam was successfully found.

The Laplacian function represents the curvature of the mode shape. This is proportional to the surface strain on a beam. This suggests that modal data obtained using strain gauges may be used directly in place of the Laplacian. While this was not verified experimentally, measuring strain mode shapes potentially may further improve the sensitivity of the proposed damage detection procedure.

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