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EXPLICIT COMPUTATION OF WEIGHTING COEFFICIENTS IN THE HARMONIC DIFFERENTIAL QUADRATURE

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1. INTRODUCTION

Recently, the method of differential quadrature (DQ) proposed by Bellman *et al*. [1] has been increasingly applied to solve many engineering problems such as fluid mechanics problems [2–4] and structural problems [5–7]. The key procedure in DQ is the determination of weighting coefficients for any order derivative discretization. Based on the analysis of high order polynomial approximation in a polynomial linear vector space, Shu [2] generalizes all the ways of computing the weighting coefficients in DQ and computes the weighting coefficients of the first order derivative by a simple algebraic formulation without any restriction on choice of grid points, and the weighting coefficients of the second and higher order derivatives by a recurrence relationship.

On the other hand, as indicated in reference [8], for some problems, especially for those with periodic behaviors, the polynomial approximation is not the best fitting. In contrast, the Fourier series expansion could be the best approximation. Following this idea, Striz *et al*. [8] choose the harmonic functions as the test functions in the DQ application. Using the same manner as in DQ, they then obtained a set of algebraic equations for determination of weighting coefficients. However, solving this algebraic equation system encounters the same difficulty as in the original DQ. In this paper, by following the Fourier series expansion and the same concept of generalized differential quadrature (GDQ) [2], one will demonstrate that the weighting coefficients can also be calculated by explicit formulations. The developed method is validated by its application to the free vibration analysis of rectangular plates which have been widely studied by many researchers [9–12].

2. HARMONIC DIFFERENTIAL QUADRATURE (HDQ)

For simplicity, the one dimensional problem is chosen to demonstrate the HDQ method. Following the idea of DQ, any derivative at a grid point is approximated by a linear summation of all the functional values in the whole computational domain. For example, the first and second order derivatives of $f(x)$ at a point x_i can be approximated by

$$f_x(x_i) = \sum_{j=1}^{N} a_{ij} f(x_j), \qquad \text{for } i = 1, 2, \ldots, N, \tag{1}$$

$$f_{xx}(x_i) = \sum_{j=1}^{N} b_{ij} f(x_j), \qquad \text{for } i = 1, 2, \ldots, N, \tag{2}$$

where N is the number of grid points, and a_{ij}, b_{ij} are the weighting coefficients. To determine a_{ij} and b_{ij}, one follows the same procedure as in GDQ.

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It is supposed that a function $f(x)$ in the interval $0 \leqslant x \leqslant 1$ is approximated by a Fourier series expansion in the form

$$f(x) = c_0 + \sum_{k=1}^{N/2} (c_k \cos k\pi x + d_k \sin k\pi x). \tag{3}$$

It is easy to show that $f(x)$ in equation (3) constitutes a $(N+1)$ dimensional linear vector space with respect to the operation of addition and multiplication. From the concept of linear independence, the bases of a linear vector space can be considered as a linearly independent subset which spans the entire space. Here if $r_k(x)$, $k = 0, 1, \ldots, N$, are the base functions, any function in the space can be expressed as a linear combination of $r_k(x)$, $k = 0, 1, \ldots, N$. And if all the base functions satisfy a linear constrained relationship such as equation (1) or (2), so does any function in the space. In the linear vector space, there may exist several sets of base functions. Each set of base functions can be expressed uniquely by another set of base functions. It is obviously observed from equation (3) that one set of base functions is 1, $\sin \pi x$, $\cos \pi x$, $\sin 2\pi x, \ldots, \sin (N\pi x/2)$, $\cos (N\pi x/2)$ which has been used in reference 8. Here for generality, two sets of base functions will be used in HDQ. Firstly, the Lagrange interpolated trigonometric polynomials are taken as one set of base functions,

$$r_k(x) = \frac{\sin \frac{x - x_0}{2}\pi \cdots \sin \frac{x - x_{k-1}}{2}\pi \sin \frac{x - x_{k+1}}{2}\pi \cdots \sin \frac{x - x_N}{2}\pi}{\sin \frac{x_k - x_0}{2}\pi \cdots \sin \frac{x_k - x_{k-1}}{2}\pi \sin \frac{x_k - x_{k+1}}{2}\pi \cdots \sin \frac{x_k - x_N}{2}\pi},$$

$$k = 0, 1, \ldots, N. \tag{4}$$

Setting

$$M(x) = \prod_{k=0}^{N} \sin \frac{x - x_k}{2}\pi = N(x, x_k) \sin \frac{x - x_k}{2}\pi, \tag{5}$$

where

$$N(x_i, x_i) = \prod_{k=0, k \neq i}^{N} \sin \frac{x_i - x_k}{2}\pi = P(x_i), \tag{6}$$

$$N(x_i, x_j) = N(x_i, x_i)\delta_{ij}, \; \delta_{ij} \text{ is the Kronecker operator,}$$

equation (4) can then be reduced to

$$r_k(x) = N(x, x_k)/P(x_k). \tag{7}$$

Using the same fashion as in GDQ, one lets all the base functions given by equation (7) satisfy two linear constrained relations (1) and (2). This results in the following two formulations

$$a_{ij} = N^{(1)}(x_i, x_j)/P(x_j), \qquad b_{ij} = N^{(2)}(x_i, x_j)/P(x_j), \tag{8, 9}$$

where $N^{(1)}(x, x_k)$ and $N^{(2)}(x, x_k)$ are the first and second order derivatives of the function $N(x, x_k)$. It is observed from equations (8) and (9) that the computation of a_{ij} and b_{ij} is equivalent to the evaluation of $N^{(1)}(x_i, x_j)$ and $N^{(2)}(x_i, x_j)$ since $P(x_j)$ can be easily calcualted by equation (6). To evaluate $N^{(1)}(x_i, x_j)$ and $N^{(2)}(x_i, x_j)$, equation (5) is differentiated successively to obtain

$$M^{(1)}(x) = N^{(1)}(x, x_k) \sin \frac{x - x_k}{2} \pi + \frac{\pi}{2} N(x, x_k) \cos \frac{x - x_k}{2} \pi, \tag{10}$$

$$M^{(2)}(x) = N^{(2)}(x, x_k) \sin \frac{x - x_k}{2} \pi + \pi N^{(1)}(x, x_k) \cos \frac{x - x_k}{2} \pi - \frac{\pi^2}{4} N(x, x_k) \sin \frac{x - x_k}{2} \pi, \tag{11}$$

$$M^{(3)}(x) = N^{(3)}(x, x_k) \sin \frac{x - x_k}{2} \pi + \frac{3\pi}{2} N^{(2)}(x, x_k) \cos \frac{x - x_k}{2} \pi - \frac{3\pi^2}{4} N^{(1)}(x, x_k) \sin \frac{x - x_k}{2} \pi - \frac{\pi^3}{8} N(x, x_k) \cos \frac{x - x_k}{2} \pi. \tag{12}$$

From the above equations, one can obtain the following results

$$N^{(1)}(x_i, x_j) = \pi P(x_i) \Big/ 2 \sin \frac{x_i - x_j}{2} \pi, \qquad \text{when} \quad j \neq i, \tag{13}$$

$$N^{(1)}(x_i, x_i) = M^{(2)}(x_i)/\pi, \tag{14}$$

$$N^{(2)}(x_i, x_j) = M^{(2)}(x_i) - \pi N^{(1)}(x_i, x_j) \cos \frac{x_i - x_j}{2} \pi \Big/ \sin \frac{x_i - x_j}{2} \pi, \qquad \text{when} \quad j \neq i, \tag{15}$$

$$N^{(2)}(x_i, x_i) = \frac{2}{3\pi} \left[M^{(3)}(x_i) + \frac{\pi^3}{8} N(x_i, x_i) \right]. \tag{16}$$

Substituting equations (13), (14) into equation (8) one obtains

$$a_{ij} = \frac{\pi}{2} P(x_i) \Big/ P(x_j) \sin \frac{x_i - x_j}{2} \pi, \qquad \text{when} \quad j \neq i, \qquad a_{ii} = \frac{M^{(2)}(x_i)}{\pi P(x_i)}. \tag{17, 18}$$

Similarly, by substituting equations (15), (16) into equation (9) and using equations (17), (18),

$$b_{ij} = a_{ij} \left[2a_{ii} - \pi \operatorname{ctg} \frac{x_i - x_j}{2} \pi \right], \qquad \text{when} \quad j \neq i, \qquad b_{ii} = \frac{2}{3\pi} \left[\frac{M^{(3)}(x_i)}{P(x_i)} + \frac{\pi^3}{8} \right]. \tag{19, 20}$$

From equations (17), (19), a_{ij}, b_{ij} ($i \neq j$) can be easily computed. However, the calculation of a_{ii} (equation (18)) and b_{ii} (equation (20)) involves the computation of $M^{(2)}(x_i)$ and $M^{(3)}(x_i)$ which are not easy to compute. This difficulty can be removed by the following analysis. According to the analysis of a linear vector space, one set of base functions can be expressed uniquely by a linear sum of another set of base functions. Thus, if one set of base functions satisfies a linear equation like equation (1) or (2), so does another set of base functions. Therefore, a_{ij} and b_{ij} should also satisfy the following equations which are derived by using the base function 1 among the set of base functions $1, \sin x, \cos x, \sin 2x, \ldots, \sin(Nx/2), \cos(Nx/2)$

$$\sum_{j=1}^{N} a_{ij} = 0, \qquad \sum_{j=1}^{N} b_{ij} = 0. \tag{21, 22}$$

From equations (21) and (22), a_{ii} and b_{ii} can be easily calculated from $a_{ij}(i \neq j)$ and $b_{ij}(i \neq j)$. The weighting coefficient of the third and fourth order derivatives can be computed easily from a_{ij} and b_{ij} by

$$c_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}, \qquad d_{ij} = \sum_{k=1}^{N} b_{ik} b_{kj}, \tag{23, 24}$$

where c_{ij} and d_{ij} are the weighting coefficients of the third and fourth order derivatives, respectively.

3. FREE VIBRATION ANALYSIS OF RECTANGULAR PLATES

The non-dimensional equation for a thin uniform thickness, rectangular plate may be written as

$$\partial^4 W/\partial X^4 + 2\lambda^2\, \partial^4 W/\partial X^2\, \partial Y^2 + \lambda^4\, \partial^4 W/\partial Y^4 = \Omega^2 W \tag{25}$$

where W is the dimensionless mode function; Ω is the dimensionless frequency; $X = x/a$, $Y = y/b$ are dimensionless co-ordinates, a and b are the lengths of the plate edges; $\lambda = a/b$ is the aspect ratio. Further, $\Omega = \omega a^2 \sqrt{\rho/D}$, where ω is the dimensional circular frequency, $D = Eh^3/[12(1 - v^2)]$ is the flexural rigidity, E, v, ρ and h are Young's modulus, Poisson ratio, density of the plate material, and the plate thickness, respectively. Equation (25) is a fourth order partial differential equation with respect to X and Y. Thus, it requires two boundary conditions at each edge. The following three types of boundary conditions are considered.

3.1. *Simply–supported edge* (*SS*)

$$W = 0, \quad \frac{\partial^2 W}{\partial X^2} = 0, \quad \text{at} \quad X = 0 \quad \text{or} \quad X = 1, \tag{26a}$$

$$\text{and} \qquad W = 0, \quad \frac{\partial^2 W}{\partial Y^2} = 0, \quad \text{at} \quad Y = 0 \quad \text{or} \quad Y = 1. \tag{26b}$$

3.2. *Clamped edge* (*C*)

$$W = 0, \quad \frac{\partial W}{\partial X} = 0, \quad \text{at} \quad X = 0 \quad \text{or} \quad X = 1, \tag{27a}$$

$$\text{and} \qquad W = 0, \quad \frac{\partial W}{\partial Y} = 0, \quad \text{at} \quad Y = 0 \quad \text{or} \quad Y = 1. \tag{27b}$$

3.3. *Free edge* (*F*)

$$\frac{\partial^2 W}{\partial X^2} + v\lambda^2 \frac{\partial^2 W}{\partial Y^2} = 0, \quad \frac{\partial^3 W}{\partial X^3} + (2 - v)\lambda^2 \frac{\partial^3 W}{\partial X\, \partial Y^2} = 0, \qquad \text{at} \quad X = 0 \text{ or } 1 \tag{28a}$$

$$\text{and} \qquad \lambda^2 \frac{\partial^2 W}{\partial Y^2} + v \frac{\partial^2 W}{\partial X^2} = 0, \quad \lambda^2 \frac{\partial^3 W}{\partial Y^3} + (2 - v) \frac{\partial^3 W}{\partial X^2\, \partial Y} = 0, \qquad \text{at} \quad Y = 0 \text{ or } 1 \tag{28b}$$

$$\text{and} \quad \partial^2 W / \partial X \, \partial Y = 0, \tag{28c}$$

at the corner of two adjacent free edges.

By applying the HDQ or GDQ method, equation (25) can be discretized as

$$\sum_{k=1}^{N} c_{i,k}^{(4)} W_{k,j} + 2\lambda^2 \sum_{k1=1}^{N} \sum_{k2=1}^{M} c_{i,k1}^{(2)} \bar{c}_{j,k2}^{(2)} W_{k1,k2} + \lambda^4 \sum_{k=1}^{M} \bar{c}_{j,k}^{(4)} W_{i,k} = \Omega^2 W_{i,j}, \tag{29}$$

where N, M are the number of grid points in the X and Y directions, $c_{i,k}^{(n)}$, $\bar{c}_{j,k}^{(m)}$ are the HDQ or GDQ weighting coefficients related to the derivatives $\partial^n W / \partial X^n$, $\partial^m W / \partial Y^m$, respectively. Similarly, the derivatives in the boundary conditions (26), (27) and (28) can be discetized by the HDQ or GDQ method. Substituting the discretized boundary conditions into equation (29) gives the following eigenvalue equation system

$$[\mathbf{A}]\{\mathbf{W}\} = \Omega^2 \{\mathbf{W}\}. \tag{30}$$

Obviously, the frequencies Ω can be given from the eigenvalue of matrix $[\mathbf{A}]$.

For the rectangular plate, the co-ordinates of grid points are chosen as

$$X_i = \left[1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right] \Big/ 2, \qquad i = 1, 2, \ldots, N, \tag{31}$$

$$Y_j = \left[1 - \cos\left(\frac{j-1}{M-1}\pi\right)\right] \Big/ 2, \qquad j = 1, 2, \ldots, M. \tag{32}$$

The free vibration analysis of rectangular plates has been studied by many researchers. There is a variety of publications available [9–12]. Among those, the work of Leissa [12] is most complete in that it presents the frequency data of all twenty-one plate configurations for the first nine modes and for a wide range of aspect ratios. In this study, the developed HDQ method and the previously developed GDQ method are applied to the free vibration analysis of rectangular plates with the above mentioned three types of boundary conditions and their results are compared to Leissa's data [12]. The numerical results are presented for aspect ratios of $\lambda = a/b = 2/5$, 2/3, 1, 3/2, 5/2. Table 1 shows the natural frequencies of the first five modes for a plate with all four edges simply–supported (SS–SS–SS–SS). The HDQ, GDQ and Leissa's results [12] are included in the table. The HDQ results are obtained by the mesh size of 9×9 while the GDQ results are given from the mesh size of 15×15. For the SS–SS–SS–SS boundary condition, Leissa's results are the exact solutions. It can be observed that for this case, the HDQ results are almost identical to the exact solutions even though very few grid points are used. Actually, the HDQ results using the mesh size of 9×9 have better accuracy than the GDQ results using the mesh size of 15×15. It is indicated that the SS–SS–SS–SS plate configuration has periodic behaviors. Thus, for this case, the HDQ results are much more accurate than the GDQ results. Table 2 lists the natural frequencies of the first five modes for a plate with all four edges clamped (C–C–C–C). The HDQ, GDQ and Leissa's results [12] are included in the table for comparison. For this case, the HDQ and GDQ results are given from the mesh size of 15×15. It can be seen that, by comparison with the Leissa data [12], the HDQ results are slightly better than the GDQ results for the C–C–C–C boundary conditions. Table 3 displays the natural frequencies of the first five modes for a plate configuration of C–F–SS–F. The HDQ, GDQ and Leissa's results [12] are shown in the table for comparison. The HDQ and GDQ results are obtained by using a mesh size of 15×15. It can be observed from Table 3 that for the fundamental frequency, the HDQ results are

TABLE 1

Natural frequencies of a rectangular plate (SS–SS–SS–SS)

$\lambda = a/b$	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2/5	Leissa [12]	11·4487	16·1862	24·0818	35·1358	41·0576
	HDQ (9 × 9)	11·4487	16·1862	24·0818	35·1358	41·0575
	GDQ (15 × 15)	11·4487	16·1862	24·0817	35·1526	41·0575
2/3	Leissa [12]	14·2561	27·4156	43·8649	49·3480	57·0244
	HDQ (9 × 9)	14·2561	27·4156	43·8649	49·3481	57·0244
	GDQ (15 × 15)	14·2561	27·4156	43·8649	49·3475	57·0244
1	Leissa [12]	19·7392	49·3480	49·3480	78·9568	98·6960
	HDQ (9 × 9)	19·7392	49·3480	49·3480	78·9568	98·6960
	GDQ (15 × 15)	19·7414	49·3480	49·3481	78·9568	98·6947
3/2	Leissa [12]	32·0762	61·6850	98·6960	111·0330	128·3049
	HDQ (9 × 9)	32·0762	61·6850	98·6960	111·0331	128·3049
	GDQ (15 × 15)	32·0762	61·6850	98·6960	111·0318	128·3048
5/2	Leissa [12]	71·5564	101·1634	150·5115	219·5987	256·6097
	HDQ (9 × 9)	71·5564	101·1634	150·5115	219·5986	256·6097
	GDQ (15 × 15)	71·5564	101·1634	150·5106	219·7034	256·6096

more accurate than the GDQ results. However, for other frequencies, the HDQ results are less accurate than the GDQ results.

4. CONCLUSIONS

The explicit formulations for computing the weighting coefficients in the harmonic differential quadrature (HDQ) have been developed. In HDQ, the solution of a differential equation is approximated by a Fourier series expansion. For the free vibration analysis of rectangular plates with all edges simply-supported (SS–SS–SS–SS), it was found that the HDQ method is very efficient and its results are much more accurate than the GDQ results. For the C–C–C–C plate configuration, the HDQ results are slightly better than the GDQ results. For the plate configuration with at least one free edge, it was found that

TABLE 2

Natural frequencies of a rectangular plate (C–C–C–C)

$\lambda = a/b$	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2/5	Leissa [12]	23·648	27·817	35·446	46·702	61·554
	HDQ (15 × 15)	23·644	27·810	35·422	46·687	61·520
	GDQ (15 × 15)	23·644	27·807	35·418	46·681	61·592
2/3	Leissa [12]	27·010	41·716	66·143	66·552	79·850
	HDQ (15 × 15)	27·006	41·709	66·132	66·528	79·823
	GDQ (15 × 15)	27·005	41·704	66·125	66·522	79·806
1	Leissa [12]	35·992	73·413	73·413	108·270	131·640
	HDQ (15 × 15)	35·986	73·402	73·402	108·241	131·591
	GDQ (15 × 15)	35·986	73·394	73·394	108·217	131·580
3/2	Leissa [12]	60·772	93·860	148·820	149·740	179·660
	HDQ (15 × 15)	60·763	93·844	148·796	149·688	179·601
	GDQ (15 × 15)	60·761	93·834	148·780	149·674	179·564
5/2	Leissa [12]	147·800	173·850	221·540	291·890	384·710
	HDQ (15 × 15)	147·778	173·812	221·385	291·794	384·437
	GDQ (15 × 15)	147·772	173·796	221·363	291·756	384·951

TABLE 3

Natural frequencies of a rectangular plate (C–F–SS–F) ($\nu = 0{\cdot}3$)

$\lambda = a/b$	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2/5	Leissa [12]	15·382	16·371	19·656	25·549	34·507
	HDQ (15 × 15)	15·367	16·530	19·804	26·048	34·740
	GDQ (15 × 15)	15·347	16·357	19·711	25·647	34·514
2/3	Leissa [12]	15·340	17·949	26·734	43·190	49·840
	HDQ (15 × 15)	15·342	18·333	27·043	44·118	49·785
	GDQ (15 × 15)	15·319	18·018	26·908	43·383	49·637
1	Leissa [12]	15·285	20·673	39·775	49·730	56·617
	HDQ (15 × 15)	15·252	21·373	40·092	49·615	57·253
	GDQ (15 × 15)	15·232	20·693	39·882	49·500	56·393
3/2	Leissa [12]	15·217	25·711	49·550	64·012	68·126
	HDQ (15 × 15)	15·180	26·865	49·382	65·384	68·573
	GDQ (15 × 15)	15·154	25·750	49·269	63·802	68·208
5/2	Leissa [12]	15·128	37·294	49·226	83·325	103·140
	HDQ (15 × 15)	15·119	39·218	49·051	85·889	102·426
	GDQ (15 × 15)	15·055	37·365	48·896	83·177	102·687

the HDQ method provides more accurate fundamental frequency than the GDQ method. However, for other frequencies, the GDQ results are more accurate than the HDQ results.

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