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EXACT MIMICRY OF NONLINEAR OSCILLATORY POTENTIAL MOTION: NONUNIQUENESS OF ISODYNAMICAL TRACKS

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The relationship is investigated between a one-dimensional potential and a track in a vertical plane along which a bead is constrained to slide freely under the influence of gravity, such that the motion of the bead, projected onto the horizontal axis, is exactly the same as (i.e., is isodynamical to) the one-dimensional oscillatory motion due to the potential. For a given potential, the isodynamical track is specified in terms of a non-linear first-order ordinary differential equation which depends on the amplitude, and whose relevant solutions may be neither unique nor smooth. Several cases of quadratic and quartic convex functions are solved numerically and displayed. For a given amplitude of oscillation, only the track shape of minimum height is smooth at the origin. The track shapes isodynamical to a double-well (Duffing oscillator) potential for the symmetrical cross-well oscillations are all found to have a kink at the origin. Corresponding to a V-shaped potential, there is a variety of track shapes including one of minimum height which is smooth at the origin.

1. INTRODUCTION

In a previous paper [1], the problem of finding a one-dimensional potential, isodynamical to (i.e., producing exactly the same time dependence of displacement as) the horizontally projected motion of a particle moving on a given vertically planar curved track under the influence of gravity, was solved. In this paper, the converse problem is addressed: given the potential, what is the shape of the isodynamical track?

These problems were motivated by the theory and experiments of Gottwald *et al*. [2], who constructed a track with a quartic-minus-quadratic double-valley shape, upon which moved a "cart". It was shown in [1] that the isodynamical potential to this for free undamped large amplitude oscillations was actually a triple-well function.

Subsequent to the work of [2], Shaw and Haddow [3] were apparently the first to tackle systematically the problem of choosing a track shape to achieve a desired oscillator equation. However, they concentrated on the acceleration equation, which more naturally incorporated the damping and driving forces which were crucial to the bulk of the work of Gottwald *et al*. [2]. By contrast, the present paper is concerned strictly with the freely moving situation as in the earlier part of [2], i.e., with a given one-dimensional *potential* problem. Thus the procedure here is via the energy (first integral) equation: the resulting simple relationship (equation (2.3b) in the following) seems to have been overlooked in [3], but forms the basis of the present work.

Moveover, as stated in the Discussion in Reference [3], that work emphasised the use of the arc length co-ordinate, with motion along the track having the desired oscillator

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equation. By contrast, the present paper specifies the motion projected onto the horizontal Cartesian co-ordinate as the direct analogy of the given one-dimensional potential motion. Thus, amongst the interesting examples detailed in Shaw and Haddow [3], those examples common with this paper, viz., harmonic oscillator and Duffing double well potentials, exhibit substantial differences in the appearances of the track shapes, not the least of which are the non-uniqueness (even for given amplitude) and the possibility of a kink at the origin in the results of this paper.

Unlike the first problem [1] which generated the isodynamical potential as an explicit formula involving the given track shape function and its derivative, the problem of finding the isodynamical track shape from a given potential will be shown to involve the solution of a first order non-linear ordinary differential equation. This requires a numerical approach, and moreover there may be non-unique solutions corresponding to a range of track "heights" at fixed amplitude. Thus there may be a family of tracks isodynamical to the potential *and* with each other.

With reference to [2], there does not appear to be any completely smooth track shape curve which is isodynamical to a Duffing double-well potential for the symmetric "large orbit" oscillations: a V like kink in the track shape is always found at the origin.

Examples are also given for tracks corresponding to various quadratic and quartic potentials with positive curvature. Whilst there may be isodynamical track curves with a kink at the origin, for a given potential of these types and given amplitude of oscillation there is a unique isodynamical track curve, of minimum height, which has a continuous derivative (zero slope) minimum, i.e., it is smooth at the origin.

Finally, non-linear tracks of various shapes isodynamical to V-shaped potential and tracks are found. An Appendix deals with a simpler non-linear comparison differential equation which exhibits non-unique solutions but possesses some known exact analytical solutions which consequently help in the understanding of the more complicated differential equations in the body of the paper.

2. PATHS CORRESPONDING TO A GIVEN POTENTIAL

One-dimensional oscillatory motion for a particle of mass m due to a potential $v(x)$ is described by (equation (2.2b) of [1])

$$\dot{x}^2 = (2/m)[v(a) - v(x)], \tag{2.1}$$

where an overdot denotes differentiation with respect to time, and $\dot{x} = 0$ at $x = a$ (the amplitude: $a > 0$). For the horizontal co-ordinate X-motion of some other particle sliding freely along a curved track with shape equation $Y = Y(X)$ (geometric path in a vertical plane) under the influence of gravity (constant g) acting vertically downwards, the following equation was found ((2.4b) in [1]):

$$\dot{X}^2 = 2g[Y(A) - Y(X)]/[1 + (Y'(X))^2], \tag{2.2}$$

where $\dot{X} = 0$ at $X = A > 0$, and a prime denotes differentiation with respect to X. (Upper case co-ordinates refer to this constrained two-dimensional motion along the track.)

If the potential $v(x)$ and amplitude a are given, then a curve $Y(X)$ whose projected motion $X(t)$ onto the X-axis is isodynamical to $x(t)$ must, by equations (2.1) and (2.2) with $A = a$, satisfy the ordinary first order non-linear differential equation

$$(Y'(X))^2 + 1 = mg[Y(a) - Y(X)]/[v(a) - v(X)]. \tag{2.3a}$$

ie.,

$$\mathrm{d}Y/\mathrm{d}X = \pm[(Y(a) - Y(X))/(\hat{v}(a) - \hat{v}(X)) - 1]^{1/2}, \tag{2.3b}$$

where

$$\hat{v}(x) = v(x)/(mg). \tag{2.3c}$$

In general, equation (2.3b) is not first order separable, and has an (integrable) singularity as $X \to a - 0$. Because of the form and non-linearity of this d.e., the solutions in general depend on the amplitude a.

The special case of constant track slope $Y'(X) =$ constant, i.e., $v'(x) =$ constant, was dealt with in the Appendix to Reference [1]. This provided a simple explicit example of the possibility of non-uniqueness of $Y(X)$ due to the projected motion property. There were two distinct constant track slopes (if unequal to $\pi/4$) yielding motion isodynamical to a constant slope potential. However, this was found under the "direct" framework of Reference [1]. Solution of the "converse" problem (2.3), undertaken later in this paper (section 5), shows that, corresponding to $v'(x) =$ constant, there is a continuum of isodynamical track shapes $Y(X)$ which have negative, zero, or positive curvature depending on their "height" at a specified amplitude.

It may be assumed in general that axes are chosen such that

$$\hat{v}(0) = 0 = Y(0). \tag{2.4}$$

The common case that $v'(0) = 0$ does not necessarily imply that $Y'(0) = 0$. If $v(x)$ is symmetrical about the origin then evidently so is $Y(X)$, and equation (2.3b) need to be solved only in $0 \leqslant \mathrm{X} \leqslant \mathrm{a}$. In that case, for a track solution which is smooth at the origin, $Y'(0) = 0$ by equation (2.3b). Then, via equation (2.4), a necessary and sufficient condition for a track to be smooth at the origin (if a solution exists) is

$$Y(a) = \hat{v}(a). \tag{2.5}$$

In general, setting

$$\tilde{Y}(X) = Y(a) - Y(X) \tag{2.6}$$

there results the differential equation

$$\mathrm{d}\tilde{Y}/\mathrm{d}X = \mp[(\tilde{Y} - [\hat{v}(a) - \hat{v}(X)])/(\hat{v}(a) - \hat{v}(X))]^{1/2}, \tag{2.7}$$

which is to be solved subject (by equation (2.6)) to

$$\tilde{Y}(a) = 0. \tag{2.8}$$

Then, since $Y(0) = 0$,

$$Y(a) = \tilde{Y}(0) \tag{2.9}$$

and so the actual track shape is given by the curve

$$Y(X) = \tilde{Y}(0) - \tilde{Y}(X). \tag{2.10}$$

For *real* solutions,

$$\tilde{Y}(X) \geqslant \tilde{Z}(X), \tag{2.11a}$$

where, by equation (2.7),

$$\tilde{Z}(X) = \hat{v}(a) - \hat{v}(X). \tag{2.11b}$$

$\tilde{Z}(X)$ is not a solution, but by equation (2.7) $\tilde{Y}'(X) = 0$ at points where $\tilde{Y}(X) = \tilde{Z}(X)$. Thus (using equation (2.4)) for real solutions $\tilde{Y}(0) \geqslant \hat{v}(a)$, i.e., $Y(a) \geqslant \hat{v}(a)$, and for a track solution $Y(X)$ smooth at $X = 0$, (by (2.5) and (2.9))

$$\tilde{Y}(0) = \hat{v}(a). \tag{2.12}$$

The first order ODE (2.7) may present difficulties in its numerical solution. Even if the aforementioned integrable singularity can be removed by suitable analytical change of independent variable, the right side of equation (2.7) may not satisfy the Lipschitz condition at $X = a$: $(\partial/\partial\tilde{Y})$ (R.S) may$\to\infty$ as $\tilde{Y}\to 0$ (e.g., see [4], p. 276). Thus its solution subject to the single condition (2.8) at $X = a$ may not be unique, and there may correspondingly be different physical track solutions through the origin with a range of acceptable values of $Y(a)$. This is actually not surprising, since it is the *projected* X-motion which is being considered here, so different track curves might still yield the same X-motion and hence correspond to the same $v(x)$. The class of solutions to equations (2.7) with (2.8) would then lead via equation (2.10) to curves $Y(X)$ which are also isodynamical *with each other*. However, as indicated by equation (2.5), only a track solution with $Y(a) = \hat{v}(a)$, i.e., its minimum value for reality at $X = a$, would be smooth at the origin $X = 0$.

3. QUADRATIC EXAMPLES

3.1. HARMONIC OSCILLATOR POTENTIAL

For the quadratic (one-dimensional harmonic oscillator) potential

$$v(x) = \tfrac{1}{2}kx^2, \tag{3.1}$$

the projected X-motion isodynamical track curves $Y(X)$ are given for $X \geqslant 0$ by equation (2.10), with $\tilde{Y}(X)$ satisfying the ODE

$$\mathrm{d}\tilde{Y}/\mathrm{d}X = -\left[\frac{((2mg/k)\tilde{Y} - (a^2 - X^2))}{(a^2 - X^2)}\right]^{1/2}, \tag{3.2}$$

with $\tilde{Y}(a) = 0$, equation (2.8). In this case the differential equation for $\tilde{Y}$ is solved with the negative square root (c.f. equation (2.10)) giving positive slope for $Y(X)$ for $0 < X \leqslant a$, the curve for negative X being completed by symmetry. This problem corresponds to finding those tracks with motion due to gravity whose X-projections obey simple harmonic motion.

Upon making the standard trigonometric transformation

$$X = a\cos\theta \tag{3.3}$$

to remove the integrable singularity in equation (3.2), there results

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta = [(2mg/k)\tilde{Y} - a^2\sin^2\theta]^{1/2}, \tag{3.4}$$

subject to

$$\tilde{Y}(\theta = 0) \equiv \tilde{Y}(X = a) = 0 \tag{3.5}$$

and then by equation (2.9)

$$Y(X = a) = \tilde{Y}(X = 0) \equiv \tilde{Y}(\theta = \pi/2). \tag{3.6}$$

Since equation (3.4) may not have a unique solution through the point $(\theta = 0,\ \tilde{Y} = 0)$, as mentioned above for $X = a$, it cannot be integrated forwards from $\theta = 0$ using equation

(3.5). Instead, it must be integrated *backwards* from the other endpoint $\theta = \pi/2$, for selected values of $\tilde{Y}(\theta = \pi/2)$, to find which *range* of "heights" $\tilde{Y}(\theta = \pi/2) \equiv Y(X = a)$ lead to $\tilde{Y}(\theta = 0) = 0$.

Merely from equation (2.6) with equation (3.3),

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta = a \sin\theta \, \mathrm{d}Y/\mathrm{d}X \tag{3.7}$$

so $\mathrm{d}\tilde{Y}/\mathrm{d}\theta$ and $\mathrm{d}Y/\mathrm{d}X$ have the same sign for $X > 0$. Then (provided $\mathrm{d}Y/\mathrm{d}X$ is finite) $d\tilde{Y}/\mathrm{d}\theta|_{\theta=0} = 0$. Furthermore,

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta|_{\theta=\pi/2} = a \, \mathrm{d}Y/\mathrm{d}X|_{X=0}. \tag{3.8}$$

Thus for a track curve $Y(X)$ which is smooth at the origin, i.e., with $Y'(0) = 0$, solutions to equation (3.4) would be required with

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta|_{\theta=\pi/2} = 0. \tag{3.9}$$

Consequently, by equation (3.4), or more generally by equation (2.12), the criterion for a completely smooth curve $Y(X)$ is

$$\tilde{Y}(X = 0) \equiv \tilde{Y}(\theta = \pi/2) \equiv Y(X = a) = \hat{v}(a) \tag{3.10}$$

with value $\frac{1}{2}ka^2$ for this quadratic potential, provided that equation (3.5) is also satisfied. Larger values of $\tilde{Y}(\theta = \pi/2)$ may still yield equation (3.5), but then equation (3.9) is no longer satisfied, so the corresponding isodynamical paths $Y(X)$ would, by equation (3.8), have a kink at the origin when extended by symmetry (c.f. equation (2.3b)) to $-\mathrm{a} \leqslant X \leqslant 0$.

For reality of $\tilde{Y}(\theta)$, from equation (3.4) the solutions must satisfy $\tilde{Y}(\theta) \geqslant \tilde{Z}(\theta)$, where $\tilde{Z}(\theta) = [ka^2/(2mg)] \sin^2\theta$ here (consistent with equation (2.11b)). The curve $\tilde{Z}(\theta)$, provides a lower bound to solution values, and bounds a direction field plot which may be useful in obtaining a qualitative picture of the nature of the relevant solution curves.

With $a = 1$ (or non-dimensionalising $\tilde{Y}$ with respect to a and scaling up k by a factor a), equation (3.4) behaves for small θ like

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta \approx [(2mg/k)\tilde{Y} - \theta^2]^{1/2}. \tag{3.11}$$

Thus a behaviour similar to the comparison d.e. dealt with in the Appendix may be expected. With constants $a = 1$ and $(2mg/k) = 5$ chosen in the full equation (3.4), there must be at least be a range of solutions through $\tilde{Y}(\theta = 0) = 0$ between $(\frac{1}{4})\theta^2$ and θ^2 for θ small (see equation (A.3)). Backwards numerical integration of equation (3.4) yields a range for $\tilde{Y}(\theta = \pi/2)$ between 0·2 and about 2·6 such that $\tilde{Y}(\theta = 0) = 0$. The lower limit $Y(1)_{[MIN]} = 0{\cdot}2$ is just equation (3.10), i.e., $\hat{v}(1)$ in this case, because $\tilde{Y} \geqslant \frac{1}{5}\sin^2\theta$ in equation (3.4) for real derivative, and $Y(1)_{[MIN]}$ corresponds to the smooth solution for $Y(X)$ in $-1 \leqslant X \leqslant 1$.

Points of inflexion, with $\mathrm{d}^2\tilde{Y}/\mathrm{d}\theta^2 = 0$, lie on the curve $\tilde{Y}_I = \frac{1}{5}\sin^2\theta + \frac{1}{125}\sin^2 2\theta$ which *itself* has a point of inflexion at about $\theta = 0{\cdot}23\pi$. Thus, for instance, the solution $\tilde{Y}(\theta)$ with $\tilde{Y}(\pi/2) = 0{\cdot}3$ decreases (concave up) until it crosses the curve $\tilde{Y}_I(\theta)$ from above and becomes concave down but recrosses $\tilde{Y}_I$ for smaller θ and finally tends towards $\theta = 0$ as a concave up curve. On the other hand, solutions $\tilde{Y}(\theta)$ with $\tilde{Y}(\pi/2)$ near the upper end of the range are concave up, right down to the origin.

Thus the range $Y(a)$ of possible heights of the track at amplitude $a = 1$ (for $(2mg/k) = 5$ in equation (3.4)) of the oscillatory motion under gravity of isodynamical paths is between 0·2 and about 2·6, and the X projected motion must, by construction, be standard simple harmonic as resulting from equation (3.1).

The resulting actual track shape $Y(X)$ is given by equation (2.10) with equation (3.3). For height $Y(1) = 2{\cdot}5$, the track, for X decreasing from 1, appears concave down, all the

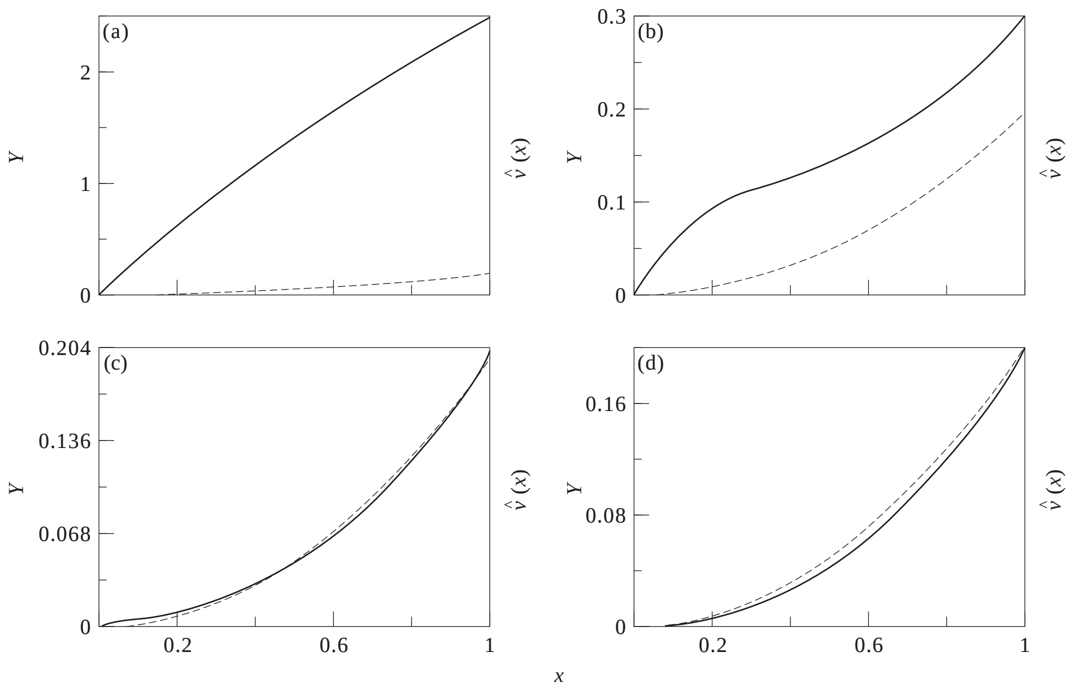

Figure 1. Track shape $Y(X)$ (solid curve) isodynamical to quadratic (harmonic oscillator) potential $v(x)$ (normalized with respect to mg) (dashed curve). See equations (3.4) (with (3.3), (2.10)) and (3.1) with $a = 1$ and $(2mg/k) = 5$. The curves are extended to negative $X = x$ by symmetry. See text, sub-section 3.1, for discussion. (a) $Y(1) = 2{\cdot}5$; (b) $Y(1) = 0{\cdot}3$; (c) $Y(1) = 0{\cdot}204$; (d) $Y(1) = 0{\cdot}20001$.

way to the origin: see Figure 1a; (these figures were prepared using Mathcad Plus 5.0). For height $Y(1) = 0{\cdot}3$, as X decreases the track curve $Y(X)$ is concave up until about $X = 0{\cdot}3$ and then is concave down, always being above $\hat{v}(X) = 0{\cdot}2X^2$; (see Figure 1b). It is therefore still quite different in shape from its isodynamical parabolic potential (3.1). As $Y(1)$ approaches its lower value 0·2, the curve $Y(X)$ crosses $\hat{v}(X)$ just below $X = 1$ and remains below it until X gets very small; the portion of the curve which is concave down occupies a smaller and smaller region near the origin. (See Figure 1c for $Y(1) = 0{\cdot}204$). Thus $Y(X)$ tends to a concave up curve with zero slope at the origin as $Y(1)$ approaches $Y(1)_{[MIN]} = \hat{v}(1) = 0{\cdot}2$. In Figure 1d, which has $Y(1) = 0{\cdot}20001$, there is still a small region near the origin in which Y is concave down, but it is not visible on the scale of the figure. (It did not seem feasible to integrate equation (3.4) numerically from $\tilde{Y}(\theta = \pi/2)$ exactly equal to $\hat{v}(1) = 0{\cdot}2$, because the numerical routine used gave small negative values inside the square root even for very small step size.)

This smooth limiting solution track curve, with $Y(1) = \hat{v}(1) = 0{\cdot}2$ here and $Y'(0) = \hat{v}'(0) = 0$, is not, it is emphasized, of the same shape as the potential curve (c.f. Figure 1d). This can easily be proved for general non-linear curves as follows. From equation (2.3a), if $Y(X) = Kv(X)/(mg)$, where K is a constant, then $(Y'(X))^2 = K - 1$, so Y' is constant. This is just a linear case for $Y(X)$ (considered in the Appendix), or the irrelevant constant case if $K = 1$. Similar numerical behaviour was found (with $a = 1$) for $(2mg/k) = 4$, for $Y(1)$ between $0{\cdot}25 = \hat{v}(1)$ and about 1·6.

These tracks are symmetric about $X = 0$, and for fixed mg/k the unique solution $Y(X)$ with continuous (zero) slope at the origin, viz., having $Y(1) = Y(1)_{[MIN]} = \hat{v}(1)$, is a smooth track shape sustaining oscillatory motion of a sliding bead; a track-and-cart system such as in [2] could be used. For solutions with $Y(1) > Y(1)_{[MIN]}$ there is a kink at $X = 0$, so the motion would be that of a "point" bead: there could then be difficulties in making an actual mechanical model which would exhibit full oscillatory motion. Nevertheless, the motion of a bead along these tracks from $X = 1$ to $X = 0$, projected onto the X-axis, would be exactly the same as the one-dimensional motion due to the harmonic oscillator potential (3.1), from $x = 1$ to $x = 0$. The tracks for fixed mg/k (and fixed $a = 1$) in the family with allowable heights $Y(1)$ are isodynamical to the potential (3.1) *and with each other*.

3.2. PATHS ISODYNAMICAL TO A PARABOLIC PATH

In sub-section 3.1.1 of Reference [1], the parabolic track shape

$$Y(X) = (1/2)KX^2 \qquad (K > 0) \tag{3.12}$$

was isodynamical to the (explicit, unique) potential

$$\hat{v}(x) = (1/2)K(1 + K^2A^2)x^2/[1 + K^2x^2] \tag{3.13}$$

for oscillation amplitude A. In view of the non-uniqueness of the inverse correspondence as discussed above, one may now start with potential $v(x)$ given by equation (3.13) with $A = a$ and seek those track curves $Y(X)$ which are isodynamical with it; $Y(X) = \frac{1}{2}KX^2$ will be one known exact solution.

The differential equation (2.7) is in this case

$$\mathrm{d}\tilde{Y}/\mathrm{d}X = -[(2(1 + K^2X^2)\tilde{Y} - K(a^2 - X^2))/K(a^2 - X^2)]^{1/2}. \tag{3.14}$$

With $a = 1$, this becomes, after transformation $X = \cos\theta$,

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta = +[(2/K)(1 + K^2\cos^2\theta)\tilde{Y} - \sin^2\theta]^{1/2}, \tag{3.15a}$$

subject to

$$\tilde{Y}(\theta = 0) = 0. \tag{3.15b}$$

A known exact solution is $\tilde{Y} = \frac{1}{2}K\sin^2\theta$ with $\tilde{Y}(\theta = \pi/2) = \frac{1}{2}K$, giving the original parabolic path.

Near $\theta = 0$, (3.15a) behaves like

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta \approx [(2/K)(1 + K^2)\tilde{Y} - \theta^2]^{1/2}, \tag{3.16}$$

which according to the Appendix has exact parabolic solutions for $(2/K)(1 + K^2) \geqslant 4$, i.e., $(K - 1)^2 \geqslant 0$, i.e., any positive K.

If $K = 2$, $\tilde{Y} = \sin^2\theta$ is the known exact solution, with $\tilde{Y}(\theta = \pi/2) = 1$, to the differential equation (3.15) which now reads:

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta = [(1 + 4\cos^2\theta)\tilde{Y} - \sin^2\theta]^{1/2}. \tag{3.17}$$

For real solutions, $\tilde{Y} \geqslant \tilde{Z}(\theta) = \sin^2\theta/(1 + 4\cos^2\theta)$, so $\tilde{Y}(\pi/2) \geqslant 1$. In fact, although it has not been shown analytically, backwards numerical integration indicates that $\tilde{Y}(\theta = \pi/2) = 1$ is the only value yielding $\tilde{Y}(\theta = 0) = 0$. Thus $Y = X^2$ is the only track (with $a = 1$) having its X motion.

Similar results were found for $K = 4$, with $Y = 2X^2$. Then the differential equations is

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta = [(\tfrac{1}{2})(1 + 16\cos^2\theta)\tilde{Y} - \sin^2\theta]^{1/2}. \tag{3.18}$$

$\tilde{Y} = 2\sin^2\theta$ is the known exact solution, with $\tilde{Y}(\theta = \pi/2) = 2$, and this appears to be the only solution.

If $K = 1$, $\tilde{Y} = \frac{1}{2}\sin^2\theta$ is the known exact solution, with $\tilde{Y}(\pi/2) = 0{\cdot}5$, to the differential equation (3.15)

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta = [2(1+\cos^2\theta)\tilde{Y} - \sin^2\theta]^{1/2}. \tag{3.19}$$

For real solutions, $\tilde{Y} \geqslant \tilde{Z}(\theta) = \sin^2\theta/[2(1+\cos^2\theta)]$, so $\tilde{Y}(\pi/2) \geqslant 0{\cdot}5$. Backwards integration gives a range of $\tilde{Y}(\theta = \pi/2)$, yielding $\tilde{Y}(0) = 0$, from 0·5 to about 0·7. Thus $Y = \frac{1}{2}X^2$ is isodynamical with a continuum of other track shapes with $a = 1$ and $0{\cdot}5 \leqslant Y(1) \leqslant 0{\cdot}7$. For example, with $\tilde{Y}(\pi/2) = Y(1) = 0{\cdot}7$, the track shape $Y(X)$ is depicted in Figure 2 (together with its isodynamical parabola $Y = \frac{1}{2}X^2$). It has positive slope at the origin, and so again refers, for oscillatory motion, to a point bead; but the motion of a macroscopic bead from $X = 1$ to $X = 0$, projected onto the X-axis, would still be the same as for the "parent" parabola $Y(X) = \frac{1}{2}X^2$, which is the unique smooth solution with continuous (zero) slope at the origin.

4. QUARTIC POTENTIAL EXAMPLES

4.1. POSITIVE CURVATURE

4.1.1. *Duffing oscillator—hard spring*

For the potential

$$v(x) = k_4 x^4 + k_2 x^2, \qquad k_i > 0, \tag{4.1}$$

the differential equation (2.7) is again subjected to the co-ordinate transformation (3.3) and then reads

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta = [\gamma\tilde{Y}/(a^2(1+\cos^2\theta)+\kappa) - a^2\sin^2\theta]^{1/2}, \tag{4.2}$$

where

$$\kappa = k_2/k_4, \qquad \gamma = mg/k_4. \tag{4.3a, b}$$

A numerical example is described with $a = 1$, $\kappa = 1/2$, $\gamma = 10$, and attention is now restricted to solutions near the smooth track case $\tilde{Y}(\theta = \pi/2) = Y(X = 1) = \hat{v}(1) = 3/20 = 0{\cdot}15$. For $\tilde{Y}(\theta = \pi/2) = 0{\cdot}2$, it was evident that equation (3.9) for $\tilde{Y}$ was not satisfied, and a track shape Y akin to Figure 1b was found. For $\tilde{Y}(\theta = \pi/2) = Y(X = 1) = 0{\cdot}1501$, a situation similar to Figure 1d was found. The eventual smooth track shape $Y(X)$ with $Y(X = 1) = 0{\cdot}15$ is evidently different from $\hat{v}(X)$, being mainly below it, whilst the conditions (3.10) and (2.4) are satisfied.

4.1.2. *Pure quartic potential* (*cubic oscillator*)

With $k_2 = 0$ in Equation (4.1), so $\kappa = 0$, and choosing a numerical example with $a = 1$, $\gamma = 8$, the differential equation is

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta = [8\tilde{Y}/(1+\cos^2\theta) - \sin^2\theta]^{1/2} \tag{4.4}$$

with $\hat{v}(1) = 1/8 = 0{\cdot}125$. The slope of the lower bounding curve $\tilde{Z}(\theta) = (1/8)\sin^2\theta\,(1+\cos^2\theta)$ is proportional to $\sin\theta\cos^3\theta$, which is positive but very small near $\theta = \pi/2$. Thus it was difficult to integrate numerically below about $\tilde{Y}(\theta = \pi/2) = 0{\cdot}13$; a situation for Y similar to Figure 1c was found.

4.2. SYMMETRIC DOUBLE-WELL POTENTIAL

Of particular interest in this paper, in view of the spirit of Reference [2], is the question: what *track* shapes are actually isodynamical to a double-well Duffing *potential*? For

$$v(x) = k_4 x^4 - k_2 x^2, \qquad k_i > 0 \tag{4.5}$$

Equation (4.2.) now becomes

$$\mathrm{d}\tilde{Y}/\mathrm{d}\theta = [\gamma \tilde{Y}/(a^2(1+\cos^2\theta) - \kappa) - a^2 \sin^2\theta]^{1/2} \tag{4.6}$$

with κ and γ still as in equations (4.3a, b), so

$$\hat{v}(x) = (1/\gamma)(x^4 - \kappa x^2). \tag{4.7}$$

For the interesting case of symmetric ("large") oscillations across both wells ("large orbits"), by equation (4.7) the amplitude satisfies

$$a > \sqrt{\kappa} \tag{4.8}$$

and the integrand in (4.6) is then in fact not singular. The curve below which solutions are not real is

$$\tilde{Z}(\theta) = (a^2/\gamma)\sin^2\theta[a^2(1+\cos^2\theta) - \kappa] \tag{4.9}$$

so

$$\tilde{Y}(\theta = \pi/2) \geqslant \tilde{Z}(\theta = \pi/2) = (a^2/\gamma)(a^2 - \kappa) = \hat{v}(x = a). \tag{4.10}$$

$\mathrm{d}\tilde{Z}/\mathrm{d}\theta$ is zero at $\theta = 0$ and $\pi/2$ *and* at $\theta = \pi/2 - \alpha/2$, $\alpha = \arccos(1 - \kappa/a^2)$. Unlike previous cases, $\tilde{Z}(\theta)$ *decreases* between $\theta = \theta_3 \equiv \pi/2 - \alpha/2$ and $\theta = \pi/2$, where

$$\cos(2\theta_3) = -(1 - \kappa/a^2). \tag{4.11}$$

Thus it is not possible to have a smooth decreasing solution curve $\tilde{Y}(\theta)$ satisfying both $\tilde{Y}(\theta = \pi/2) = \tilde{Z}(\theta = \pi/2) = \hat{v}(X = a)$ and $\tilde{Y}(\theta = 0) = 0$, so there is no corresponding smooth track shape $Y(X)$. This is because such a solution to equation (4.6) has zero slope at $\theta = \pi/2$, and so becomes imaginary as soon as θ becomes less than $\pi/2$, since $\tilde{Z}(\theta)$ initially increases as θ decreases from $\pi/2$.

There is still the possibility that there is a solution to the differential equation with negative sign outside the square root in equation (4.6) (c.f. (2.3b)), for some region near $\theta = \pi/2$. (After all, if an attempt was made to recover a track by trying to solve the differential equation (2.3) using the complicated isodynamical potential (3.16) of Reference [1] corresponding to the hill-and-two-valleys track (3.15) of Reference [1], it was evident in Figure 2 of Reference [1], that $\mathrm{d}Y/\mathrm{d}X$ was negative in a region for that problem.)

Indeed, from the general result (3.7) which follows from the dependent variable redefinition (2.16) and the independent co-ordinate transformation (3.3), one obtains

$$\mathrm{d}^2\tilde{Y}/\mathrm{d}\theta^2|_{\theta=\pi/2} = -a^2\,\mathrm{d}^2 Y/\mathrm{d}X^2|_{X=0}. \tag{4.12}$$

This shows that if $\tilde{Y}(\theta)$ is concave up at $\theta = \pi/2$ then $Y(X)$ must be concave down at $X = 0$. This was borne out by Figure 2 of Reference [1] for the problem there.

In the present example with equation (4.5), there seems to be the possibility of a solution $\tilde{Y}(\theta)$ which starts at $\theta = \pi/2$, $\tilde{Y} = \tilde{Z}(\theta = \pi/2)$, increases as θ decreases (negative slope), and then bends over to tend towards the origin (positive slope). For a smooth curve, the transition from negative to positive sign outside the square root in equation (4.6) can only occur when $\mathrm{d}\tilde{Y}/\mathrm{d}\theta = 0$, and hence only for a value of θ and $\tilde{Y}$ such that $\tilde{Y}(\theta) = \tilde{Z}(\theta)$ (see

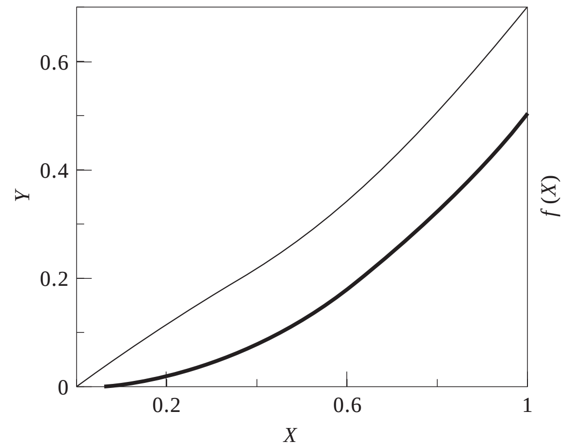

Figure 2. Track shape $Y(X)$ with $Y(1) = 0{\cdot}7$ (upper curve) isodynamical to parabolic track (lower curve) $f(X) = X^2/2$, for motion $X(t)$. See equations (3.19) (with (3.3), (2.10)) and (3.12). The curves are extended to negative X by symmetry. These are both isodynamical to the potential (3.13), with $A = 1$ and $K = 1$. (c.f. Figure 1 in Reference [1].)

equation (4.9)). Furthermore, to maintain the reality of the solution, this could only occur at the maximum point of $\tilde{Z}(\theta)$, i.e., at $\theta = \theta_3$ given by equation (4.11), where

$$\tilde{Z}(\theta_3) = (1/\gamma)(a^2 - \kappa/2)^2. \tag{4.13}$$

However, some mathematical working shows that for this problem

$$\mathrm{d}^2\tilde{Y}/\mathrm{d}\theta^2[\text{at } \theta = \theta_3, \text{ as } \tilde{Y}\to\tilde{Z}] = (\gamma/2)/(a^2 - \kappa/2). \tag{4.14}$$

By equation (4.8), this is positive ($\tilde{Y}(\theta)$ is concave up). Thus it is not possible for a smooth solution $\tilde{Y}(\theta)$ such as that just described to osculate, concave down, over the "hump" in $\tilde{Z}(\theta)$ and decrease to the origin. Similar reasoning shows that a curve for $\tilde{Y}(\theta)$ which is smooth from $\theta = \pi/2$ to $\theta = 0$ cannot have negative slope anywhere in that domain.

For a numerical example, $a = 1$, $\kappa = 1/2$, $\gamma = 6$ were chosen. A range of $Y(1)$ (such that

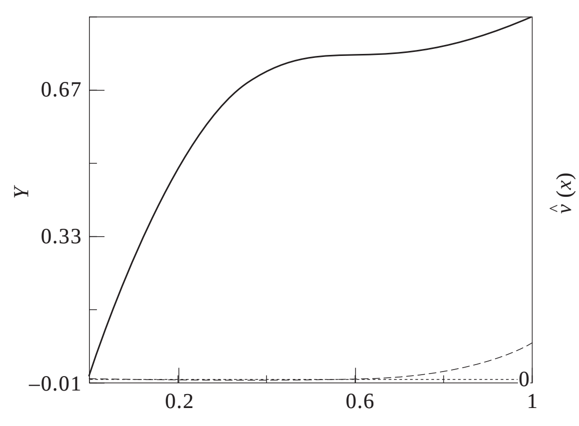

Figure 3. Track shape $Y(X)$ (solid curve) isodynamical to double-well potential $v(x)$ (normalized with respect to mg) (dashed curve). See equations (4.6) (with (3.3), (2.10)) and (4.5), (4·7) (with 4.3a, b), with $a = 1$, $\gamma = 6$, $\kappa = 1/2$, and $Y(1) = 0{\cdot}84$. The curves are extended to negative $X = x$ by symmetry.

$Y(0) = 0$) from about 0·84 to 3·3 was found. The lowest value is still well above $\hat{v}(1) = 1/(2\gamma) = 1/12 = 0{\cdot}0833$, as explained by the discussion above. For $Y(1) = 3$, the track curve looks much like Figure 1a. For $Y(1) = 0{\cdot}84$, the track curve is shown in Figure 3. The whole (symmetric) curves from -1 to $+1$ thus have a kink at the origin. A similar behaviour was found for $\gamma = 8$.

There is therefore (perhaps unfortunately in view of [2]) no completely smooth shape of track isodynamical to a Duffing-type double-well potential as regards symmetric (cross-well) oscillations.

5. LINEAR POTENTIAL

These investigations are concluded by reconsidering the simple case of a linear potential/straight-line track, in particular a V-shape. In the case that $v'(x) = \text{constant}$, then $v'(0) \neq 0$. With

$$v(x) = kx, \qquad \hat{v}(x) = \lambda x, \qquad \lambda = k/(mg), \tag{5.1a, 1b, 1c}$$

equation (2.7) reads

$$\mathrm{d}\tilde{Y}/\mathrm{d}X = -[(\tilde{Y} - \lambda(a - X))/(\lambda(a - X))]^{1/2}. \tag{5.2}$$

This is known to have solutions (see [1])

$$\tilde{Y} = K(a - X), \qquad Y(X) = KX, \qquad K = K_{\pm} = [1 \pm (1 - 4\lambda^2)^{1/2}]/(2\lambda). \tag{5.3a, 3b, 3c}$$

To find more general solutions, let

$$X = a - \eta^2/2; \qquad \eta = [2(a - X)]^{1/2} \tag{5.4}$$

so equation (5.2) becomes

$$\mathrm{d}\tilde{Y}/\mathrm{d}\eta = [(2/\lambda)\tilde{Y} - \eta^2]^{1/2} \tag{5.5a}$$

with (c.f. equation (2.8))

$$\tilde{Y}(\eta = 0) = 0. \tag{5.5b}$$

By inspection, this has exact parabolic solutions

$$\tilde{Y}(\eta) = \{(2/\lambda) \pm [(4/\lambda^2) - 16]^{1/2}\}\eta^2/8 \tag{5.6}$$

corresponding to the linear solutions (5.3) for $Y(X)$. More generally, for $\tilde{Y}(\eta)$ satisfying the non-linear differential equation (5.5) (see equation (2.10))

$$Y(X) = \tilde{Y}(\sqrt{(2a)}) - \tilde{Y}(\eta) \tag{5.7a}$$

with (see equation (2.9))

$$Y(X = a) = \tilde{Y}(\eta = \sqrt{(2a)}). \tag{5.7b}$$

Equation (5.5) for $\tilde{Y}(\eta)$ is actually exactly of the form of the "comparison" differential equation (A1) for $y(x)$ of the Appendix, with

$$\alpha = 2/\lambda. \tag{5.8}$$

Thus the behaviour of its non-analytical continuum of non-unique solutions is similarly understood, and it only remains, after computing solutions to equation (5.5a) satisfying (5.5b), to depict corresponding track shape solutions $Y(X)$ via equations (5.7).

The case $\alpha = 5$ ($\lambda = 0{\cdot}4$) is chosen as a typical example, with exact solutions $\tilde{Y} = \eta^2$ and $\tilde{Y} = \eta^2/4$, i.e., track functions $Y = 2X$ and $Y = X/2$. With amplitude $a = A = 1$, the

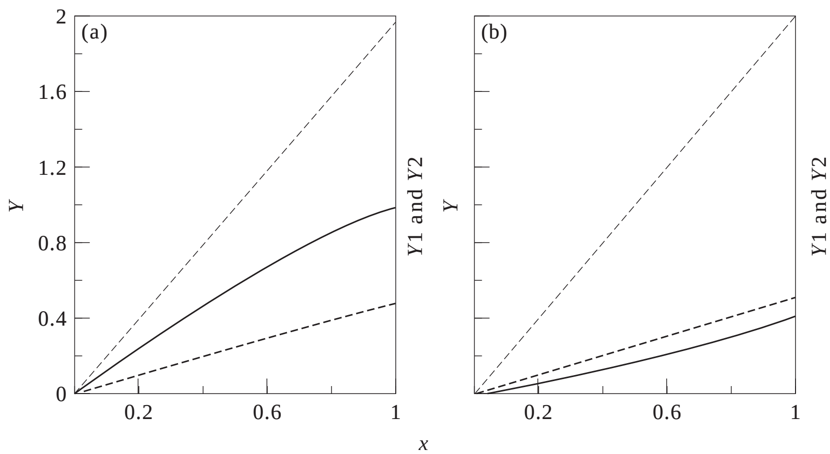

Figure 4. Track shape $Y(X)$ (solid curve) isodynamical to the linear potential $\hat{v}(x) = 0{\cdot}4x$ See equations (5.5) (with (5.4), (5.7)) and (5.1) with $a = 1$ and $\lambda = 2/5$: (a) $Y(1) = 1$; (b) $Y(1) = 0.401$. The isodynamical exact solutions $Y1 = 2X$ $(Y(1) = 2)$ and $Y2 = X/2$ $(Y(1) = 0{\cdot}5)$ are also shown (upper light and lower bold dashed straight lines respectively). For a V-shaped track, the curves are extended to negative $X = x$ by symmetry.

"heights" $Y(X = 1) = \tilde{Y}(\eta = \sqrt{2})$ range from the maximum solution satisfying equation (5.5b), which is known from the Appendix to be $(\sqrt{2})^2 = 2$, down to the minimum for reality of equation (5.5a), viz., $(\sqrt{2})^2/\alpha = 0{\cdot}4$.

For $Y(X = 1) = 2$, the solution is exact: the straight line $Y = 2X$; for $Y(X = 1) = 0{\cdot}5$, the solution is the exact straight line $Y = X/2$. For $Y(X = 1)$ between 2 and 0·5, the solutions $Y(X)$ are concave-down curves. For $Y(X = 1)$ between 0·5 and 0·4, the solutions are concave-up curves. Figure 4a shows the track shape $Y(X)$ for $Y(X = 1) = 1$, together with its two isodynamical straight line tracks as described above. Figure 4b shows the track shape for $Y(X = 1) = 0{\cdot}401$. The track solution satisfying equation (2.5), i.e., here the limiting curve as $Y(1) \to \hat{v}(1) = \lambda = 0{\cdot}4$, is smooth at the origin, with $Y'(0) = 0$, and looks like Figure 4b on the scale of that figure.

For a V-shaped potential, therefore, the isodynamical track shape curves are symmetrical but (except for two special "heights") are not in general of a simple V-shape. There is even a smooth U-shaped track, isodynamical to the V-shaped track, upon which a macroscopic "cart" (c.f. [2]) could move.

6. CONCLUSION

One-dimensional potential motion may in many cases be realized mechanically by projection, onto the horizontal axis, of motion due to gravity on an isodynamical track shape which depends on the amplitude and is not in general of the same functional form and may not be unique. In Reference [1], for the direct problem, for a given track function, the potential was explicitly and uniquely specified. In particular, corresponding to a given double-well shaped track (c.f. [2]) (with motion under gravity, projected onto the horizontal axis), the (scaled) potential for large amplitude oscillations had a triple-well form, with minima at the turning points of the track curve.

For the converse problem dealt with here, for given parabolic and positive curvature quartic potentials, there is a continuum of isodynamical track shapes with a kink at the origin. Only the track shape with minimum "height" is smooth at the origin (but of different shape from the corresponding potential). For a given Duffing-type double-well potential, corresponding to the "large orbits" there are no completely smooth track shapes—they are something like a florid V or wings shape.

For an ordinary V-shaped potential, the isodynamical track shapes include two V-shapes but otherwise consist of a continuum of symmetrical curves not of a simple V-shape; the shape with minimum "height" is in fact smooth at the origin.

Thus the isodynamical mechanical analogues of some potentials have a kinked track shape and so for oscillations correspond to the concept of only a point bead sliding along a wire. In other cases, a unique smooth track shape may exist which may therefore be amenable to construction and experimentation.

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APPENDIX: A COMPARISON EQUATION

The technique of backwards-integration to find the range of "heights" of solutions to a differential equation with non-unique solutions through the origin, as used in sections 3 and 4 above, may be illustrated by a related but simpler "comparison" equation for which some exact analytical solutions are known. Consider the ordinary non-linear differential equation (whose variables are unrelated to those in the body of this paper)

$$\mathrm{d}y/\mathrm{d}x = [\alpha y - x^2]^{1/2}, \tag{A1}$$

with

$$y(0) = 0. \tag{A2}$$

Inspection of (A1) suggests seeking solutions of the parabolic form $y = \beta x^2$. Then if $\alpha = 4$ an exact solution is $y = \frac{1}{2}x^2$; and if $\alpha > 4$ there are *two* known real exact solutions to (A1, 2), viz.,

$$y_i = \beta_i x^2, \qquad \beta_{1,2} = [\alpha \mp (\alpha^2 - 16)^{1/2}]/8. \tag{A3a, b}$$

By Osgood's theorem ([5], p. 67) there are also solutions satisfying (A2) through any point between the curves $\beta_1 x^2$ and $\beta_2 x^2$. This does not preclude there being other solutions outside this region but still satisfying (A2). Because solutions are unique for $xy \neq 0$, none of this continuum of solutions cross or touch for $x, y > 0$.

An advantage of having these explicit solutions (A3) available is that they assist in understanding the solutions to the more complicated differential equations in the main part of this paper. In addition, the numerical technique of backward integration can be checked down to small values of x, starting with $y(1)$ in $[\beta_1, \beta_2]$ which is known to satisfy all the conditions.

For $\alpha = 4$, the solution $y = \frac{1}{2}x^2$ to equations (A1, 2) has $y(1) = 0{\cdot}5$. Backward integration of equation (A1) yields a range of $y(1)$ (such that equation (A2) is satisfied) from 0·25 to about 0·53. (It is actually quite difficult to decide when (A2) is satisfied because of the fine scale needed near $x = 0$, where the term inside the square root on the right side of (A1) may computationally become negative.) The lower limit of 0·25 for $y(1)$ simply corresponds to the requirement that $\mathrm{d}y/\mathrm{d}x$ in equation (A1) is real, i.e., $y \geqslant \frac{1}{4}x^2$. As x increases, solutions $y(x)$ hit the curve $y = \frac{1}{4}x^2$ from the left with zero slope and then stop. Points of inflexion ($y'' = 0$) lie on the curve $y_I = (5/16)x^2$. For solutions with $1/4 < y(1) < 5/16$, as x decreases below 1 the solution curves are concave down until they cross y_I and then become concave up, decreasing to the origin. Solutions with $y(1) > 5/16$ are concave up, and decrease to the origin at least for $y(1) \leqslant \frac{1}{2}$. Inspection of plots of the gradient field confirm these considerations, and may also in general assist in determining the range of $y(1)$.

For $\alpha = 5$, the *two* explicit solutions (A3) are $y_1 = \frac{1}{4}x^2$ and $y_2 = x^2$ with $y(1) = 0{\cdot}25$ and 1 respectively. Backward integration of (A1) yields a range of possible values of $y(1)$ between 0·20 and 1·0. This situation may again be described in some detail. For reality of solutions, $y \geqslant (1/5)x^2$, $y(1) \geqslant 1/5$ corresponding to the lower range value 0·2. Moreover, solution curves hit the parabola $y_R = (1/5)x^2$ with zero slope. Points of inflexion of solutions, with $y'' = 0$, lie on the parabola $y_I = (29/125)x^2$, which lies between the smaller known exact solution y_1 and the lower bounding parabola y_R. Solutions when below the parabola y_I have negative curvature (concave down) and when above have positive curvature (concave up). Thus (as x decreases) any solution starting on $y_R = (1/5)x^2$ (with zero slope) is concave down until it crosses the parabola y_I after which it is concave up and tends towards the origin. This is the case for solutions with $1/5 \leqslant y(1) \leqslant 29/125$ (which have $y'(1) \geqslant 0$). Solutions to equation (A1) with $y(1) \geqslant 29/125 = 0{\cdot}232$ are concave up and therefore remain so as x decreases.

Solutions lying between $y_I = (29/125)x^2$ and $y_2 = x^2$ (including $y_1 = \frac{1}{4}x^2$) are therefore concave up and decrease to the origin as $x \to 0$. (They do not cross each other because the solution of the d.e. (A1) through a point with $xy \neq 0$ is unique.) Solutions to (A1) with $y(1) > 1 = y_2(1)$ appear to curve away from the exact solution $y_2(x) = x^2$ and cross the y-axis at positive values, so they do not satisfy (A2), i.e., they have $y(0) \neq 0$.