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LETTERS TO THE EDITOR

A NOTE ON THE STRUCTURE OF THE ACOUSTIC FIELD EMITTED BY A WAVE PACKET

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1. INTRODUCTION

The sound field created by wave packets has been subject to rigorous experimental, analytical and numerical research in the last two decades. It has been argued that the wave packet can model a large scale structure in the form of an instability wave, initially growing, then saturating and decaying. These structures were found to be dominant sound producers in the mixing region of forced jets [1, 2]. Several analytical models for analyzing the emitted sound by these structures have been offered over the years. Ffowcs Williams and Kempton [3] used Lighthill's analogy to determine the magnitude of the radiated sound. Tam and Burton [4, 5] used a linear stability analysis for calculating the sound field emitted by a slowly expanding supersonic shear layer and Crighton and Huerre [6] looked at the appearance of superdirectivity in the sound field of low subsonic shear layers.

The purpose of this study is to produce simple analytical approximations for the sound field, which the previous studies seem to lack. There is a need for such approximations not just for the physical understanding of the sound field but also as a design and validation tool for computational and experimental aeroacoustics. The model suggested by Crighton and Huerre [5] seems to be the most appropriate for this aim. It treats the problem as a boundary value problem where the effect of the wave packet comes from the boundary condition. They have already succeeded in estimating the penetration distance of the transition region between the near field and the far field of a superdirective sound field of a low subsonic Gaussian wave packet. This study will give simple approximations for the near field and far field pressure for a subsonic and a supersonic wave packet. It will investigate the effect of the convective Mach number and different wave packet shapes on the penetration distance of the transition region in the transverse and longitudinal directions.

2. THE MODEL

The analysis will follow the model suggested by Crighton and Huerre [6]. The acoustic source is represented by a wave packet on the lower side of the box as in Figure 1. The space and time arguments are normalised, so the pressure fluctuations of the sound field are taken as $p(x, y)\exp(-\mathrm{i}t)$. A two dimensional wave equation of a stationary medium is assumed to be the governing equation, and the problem can be specified as

$$\partial^2 p/\partial x^2 + \partial^2 p/\partial y^2 + M_c^2 p = 0, \qquad -\infty < x < \infty, \qquad y \geqslant 0, \tag{1.1}$$

$$M_c \equiv \omega_m/k_m c_0, \qquad p(x, y = 0) = A(\varepsilon x)\,\mathrm{e}^{\mathrm{i}x}, \qquad (1.2, 1.3)$$

where M_c is the convective Mach number, ω_m and k_m are the wave packet's dimensional main frequency and wave number respectively. A is the envelope shape function and it is

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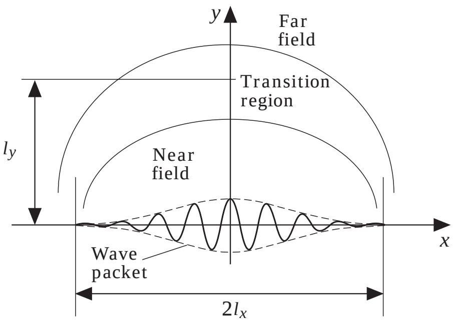

Figure 1. Schematic description of the acoustic field of a low subsonic wave packet (not to scale). The wave packet acts as a lower boundary condition on the solution of the wave equation. The lengths l_x and l_y are the penetration distances in the x and y directions respectively.

taken that $\varepsilon < O(1)$. Crighton and Huerre [6] showed that a solution of equations (1) that obeys the radiation boundary condition at infinity can be expressed as

$$p(x, y) = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \hat{A}\left(\frac{k-1}{\varepsilon}\right) \mathrm{e}^{\mathrm{i}kx - \gamma y}\, \mathrm{d}k, \tag{2.1}$$

$$\gamma = \begin{Bmatrix} \sqrt{k^2 - M_c^2}, & |k| \geqslant M_c \\ -\mathrm{i}\sqrt{M_c^2 - k^2}, & |k| \leqslant M_c \end{Bmatrix}, \qquad \hat{A}(K) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(X)\, \mathrm{e}^{-\mathrm{i}KX}\, \mathrm{d}X, \tag{2.2, 2.3}$$

where $\hat{A}$ is the Fourier transform of A.

This is a simplified model that neglects the variation of the main packet wave number due to the slow spreading of the shear layer. Nevertheless this simplicity allows one to derive simple analytic approximations for p and, although one has to assume the shape of the wave packet envelope, it will be shown that the main features of the pressure behaviour seen in experiments, and in more complicated analyses such as that of Tam and Burton [4, 5], are captured by these simple analytic approximations. Three kinds of wave packets will be considered, the Gaussian packet

$$A(X) = \mathrm{e}^{-X^2}, \qquad \hat{A}(K) = \mathrm{e}^{-K^2/4}/(2\sqrt{\pi}), \tag{3}$$

the algebraic packet

$$A(X) = 1/(1 + X^2), \qquad \hat{A}(K) = \mathrm{e}^{-K^2}/2, \tag{4}$$

and the exponential packet

$$A(X) = \mathrm{e}^{-|X|}, \qquad \hat{A}(K) = 1/[\pi(1 + K^2)]. \tag{5}$$

3. THE NEAR FIELD

The near field solution can be derived by using a multiple scale method as done by Tam and Burton [4, 5]. One takes x as the slow variable,

$$p(x, y) = \sum_{n=0}^{\infty} \varepsilon^n p^{(n)}(\varepsilon x, y, \varepsilon y) \exp(\mathrm{i}x), \tag{6}$$

assumes that $A(\varepsilon x)$ is an analytic function and, after some arithmetic manipulations, obtains

$$p = A(Z)\,\mathrm{e}^{\mathrm{i}x - y\sqrt{1-M_c^2}}\left[1 - \frac{\varepsilon^2 M_c^2 y A''(Z)}{2(1-M_c^2)^{1\cdot 5}A(Z)} + \cdots\right], \qquad Z \equiv \varepsilon x + \frac{\mathrm{i}\varepsilon y}{\sqrt{1-M_c^2}}, \quad (7)$$

for $M_c < O(1)$ and

$$p = A(Z)\,\mathrm{e}^{\mathrm{i}(x + y\sqrt{M_c^2-1})}\left[1 + \frac{\mathrm{i}\varepsilon^2 M_c^2 y A''(Z)}{2(M_c^2-1)^{1\cdot 5}A(Z)} + \cdots\right], \qquad Z \equiv \varepsilon x - \frac{\varepsilon y}{\sqrt{M_c^2-1}}, \quad (8)$$

for $M_c > O(1)$. Expression (7) converges to the expansion of Crighton and Huerre [6] for the incompressible near field of the Gaussian and algebraic wave packets (3) and (4) when $M_c = 0$. It shows that an exponential decay in the amplitude dominates the near field and for a Gaussian packet (3) the lines of constant amplitude are the hyperbolic curves

$$\varepsilon^2[x^2 - y^2/\sqrt{1-M_c^2}] + y\sqrt{1-M_c^2} = const., \quad (9)$$

where the second term inside the square brackets of equation (7) was neglected. This fits the experimental results of Gutmark and Ho [7] for a low Mach number jet. In Figure 2(a) the curves of equation (9) are compared to the numerical solution of equation (2) derived by numerical integration, for a Gaussian wave packet with $M_c = 0{\cdot}1$ and $\varepsilon = 0{\cdot}2$ which is a typical value for ε given by Laufer and Yen [2]. There is an excellent agreement between the two solutions up to the transition region where the radiating modes start to play an important role. Thus the near field is dominated by the hydrodynamic field, mainly generated by the mode $k = 1$ (see equation (2)).

The supersonic solution (8) shows the generation of a Mach wave. It shows that the slowest decay in the amplitude is in the Mach angle direction as in Figure 2(b), which compares the numerical solution of equation (2) with the analytical approximation $A(Z)$ of equation (8) for a supersonic Gaussian wave packet with $M_c = 2$ and $\varepsilon = 0{\cdot}2$. The directivity shown in Figure 2(b) is similar to the one found experimentally by Troutt and

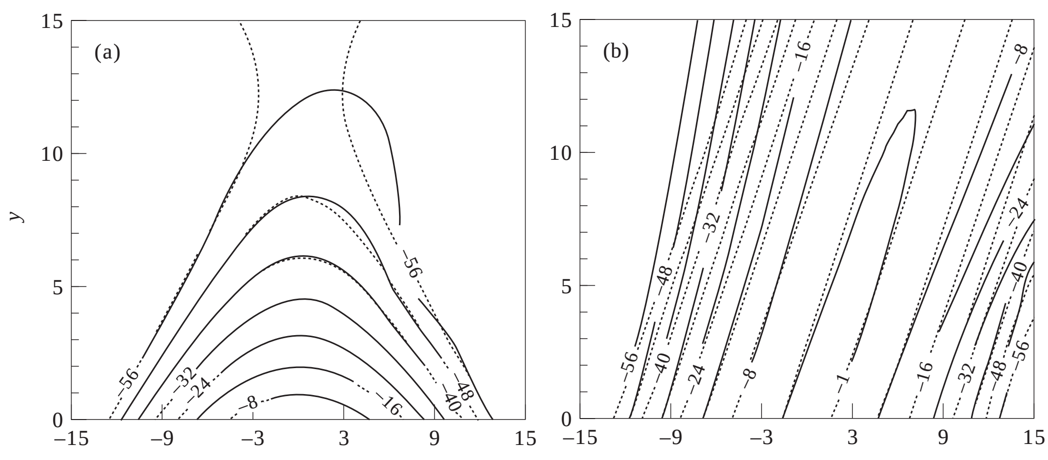

Figure 2. Lines of constant pressure amplitude for a Gaussian wave packet, $\varepsilon = 0{\cdot}2$, for (a) $M_c = 0{\cdot}1$ (b) $M_c = 2$, in dB relative to the maximum amplitude. Solid lines are from the numerical solution and dotted lines from the analytic approximation.

McLaughlin [8] and numerically by Mankbadi [9] who used the Tam and Burton [5] method. Expressions (2) and (8) show that the near field is dominated again by the mode $k = 1$ which is this time a radiating mode.

The solutions (7) and (8) are not valid for a wave packet with a non-analytic shape function such as the exponential wave packet (5), but since the mode $k = 1$ dominates the near field, a simple asymptotic evaluation expression of (2.1) leads to

$$p \sim A(O)\exp[\mathrm{i}x - \gamma(k=1)y][1 + O(\varepsilon)]. \tag{10}$$

Thus the features of an exponential decay in the subsonic case and a Mach wave in the supersonic case are expected for any integrable wave packet shape as long as $\varepsilon < O(1)$.

4. THE FAR FIELD

The last section showed that the near field is dominated by the mode $k = 1$, Crighton and Huerre [6] showed that in the far field the dominant mode is of $k = M_c \cos\theta$, where θ is the downstream angle. The solution for p can be derived by an asymptotic evaluation of the integral (2.1). Such an evaluation can be achieved by using the stationary point method of Dingle [10]. This method is an expansion of the stationary point method that can be derived by expanding the slow variable function of the integrand in a Taylor series. Taking $\hat{A}$ in equation (2.1) as the slow variable function gives, after some manipulations,

$$p(x, y) \sim \frac{1}{\varepsilon}\sqrt{\frac{2\pi M_c}{r}}\,\frac{y}{r}\,\mathrm{e}^{\mathrm{i}(M_c r - \pi/4)}\hat{A}\left(\frac{M_c\cos\theta - 1}{\varepsilon}\right)(1 + Q + \cdots), \tag{11}$$

where $r \equiv \sqrt{x^2 + y^2}$. The term outside the bracket on the right hand side of (11) is exactly what Crighton and Huerre [6] obtained using the stationary phase method. The correction term Q depends on the form of $A(\varepsilon x)$. For the Gaussian wave packet (3) one gets

$$Q = \frac{\mathrm{i}M_c y^2}{8r^3}\left[\frac{3r^2}{M_c^2 y^2} - \frac{6xr(M_c x/r - 1)}{\varepsilon^2 M_c y^2} + \frac{2}{\varepsilon^2}\left(1 - \frac{(M_c x/r - 1)^2}{2\varepsilon^2}\right)\right]. \tag{12}$$

The far field or the radiation field is defined when the presure is in phase with the velocity. This means $\partial p/\partial r \cong \mathrm{i}M_c p$ which by equation (11) leads to

$$r \gg 1/(2M_c), \qquad |Q| \ll 1. \tag{13.1, 13.2}$$

The requirement (13.1) is what one would expect from classical acoustics, since M_c^{-1} is of the order of the sound wave length in the normalised co-ordinates that are used here. Requirements (13.2) and (12) lead, for a low subsonic Gaussian wave packet ($M_c < O(1)$) to two different situations. The first is when $M_c/\varepsilon^2 < O(1)$ that leads to $r \gg 3/(8M_c)$ which is included in requirement (13.1), but if $M_c/\varepsilon^2 > O(1)$ one gets

$$y \gg M_c/8\varepsilon^4, \tag{14.1}$$

for $y \gg |x|$ and

$$|x| \gg 3/4\varepsilon^2 \tag{14.2}$$

for $|x\| \gg y$. Similar calculations for a low subsonic algebraic wave packet lead to $r \gg 3/(8M_c)$ if $M_c/\varepsilon < O(1)$ which is again included in requirement (13.1), but if $M_c/\varepsilon > O(1)$ one obtains

$$y \gg M_c/2\varepsilon^2, \tag{15.1}$$

for $y \gg |x|$ and

$$|x| \gg 3/2\varepsilon, \tag{15.2}$$

for $|x| \gg y$. For a low subsonic exponential wave packet one just gets expression (13.1) as the requirement for the far field. Expressions (13)–(15) show that the far field surrounds the near field. The 'hydrodynamic' far field, which is characterized by an algebraic decay of r^{-1} as in the Crighton and Huerre paper [6], becomes a part of the transition region between the near field and the far field.

Expressions (13)–(15) are summarized in Figures 3(a) and (b). The length scales 1, ε^{-1} and ε^{-2} are the acoustic source integral length scale of the exponential, algebraic and Gaussian source respectively, by Crighton and Huerre [6], where the integral length scale Δ is the smallest scale still fulfilling

$$\int_{-\Delta}^{\Delta} A(\varepsilon x)\,\mathrm{e}^{\mathrm{i}x}\,\mathrm{d}x \sim \int_{-\infty}^{\infty} A(\varepsilon x)\,\mathrm{e}^{\mathrm{i}x}\,\mathrm{d}x. \tag{16}$$

Thus the case in which the penetration distance changes its dependence on M_c is the case where the source is no longer compact with respect to its integral length scale, since M_c^{-1} is in the order of the sound wave length. This is the same situation when the sound field

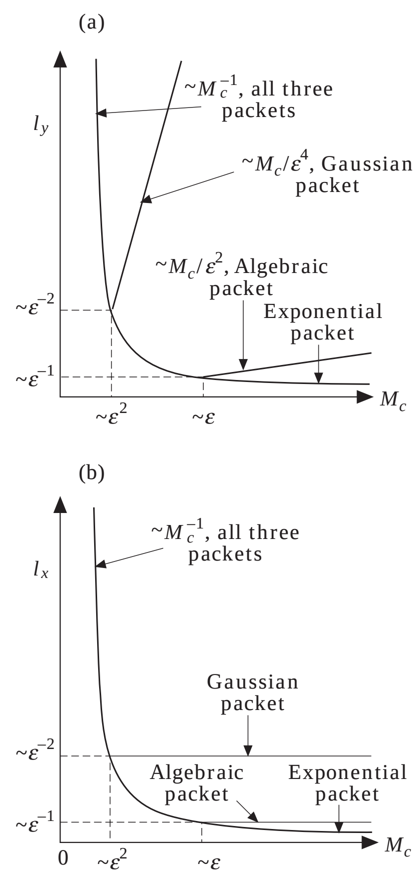

Figure 3. The asymptotic variation of the transition layer penetration distance in (a) the transverse direction and (b) the longitudinal direction for low subsonic Gaussian, algebraic and exponential wave packets, $M_c < O(1)$.

becomes superdirective as shown by Crighton and Huerre [6]. The result that the penetration distance is larger than the sound wave length scale and the source length (the integral scale) fits classical acoustics, but a somewhat surprising result is the behaviour at larger M_c where the penetration distance in the transverse direction l_y increases with increasing M_c. This increase in l_y is more due to a thickening of the transition region at the expense of the far field than to the thickening of the near field, since by equation (7) an exponential decay in the transverse direction persists in the near field of a subsonic Gaussian packet as long as

$$y \ll \varepsilon^{-2}, \tag{17.1}$$

for $M_c/\varepsilon < O(1)$ and

$$y \ll \sqrt[3]{1/(2\varepsilon^4 M_c^2)}, \tag{17.2}$$

for $M_c/\varepsilon > O(1)$. Similar calculations for the algebraic packet lead to $y < \varepsilon^{-1}$ instead of requirements (17) and to the same conclusions. This behaviour of l_y can have a significant influence on the design of a computational box for a direct simulation of the sound field.

Expressions (11) and (12) are also valid for a supersonic Gaussian wave packet ($M_c > O(1)$). The Mach wave is the main sound element, so the Mach angle direction $M_c \cos\theta = 1$ is dominant in the near field. In that direction the penetration distance is estimated as M_c/ε^2 by expressions (8) and (12). For the supersonic axisymmetric jet simulation of Mitchell *et al*. [11] ($M_c \cong 1{\cdot}5$, $\varepsilon \cong 0{\cdot}05$) this estimate yields a distance of 80 jet diameters for the wave packet of the fundamental frequency and a distance of 160 jet diameters for the first subharmonic one, which agrees with the dilatation contours from the simulation [11]. An expression similar to expression (12) cannot be derived for the algebraic source (5) in the Mach angle direction due to the singularity in $\hat{A}$, but it can be derived for the exponential source which yields again a penetration length scale of M_c/ε^2 in the Mach angle direction for $M_c > O(1)$.

5. CONCLUDING REMARKS

A wave packet model was employed to obtain simple analytic approximations for the pressure in the near field and the far field. The near field approximation showed that hydrodynamic exponential decay dominates for a subsonic source, while a Mach wave dominates for a supersonic source. The behaviour of the penetration distance of the transition region between the near field and the far field was also investigated. It was shown that for a low subsonic source the behaviour is determined by the sound wave length scale and the acoustic integral length scale. The penetration distance in the Mach angle direction for a supersonic source was also determined. The analytic approximations were compared with the exact numerical results of the present model and other experimental and numerical results reported in the literature with favourable results. It is suggested that these analytic approximations may be used as a rapid tool for preliminary design of a numerical simulation or experiment and also for validation of results.

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