Journal of Sound and Vibration (1997) **204**(4), 595–607

A COMPARISON OF DIFFERENT VERSIONS OF THE METHOD OF MULTIPLE SCALES FOR PARTIAL DIFFERENTIAL EQUATIONS

H. Boyaci and M. Pakdemirli

Department of Mechanical Engineering, Celal Bayar University, Muradiye, Manisa, Turkey

(*Received* 8 *April* 1996, *and in final form* 3 *February* 1997)

Applications of the methods of multiple scales (a perturbation method) to partial differential systems arising in non-linear vibrations of continuous systems are considered. Two different versions of the method of multiple scales are applied to two general non-linear models. In one of the models, the small parameter (ε) multiplies an arbitrary non-linear cubic operator whereas in the other model, arbitrary quadratic and cubic non-linearities exist. The linear parts of both models are represented by arbitrary operators. General solutions are found by applying different versions of the method of multiple scales. Results of the first version (reconstitution method) and the second version (proposed by Rahman and Burton [8]) are compared for both models. From the comparisons of both methods, it is found that the second version yields better results. Applications of the general models to specific problems are also presented. A final recommendation is to use the second version of the method of multiple scales combined with the direct-perturbation method in finding steady state solutions of partial differential equations.

1. INTRODUCTION

Perturbation methods are widely used for seeking approximate solutions of linear and non-linear differential equations. The simplest perturbation method is the pedestrian expansion. With this expansion, the troublesome terms such as secular, singular or resonant terms cannot be handled. Many different perturbation techniques are developed to eliminate those terms. In some of these techniques, such as Linstedt–Poincaré, harmonic balance, renormalization, one seeks periodic solutions *a priori* whereas in some others such as Lie series and transformations, generalized averaging, the Krylov–Bogoliubov–Mitropolski technique, method of multiple scales, the transient as well as the steady state response can be retrieved. The latter group provides therefore more information about the dynamics of the system. Among the latter group, the method of multiple scales is one of the frequently used methods.

Recently, there have been some studies [1–7] addressing the application of this method to partial differential equations. In these studies, comparisons are made for two different approaches. In the first approach, the method of multiple scales is directly applied to the partial differential system (direct-perturbation method). In the second approach, the partial differential system is discretized first and then the method of multiple scales is applied to the resulting ordinary differential equations (discretization-perturbation method). It is shown that for finite mode truncations [1–4, 7], the direct approach yields better approximations whereas for infinite modes [4–6] both methods yield identical results. However, as discussed in reference [5], results of the direct approach are easier to handle for infinite mode analysis. This is because the converged functions rather than the infinite

0022–460X/97/290595 + 13 $25.00/0/sv970951

set of eigenfunctions are retrieved for the mode shapes appearing at higher orders of approximation.

In an important work, Rahman and Burton [8] suggests an improvement for the method of multiple scales. They showed that the usual ordering of damping and external excitation, the usual expansion of external frequency (referred to as MMS version I in reference [8]) produces extra non-physical results for some cases. They propose a different expansion and ordering in which those results can be eliminated (referred to the MMS version II in reference [8]). Results of MMS versions I and II are compared for ordinary differential equations, specifically for a well-known problem, the Duffing oscillator. The primary resonances are considered in the analysis. Recently, Hassan [9] reinvestigated the Duffing oscillator problem, this time for the superharmonic resonances. He found that although MMS version II produces better results compared to the MMS version I, the improvement is not as good as in the case of primary resonances.

In this study, the advantages of the direct-perturbation method and MMS version II for solving partial differential equations are combined. As an illustration, two general partial differential equations are treated, one with arbitrary cubic non-linearity and the other with arbitrary quadratic and cubic non-linearities. Both equations are solved using MMS versions I and II. Results of versions I and II are compared for each equation for the primary resonance case. In all calculations, the direct-perturbation method is utilized. Since the proposed models are general, the solutions of these equations are valid for a large class of problems. The algorithm developed is applied to specific problems. Note that the models considered in this paper are weakly non-linear systems. The solutions are the perturbed solutions of the corresponding linear systems and results cease to be valid for strongly non-linear systems.

2. EQUATION WITH ARBITRARY CUBIC NON-LINEARITIES

In this section, an equation with arbitrary cubic non-linearities is considered. The equation is solved using both versions of MMS. The non-dimensional equation considered has the general form

$$\ddot{w} + \hat{\mu}\dot{w} + \mathbf{L}(w) + \varepsilon\mathbf{C}(w, w, w) = \hat{F}(x)\cos\Omega t, \tag{1}$$

where w is the response function in time (t) and spatial variable (x), ε is a small dimensionless physical parameter ($\varepsilon \ll 1$) and $\hat{\mu}$ is the damping coefficient. ($\dot{\ }$) represents differentiation with respect to time. $\mathbf{L}$ is an arbitrary linear spatial differential and/or integral operator and $\mathbf{C}$ is an arbitrary spatial differential and/or integral operator representing cubic non-linearities. $\hat{F}$ is the amplitude and Ω is the frequency of the external excitation. The cubic operator possesses the property of being multi-linear such that

$$\mathbf{C}(c_1w_1 + c_2w_2, c_3w_3 + c_4w_4, c_5w_5 + c_6w_6) = c_1c_3c_5\mathbf{C}(w_1, w_3, w_5) + c_1c_3c_6\mathbf{C}(w_1, w_3, w_6) + \cdots,$$

$$(\mathbf{C}(w_1, w_3, w_5) \neq \mathbf{C}(w_3, w_1, w_5) \neq \cdots \quad \text{in general}),$$

where c_i are arbitrary constants. To simplify the calculations, one assumes that the associated boundary conditions for equation (1) are linear, homogenous and free from time derivatives.

First, equation (1) is rewritten using the new time scale defined as $T = \Omega t$. The new equation is of the form

$$\Omega^2\ddot{w} + \bar{\mu}\dot{w} + \mathbf{L}(w) + \varepsilon\mathbf{C}(w, w, w) = \hat{F}\cos T, \tag{2}$$

where $\bar{\mu} = \Omega\bar{\mu}$ and $(\dot{})$ now represents time differentiation with respect to the new variable T.

First, the classical version of the method of multiple scales (MMS version I) and then the new version, developed by Rahman and Burton [8] (MMS version II), are applied to equation (1).

2.1. MMS VERSION I

In this section, a solution uniformly valid up to third order for equation (2) is presented. The primary resonance case is considered by assuming an expansion of the form

$$w(x, t; \varepsilon) = w_0(x, T_o, T_1, T_2) + \varepsilon w_1(x, T_0, T_1, T_2) + \varepsilon^2 w_2(x, T_0, T_1, T_2) + \cdots \tag{3}$$

where $T_0 = T$ is the fast-time scale whereas $T_1 = \varepsilon T$ and $T_2 = \varepsilon^2 T$ are the slow-time scales. The damping and the external excitation are reordered so that their effects balance the effect of non-linearity. Thus, one substitutes $\bar{\mu} = \varepsilon\mu$ and $\hat{F} = \varepsilon F$. Then, the nearness of the external frequency to one of the natural frequencies of the linear system is expressed as

$$\Omega^2 = \omega^2 + \varepsilon\sigma, \tag{4}$$

where σ is a detuning parameter of $O(1)$. The time derivatives are expressed as

$$(\dot{}) = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \qquad (\ddot{}) = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \cdots, \tag{5}$$

where $D_i = \partial/\partial T_i$.

Substituting equations (3–5) into equation (2) and equating coefficients of like powers of ε yields

order ε^0:

$$\omega^2 D_0^2 w_0 + \mathbf{L}(w_0) = 0; \tag{6}$$

order ε^1:

$$\omega^2 D_0^2 w_1 + \mathbf{L}(w_1) = -(\sigma D_0^2 + 2\omega^2 D_0 D_1 + \mu D_0)w_0 - \mathbf{C}(w_0, w_0, w_0) + F\cos T_0; \tag{7}$$

order ε^2;

$$\begin{aligned}\omega^2 D_0^2 w_2 + \mathbf{L}(w_2) = & -(2\omega^2 D_0 D_2 + \omega^2 D_1^2 + 2\sigma D_0 D_1 + \mu D_1)w_0 \\ & - (2\omega^2 D_0 D_1 + \sigma D_0^2 + \mu D_0)w_1 \\ & - \mathbf{C}(w_0, w_0, w_1) - \mathbf{C}(w_0, w_1, w_0) - \mathbf{C}(w_1, w_0, w_0).\end{aligned} \tag{8}$$

One assumes a solution at order ε^0 of the form

$$w_0 = [A(T_1, T_2)\,\mathrm{e}^{iT_0} + cc]Y(x), \tag{9}$$

where cc stands for the complex conjugate of the precedings terms and Y is defined by the equation

$$\mathbf{L}(Y) - \omega^2 Y = 0. \tag{10}$$

Clearly, equation (10) with the associated boundary conditions is an eigenvalue–eigenfunction problem where ω^2 are the eigenvalues and Y are the corresponding eigenfunctions of the system. For continuous systems there are infinitely many eigenvalues.

Substituting equation (9) to the right side of equation (7), one obtains

$$\begin{aligned}\omega^2 D_0^2 w_1 + \mathbf{L}(w_1) = & [(\sigma A - \mathrm{i}\mu A - 2\mathrm{i}\omega^2 D_1 A)\,\mathrm{e}^{\mathrm{i}T_0} + cc]Y \\ & -(A\,\mathrm{e}^{\mathrm{i}T_0} + cc)^3\mathbf{C}(Y, Y, Y) + \tfrac{1}{2}(F\,\mathrm{e}^{\mathrm{i}T_0} + cc).\end{aligned} \tag{11}$$

One assumes a solution for w_1 of the form

$$w_1 = \psi_1(x, T_1, T_2)\, e^{iT_0} + W_1(x, T_0, T_1, T_2) + cc, \tag{12}$$

where W_1 is governed by equation (11) with the terms multiplying e^{iT_0} being deleted and ψ_1 is governed by the equation

$$\mathbf{L}(\psi_1) - \omega^2\psi_1 = (\sigma A - i\mu A - 2i\omega^2 D_1 A)Y - 3A^2\tilde{A}\mathbf{C}(Y, Y, Y) + F/2. \tag{13}$$

Note that $(\tilde{\ })$ denotes complex conjugate. Recalling that the boundary conditions are linear and homogenous and assuming further that $\mathbf{L}$ is self-adjoint, one finds the solvability condition [10] for equation (13) as:

$$2i\omega^2 D_1 A + (i\mu - \sigma)A + 3\alpha_1 A^2\tilde{A} - f/2 = 0, \tag{14}$$

where α_1 and f are defined as

$$\alpha_1 = \int_D Y\mathbf{C}(Y, Y, Y)\, dx, \qquad f = \int_D YF\, dx, \tag{15}$$

where D is the domain of integration. Note that the integral $\int_D Y^2\, dx = 1$ is normalized in the calculations.

Having eliminated the secular terms, one finds that W_1 is governed by the equation

$$\omega^2 D_0^2 W_1 + \mathbf{L}(W_1) = -(A^3 e^{3iT_0} + cc)\mathbf{C}(Y, Y, Y). \tag{16}$$

A solution can be assumed of the form

$$W_1 = (A^3 e^{3iT_0} + cc)\phi(x), \tag{17}$$

where ϕ satisfies the equation

$$\mathbf{L}(\phi) - 9\omega^2\phi = -\mathbf{C}(Y, Y, Y). \tag{18}$$

At order ε^2, one substitutes solutions (9) and (17) into equation (8) and obtains the equation

$$\begin{aligned}\omega^2 D_0^2 w_2 + \mathbf{L}(w_2) = {} & -[(2i\omega^2 D_2 A + \omega^2 D_1^2 A + 2i\sigma D_1 A + \mu D_1 A)\, e^{iT_0} + cc]Y \\ & + [(9\sigma A^3 - 6i\omega^2 D_1 A^3 - 3i\mu A^3)\, e^{3iT_0} + cc]\phi - (A\, e^{iT_0} + cc)^2 \\ & \cdot (A^3 e^{3iT_0} + cc)[\mathbf{C}(Y, Y, \phi) + \mathbf{C}(Y, \phi, Y) + \mathbf{C}(\phi, Y, Y).\end{aligned} \tag{19}$$

Here, one assumes a solution for w_2 of the form

$$w_2 = \psi_2(x, T_1, T_2)\, e^{iT_0} + W_2(x, T_0, T_1, T_2) + cc, \tag{20}$$

where W_2 is governed by equation (19) with the terms multiplying e^{iT_0} being deleted and ψ_2 is governed by the equation

$$\begin{aligned}\mathbf{L}(\psi_2) - \omega^2\psi_2 = {} & -(2i\omega^2 D_2 A + \omega^2 D_1^2 A + 2i\sigma D_1 A + \mu D_1 A)Y \\ & - A^3\tilde{A}^2[\mathbf{C}(Y, Y, \phi) + \mathbf{C}(Y, \phi, Y) + \mathbf{C}(\phi, Y, Y)].\end{aligned} \tag{21}$$

The solvability condition for equation (21) is:

$$2i\omega^2 D_2 A + \omega^2 D_1^2 A + (2i\sigma + \mu)D_1 A + \alpha_2 A^3\tilde{A}^2 = 0, \tag{22}$$

where α_2 is defined as

$$\alpha_2 = \int_D Y[\mathbf{C}(Y, Y, \phi) + \mathbf{C}(Y, \phi, Y) + \mathbf{C}(\phi, Y, Y)]\,\mathrm{d}x. \tag{23}$$

Equations (14) and (22) can be combined to form a single equation of motion for the slow evaluation of A in time T,

$$\mathrm{d}A/\mathrm{d}T = \varepsilon D_1 A + \varepsilon^2 D_2 A + O(\varepsilon^3), \tag{24}$$

where equation (14) is to be used to determine the term $D_1^2 A$ which appears in equation (22). The explicit form of equation (24) is

$$\begin{aligned}2\mathrm{i}\omega^2\,\mathrm{d}A/\mathrm{d}T = {} & \varepsilon[(\sigma - \mathrm{i}\mu)A - 3\alpha_1 A^2\tilde{A} + f/2] + \varepsilon^2[(f/8\omega^2)(\mathrm{i}\mu - 3\sigma) \\ & + (1/4\omega^2)(\mu^2 - 3\sigma^2 + 4\mathrm{i}\sigma\mu)A + (3\alpha_1 f/8\omega^2)(A^2 - 2A\tilde{A}) \\ & + (3\alpha_1/2\omega^2)(\sigma + \mathrm{i}\mu)A^2\tilde{A} + (9\alpha_1^2/4\omega^2 - \alpha_2)A^3\tilde{A}^2].\end{aligned} \tag{25}$$

The steady-state solution can be obtained by setting $\mathrm{d}A/\mathrm{d}T$ to zero and substituting the polar form $A = 1/2a\,\mathrm{e}^{\mathrm{i}\beta}$ into equation (25). The result of this calculation, which defines the second order frequency amplitude (a, σ) dependence, may be expressed in the compact form

$$\Delta^2 = (R_1 a_{22} - R_2 a_{12})^2 + (R_2 a_{11} - R_1 a_{21})^2, \tag{26}$$

where

$$\begin{gathered}\Delta = a_{11}a_{22} - a_{12}a_{21}, \qquad a_{11} = f[1 - \varepsilon(3\sigma/4\omega^2 + (3\alpha_1/16\omega^2)a^2)], \\ a_{12} = a_{21} = \varepsilon\mu f/4\omega^2, \qquad a_{22} = f[-1 + \varepsilon(3\sigma/4\omega^2 + (9\alpha_1/16\omega^2)a^2)], \\ R_1 = \tfrac{3}{4}\alpha_1 a^3 - \sigma a - \varepsilon[((\mu^2 - 3\sigma^2)/4\omega^2)a + (3\sigma\alpha_1/8\omega^2)a^3 + \tfrac{1}{16}(9\alpha_1^2/4\omega^2 - \alpha_2)a^5], \\ R_2 = \mu a - \varepsilon((\mu\sigma/\omega^2)a + (3\mu\alpha_1/8\omega^2)a^3)\end{gathered} \tag{27}$$

Equation (26) is similar to equation (28) of reference [8] qualitatively. Hence, if $\Omega - a$ is plotted, there will be extra solutions which have no physical meaning. In Figures 1, 2 and 3 of reference [8], these extra solutions are shown clearly. The method outlined in this section is the usual method of reconstitution combined with the direct-perturbation method. In the next section, the same problem is solved using the method given in reference [8] combined with the direct-perturbation approach. Using discretization before perturbation produces less accurate results for finite mode truncations [1–7].

2.2. MMS VERSION II

Instead of ordering damping and frequency as in MMS version I, Rahman and Burton [8] recommended the following expansions for the damping and external excitation frequency:

$$\hat{F} = \varepsilon F, \qquad \bar{\mu} = \varepsilon\mu_1 + \varepsilon^2\mu_2, \qquad \Omega^2 = \omega^2 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2. \tag{28}$$

They also suggested that each term on the right side of equation (24) vanish separately to prevent the violation of ordering. For a detailed discussion, the reader is referred to reference [8].

Substituting the expansion (28) and (3) into equation (2), equating the coefficients of like powers of ε, one again obtains equation (6) at order ε^0. The solution at this order is given by equation (9) where the Y function also satisfies equation (10). At order ε^1, one has the equation

$$\omega^2 D_0^2 w_1 + \mathbf{L}(w_1) = -(\sigma_1 D_0^2 + 2\omega^2 D_0 D_1 + \mu_1 D_0)w_0 - \mathbf{C}(w_0, w_0, w_0) + F\cos T_0. \quad (29)$$

Using similar steps, as in the case of MMS I, one finally obtains the solvability condition for (29) as

$$2\mathrm{i}\omega^2 D_1 A + (\mathrm{i}\mu_1 - \sigma_1)A + 3\alpha_1 A^2 \tilde{A} - f/2 = 0. \quad (30)$$

The expressions α_1 and f are of the same form as in equation (15). At this order, the solvability conditions (30) and (14) are exactly the same. The solution to equation (29) after eliminating the secular terms is again W_1 given by equation (17).

In order to find steady state solutions, as mentioned earlier, Rahman and Burton suggest that each $D_i A$ in equation (24) vanish separately. Setting $D_1 A$ to zero and substituting the polar form $A = 1/2a\,\mathrm{e}^{\mathrm{i}\beta}$, one obtains the first correction to the frequency

$$\sigma_1 = \tfrac{3}{4}\alpha_1 a^2 \pm \sqrt{f^2/a^2 - \mu_1^2}. \quad (31)$$

At order ε^2, one has the equation

$$\begin{aligned}\omega^2 D_0^2 w_2 + \mathbf{L}(w_2) = {} & -(2\omega^2 D_0 D_2 + \omega^2 D_1^2 + 2\sigma_1 D_0 D_1 + \sigma_2 D_0^2 + \mu_1 D_1 + \mu_2 D_0)w_0 \\ & -(2\omega^2 D_0 D_1 + \sigma_1 D_0^2 + \mu_1 D_0)w_1 - \mathbf{C}(w_0, w_0, w_1) \\ & - \mathbf{C}(w_0, w_1, w_0) - \mathbf{C}(w_1, w_0, w_0). \end{aligned} \quad (32)$$

Setting $D_1 A$ to zero, substituting w_0 and w_1 into equation (32), carrying out the algebra as outlined in the previous section, one finally obtains the solvability condition at $O(\varepsilon^2)$ as

$$2\mathrm{i}\omega^2 D_2 A + (\mathrm{i}\mu_2 - \sigma_2)A + \alpha_2 A^3 \tilde{A}^2 = 0, \quad (33)$$

where α_2 is defined in equation (23). If one sets $D_2 A$ to zero and substitute the polar form $A = 1/2a\,\mathrm{e}^{\mathrm{i}\beta}$ into equation (33), one obtains the second correction to the frequency

$$\sigma_2 = \tfrac{1}{16}\alpha_2 a^4, \qquad \mu_2 = 0. \quad (34)$$

Finally, σ_1 and σ_2 from equations (31) and (34) are substituted into the frequency expansion given in equation (28) to obtain the frequency response equation

$$\Omega^2 = \omega^2 + \varepsilon[\tfrac{3}{4}\alpha_1 a^2 \pm \sqrt{f^2/a^2 - \mu_1^2}] + \varepsilon^2 \tfrac{1}{16}\alpha_2 a^4. \quad (35)$$

Comparing equation (35) with equation (26), one sees that there are major differences. Equation (26) leads to spurious solutions as already demonstrated in reference [8] whereas equation (35) does not include those non-physical results. An application and plot of equation (35) will be given in the next section. It is worth mentioning that those spurious solutions are also evident in reference [11] (Figure 3 of that reference).

Recently, Lee and Perkins [12] used MMS version II combined with discretization. It is worth mentioning that a direct-perturbation method, instead, would yield better results [2]. In addition to the expansions given in equations (28), they also expanded the excitation amplitude,

$$\hat{F} = \varepsilon F_1 + \varepsilon^2 F_2. \quad (36)$$

Performing the calculations for this choice, one has

$$\Omega^2 = \omega^2 + \varepsilon(\tfrac{3}{4}\alpha_1 a^2 \pm \sqrt{f_1^2/a_2 - \mu_1^2}) + \varepsilon^2(\tfrac{1}{16}\alpha_2 a^4 \pm \sqrt{f_2^2/a^2 - \mu_2^2}). \quad (37)$$

It can be shown that both square root terms in equation (37) reduce to the single square root term in equation (35) if $f = f_1 + \varepsilon f_2$ is taken. Hence, the excitation amplitude expansion is redundant. If special care is not taken, it can even lead to erroneous results. For equation (37) to produce identical results to those of equation (35), the damping and the excitation amplitude should be expanded in proportional manner:

$$f_1/\mu_1 = f_2/\mu_2. \tag{38}$$

By employing equation (38), the artificial increase in the control parameters of the problem will be avoided.

Finally, the expansion of the excitation amplitude without expanding the damping coefficient would spoil the uniqueness of the solutions.

In conclusion, MMS version I (as in the case of ordinary differential equations) produces some non-physical steady-state solutions whereas the inconsistency is eliminated in version II.

2.3. AN APPLICATION FOR MMS VERSION II

In this section, the general formalism regarding the operators is illustrated by treating an application from continuous system vibrations. Since the differences between the two versions are clearly indicated on the general equation, we consider here the solution procedure for MMS version II only. For an Euler–Bernoulli beam simply supported at both ends and resting on a non-linear elastic foundation, the non-dimensional equation of motion and the associated boundary conditions are

$$\Omega^2 \ddot{w} + \bar{\mu}\dot{w} + w^{iv} + kw + \varepsilon w^3 = \hat{F}(x)\cos T, \tag{39}$$

$$w(0, T) = w''(0, T) = w(1, T) = w''(1, T) = 0. \tag{40}$$

The linear and cubic operators have the forms

$$\mathbf{L}(w) = w^{iv} + kw, \qquad \mathbf{C}(w, w, w) = w^3. \tag{41}$$

Assuming an expansion of the form defined in equation (3) for deflection w, and expansions of the form defined in equation (28) for damping, excitation amplitude and frequency, one obtains equations (6), (29) and (32) at orders ε^0, ε^1 and ε^2 respectively. At order ε^0, solution w_0 is of the same form as in equation (9) and the Y function satisfies equation (10) or the equation

$$Y^{iv} + (k - \omega^2)Y = 0, \qquad Y(0) = Y''(0) = Y(1) = Y''(1) = 0. \tag{42, 43}$$

The solution for this system is

$$Y_n(x) = \sqrt{2}\sin(n\pi x), \qquad \omega_n^2 = n^4\pi^4 + k, \qquad n = 1, 2, \ldots \tag{44, 45}$$

At order ε^1, the solution is given in equation (17) and the function ϕ satisfies equation (18). Thus,

$$\phi_n^{iv} + (k - 9\omega^2)\phi_n = -2\sqrt{2}\sin^3(n\pi x), \tag{46}$$

$$\phi_n(0) = \phi_n''(0) = \phi_n(1) = \phi_n''(1) = 0. \tag{47}$$

A solution satisfying equations (46) and (47) is

$$\phi_n(x) = (3\sqrt{2}/16(n^4\pi^4 + k))\sin(n\pi x) + (\sqrt{2}/16(9n^4\pi^4 - k))\sin(3n\pi x). \tag{48}$$

The deflection function w up to order ε^2 is found by substituting the transformation $T = \Omega t$ and the polar form $A = 1/2a\,\mathrm{e}^{\mathrm{i}\beta}$ and using equations (9), (17), (44) and (48):

$$w_n = a\cos(\Omega t+\beta)\sqrt{2}\sin(n\pi x) + \varepsilon\{(a^3/4)\cos[3(\Omega t+\beta)]$$

$$\cdot[(3\sqrt{2}/16(n^4\pi^4+k))\sin(n\pi x) + (\sqrt{2}/16(9n^4\pi^4-k))\sin(3n\pi x)]\} + O(\varepsilon^2). \quad (49)$$

The coefficients α_1 and α_2 are calculated from equations (15) and (23), respectively:

$$\alpha_1 = 3/2, \qquad \alpha_2 = \tfrac{15}{16}(8n^4\pi^4-k)/(9n^8\pi^8+8n^4\pi^4k-k^2). \quad (50)$$

Finally, one obtains the frequency response equation from equation (35) as follows:

$$\Omega^2 = (n^4\pi^4+k) + \varepsilon[\tfrac{9}{8}a^2 \pm \sqrt{f^2/a^2-\mu_1^2}]$$

$$+\varepsilon^2[\tfrac{15}{256}[(8n^4\pi^4-k)/(9n^8\pi^8+8n^4\pi^4k-k^2)]a^4], \quad (51)$$

where $\mu_1 = \Omega\hat{\mu}/\varepsilon$. A sample plot of equation (51) is given in Figure 1 for the parameter values $n=1$, $k=10$, $\varepsilon=0{\cdot}1$, $f=2$ and $\hat{\mu}=0{\cdot}004$.

3. EQUATION WITH ARBITRARY QUADRATIC AND CUBIC NON-LINEARITIES

In this section, another type of general model having arbitrary quadratic and cubic non-linearities is treated:

$$\ddot{w} + \hat{\mu}\dot{w} + \mathbf{L}(w) + \mathbf{Q}(w,w) + \mathbf{C}(w,w,w) = \hat{F}(x)\cos\Omega t, \quad (52)$$

where $\mathbf{Q}$ is an arbitrary spatial quadratic differential and/or integral operator. Similar to the cubic operator, the quadratic operator also possesses the property of being multi-linear. To simplify the calculations, the associated boundary conditions for equation (52) are also assumed to be linear, homogenous and free from time derivatives. One rewrites equation (52) with respect to the time scale $T=\Omega t$:

$$\Omega^2\ddot{w} + \bar{\mu}\dot{w} + \mathbf{L}(w) + \mathbf{Q}(w,w) + \mathbf{C}(w,w,w) = \hat{F}\cos T, \quad (53)$$

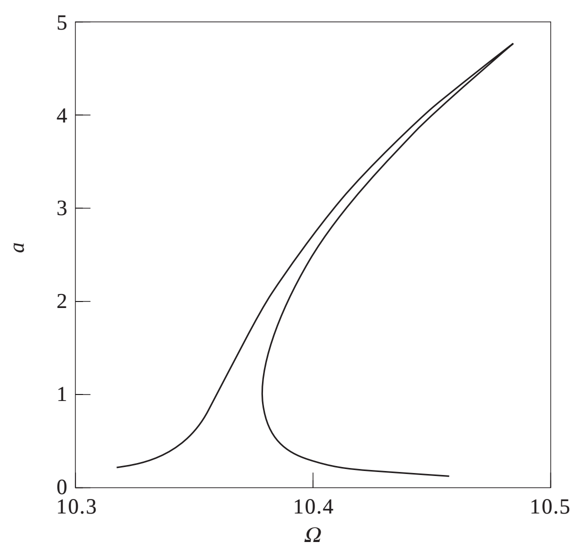

Figure 1. Frequency response curve for the example of cubic non-linearities ($n=1$, $k=10$, $\varepsilon=0{\cdot}1$, $f=2$, $\hat{\mu}=0{\cdot}004$).

where $\bar{\mu} = \Omega\hat{\mu}$ and $(\dot{})$ now represents time differentiation with respect to the new variable T. Again, equation (53) is solved by using both versions of MMS.

3.1. MMS VERSION I

A solution up to third order for equation (53) is presented:

$$w(x, t; \varepsilon) = \varepsilon w_1(x, T_0, T_1, T_2) + \varepsilon^2 w_2(x, T_0, T_1, T_2) + \varepsilon^3 w_3(x, T_0, T_1, T_2) + \ \ldots, \quad (54)$$

where ε is a small parameter used as a book keeping device. Since this parameter is artificially introduced, it can be equated to 1 at the end keeping in mind that the deflections are small. In MMS version I, the external excitation and damping are ordered as

$$\hat{F} = \varepsilon^3 F, \qquad \bar{\mu} = \varepsilon^2 \mu. \quad (55)$$

Then the nearness of external frequency to the natural frequency is expressed by requiring

$$\Omega = \omega^2 + \varepsilon^2 \sigma, \quad (56)$$

where σ is a detuning parameter of $O(1)$. Substituting the time derivatives in equation (5) and equations (54–56) into equation (53) and equating like powers of ε yield

order ε^1:

$$\omega^2 D_0^2 w_1 + \mathbf{L}(w_1) = 0; \quad (57)$$

order ε^2:

$$\omega^2 D_0^2 w_2 + \mathbf{L}(w_2) = -2\omega^2 D_0 D_1 w_1 - \mathbf{Q}(w_1, w_1); \quad (58)$$

order ε^3:

$$\omega^2 D_0^2 w_3 + \mathbf{L}(w_3) = -(2\omega^2 D_0 D_2 + \omega^2 D_1^2 + \sigma D_0^2 + \mu D_0) w_1$$
$$-2\omega^2 D_0 D_1 w_2 - \mathbf{Q}(w_1, w_2) - \mathbf{Q}(w_2, w_1) - \mathbf{C}(w_1, w_1, w_1) + F \cos T_0. \quad (59)$$

One follows the same procedure given in section 2.1. At order ε^1 one has a solution of the form

$$w_1 = [A(T_1, T_2)\, \mathrm{e}^{\mathrm{i}T_0} + cc] Y(x), \quad (60)$$

where Y function is to be determined from the equation

$$\mathbf{L}(Y) - \omega^2 Y = 0. \quad (61)$$

Substituting equation (60) into equation (58), one finds the solvability condition as $D_1 A = 0$. Therefore $A = A(T_2)$ only. At order ε^2 the solution is

$$W_2 = (A^2\, \mathrm{e}^{2\mathrm{i}T_0} + cc)\phi_1(x) + 2A\tilde{A}\phi_2(x), \quad (62)$$

where ϕ_1 and ϕ_2 functions satisfy the equations

$$\mathbf{L}(\phi_1) - 4\omega^2 \phi_1 = -\mathbf{Q}(\mathrm{Y}, \mathrm{Y}), \qquad \mathbf{L}(\phi_2) = -\mathbf{Q}(Y, Y). \quad (63, 64)$$

At order ε^3, following the same procedure as outlined in Section 2.1, we find the solvability condition

$$2\mathrm{i}\omega^2 D_2 A + (\mathrm{i}\mu - \sigma)A + \alpha A^2 \tilde{A} - f/2 = 0, \quad (65)$$

where α and f are defined as

$$\alpha = \int_D Y\{\mathbf{Q}(Y, \phi_1) + \mathbf{Q}(\phi_1, Y) + 2[\mathbf{Q}(Y, \phi_2) + \mathbf{Q}(\phi_2, Y)] + 3\mathbf{C}(Y, Y, Y)\}\, \mathrm{d}x, \quad (66)$$

$$f = \int_D YF\, \mathrm{d}x. \quad (67)$$

Now one inserts $D_1 A = 0$ and $D_2 A$ from equation (65) into equation (24), writes the polar form $A = 1/2a\, \mathrm{e}^{\mathrm{i}\beta}$, sets $\mathrm{d}A/\mathrm{d}T$ to zero, separates real and imaginary parts to obtain

$$\sigma = \tfrac{1}{4}\alpha a^2 \pm \sqrt{f^2/a^2 - \mu^2}. \quad (68)$$

The frequency response equation is

$$\Omega^2 = \omega^2 + \varepsilon^2[\tfrac{1}{4}\alpha a^2 \pm \sqrt{f^2/a^2 - \mu^2}]. \quad (69)$$

3.2. MMS VERSION II

In this section, damping and frequency are again expanded as was done in section 2.2.

$$\hat{F} = \varepsilon^2 F_1, \qquad \bar{\mu} = \varepsilon\mu_1 + \varepsilon^2\mu_2, \qquad \Omega^2 = \omega^2 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2. \quad (70)$$

When one substitutes these expansions into equation (53) and equates the coefficients of like powers of ε, one obtains equation (57) at order ε^1. The solution w_1 is also given by equation (60).

At order ε^2, one has the equation

$$\omega^2 D_0^2 w_2 + \mathbf{L}(w_2) = -(2\omega^2 D_0 D_1 + \sigma_1 D_0^2 + \mu_1 D_0)w_1 - \mathbf{Q}(w_1, w_1) + \tfrac{1}{2}(F_1\, \mathrm{e}^{\mathrm{i}T_0} + cc). \quad (71)$$

The solvability condition would be

$$2\mathrm{i}\omega^2 D_1 A + (\mathrm{i}\mu_1 - \sigma_1)A - f_1/2 = 0, \quad (72)$$

where

$$f_1 = \int_D YF_1\, \mathrm{d}x. \quad (73)$$

The spatial representation at this order will again be given by equations (62–64).

Following the same procedure of Section 2.2, one sets $D_1 A$ to zero, substitutes the polar form $A = 1/2a\, \mathrm{e}^{\mathrm{i}\beta}$, separates real and imaginary parts to obtain

$$\sigma_1 = \pm\sqrt{f_1^2/a^2 - \mu_1^2}. \quad (74)$$

The solvability condition at order ε^3 requires

$$2\mathrm{i}\omega^2 D_2 A + (\mathrm{i}\mu_2 - \sigma_2)A + \alpha A^2\tilde{A} = 0. \quad (75)$$

where α is defined by equation (66).

For steady state solutions, one sets $D_2 A$ to zero, substitutes polar form $A = 1/2a\, \mathrm{e}^{\mathrm{i}\beta}$, separates real and imaginary parts and obtains the results

$$\sigma_2 = \tfrac{1}{4}\alpha a^2, \qquad \mu_2 = 0. \quad (76)$$

Finally one substitutes σ_1 and σ_2 into the frequency expansion in equations (70) and obtains the frequency response equation

$$\Omega^2 = \omega^2 \pm \varepsilon\sqrt{f_1^2/a^2 - \mu_1^2} + \varepsilon^2 \tfrac{1}{4}\alpha a^2. \tag{77}$$

It is discussed in reference [8] that then $D_1 A = 0$, both versions of MMS provide identical results. This can be seen by comparing the external excitation expansions given by both methods, namely equations (69) and (77). Equation (69) reduces to that of equation (77) if one equates the following excitation and damping terms:

$$\varepsilon^2 f_1 = \varepsilon^3 f, \qquad \varepsilon\mu_1 = \varepsilon^2\mu \tag{78}$$

Inserting $f = f_1/\varepsilon$ and $\mu = \mu_1/\varepsilon$ into equation (69), one obtains equation (77).

3.3. AN APPLICATION FOR MMS VERSION II

In this section, to illustrate the general solution procedure for MMS version II, the following non-dimensional equation of motion and the associated boundary conditions are considered:

$$\Omega^2\ddot{w} + \bar{\mu}\dot{w} - w'' + \delta w^2 + \gamma w'^2 w'' = \hat{F}(x)\cos T, \qquad w(0, T) = w(1, T) = 0. \quad (79, 80)$$

The above equation may model non-linear string vibrations. The cubic non-linearity is similar to that given in Mote [13]. A quadratic restoring force is added to the stationary system in that reference. Applying the formalism given in the previous sections, one writes the operators as

$$\mathbf{L}(w) = -w'', \qquad \mathbf{Q}(w, w) = \delta w^2, \qquad \mathbf{C}(w, w, w) = \gamma w'^2 w''. \tag{81}$$

Assuming an expansion of the form defined in equation (54) for deflection function w and expansions of the form defined in equation (70) for excitation, damping and frequency, one obtains, at order ε^1, the following eigenvalue–eigenfunction problem:

$$Y'' + \omega^2 Y = 0, \qquad Y(0) = Y(1) = 0 \quad (82, 83)$$

for which the solution is

$$Y_n(x) = \sqrt{2}\sin(n\pi x), \qquad \omega_n = n\pi, \qquad n = 1, 2, \ldots \quad (84, 85)$$

At order ε^2, the solution is represented by equation (62) and the function ϕ_1 and ϕ_2 satisfy equations (63) and (64), respectively. Thus,

$$\phi_{1n}'' + 4n^2\pi^2\phi_{1n} = 2\delta\sin^2(n\pi x), \qquad \phi_{1n}(0) = \phi_{1n}(1) = 0, \quad (86, 87)$$

$$\phi_{2n}'' = 2\delta\sin^2(n\pi x), \qquad \phi_{2n}(0) = \phi_{2n}(1) = 0. \quad (88, 89)$$

The solutions are

$$\phi_{1n}(x) = \delta[(1 - \cos 2n\pi x)/4n^2\pi^2 - x\sin 2n\pi x/4n\pi], \tag{90}$$

$$\phi_{2n}(x) = \delta[(\cos 2n\pi x - 1)/4n^2\pi^2 + x(x-1)/2]. \tag{91}$$

The deflection function w in terms of real variables is

$$w_n = a\cos(\Omega t + \beta)\sqrt{2}\sin(n\pi x) + (a^2/2)[\cos(2\Omega t + 2\beta)\phi_{1n} + \phi_{2n}] + O(\varepsilon^3), \tag{92}$$

where ϕ_{1n} and ϕ_{2n} are given in equations (90) and (91). Since ε is artificially introduced, it is taken as 1 at the end. This solution is valid for small deflections.

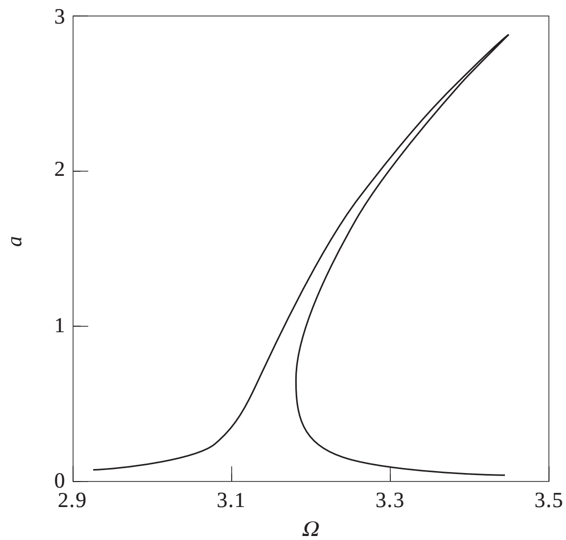

Figure 2. Frequency response curve for the example of quadratic and cubic non-linearities ($n = 1$, $\varepsilon = 1$, $f_1 = 0{\cdot}1$, $\alpha = 0{\cdot}9695$, $\hat{\mu} = 0{\cdot}01$).

To find the frequency response relation, one needs to calculate α defined in equation (66). Substituting the special forms of quadratic and cubic operators, one has

$$\alpha = \int_0^1 [2\delta Y^2(\phi_1 + 2\phi_2) + 3\gamma Y Y'^2 Y'']\,\mathrm{d}x \tag{93}$$

or, after evaluating the integrals,

$$\alpha = -2\delta^2(25/32n^2\pi^2 + \tfrac{1}{6}) - \tfrac{3}{2}\gamma n^4\pi^4. \tag{94}$$

The frequency response equation is

$$\Omega^2 = n^2\pi^2 \pm \varepsilon\sqrt{f_1^2/a^2 - \mu_1^2} + \varepsilon^2\tfrac{1}{4}\alpha a^2. \tag{95}$$

A sample plot of equation (95) is given in Figure 2 for parameter values of $n = 1$, $\varepsilon = 1$, $f_1 = 0{\cdot}1$, $\hat{\mu} = 0{\cdot}01$, and $\alpha = 0{\cdot}9695$ ($\delta = 1$, $\gamma = 0{\cdot}01$ are used in calculating α).

4. CONCLUDING REMARKS

Two general non-linear partial differential equations modelling a wide range of problems of vibrations of continuous systems have been treated. Solutions are presented using two different versions of the method of multiple scales. For the first model of cubic non-linearities, version II produces more accurate results compared to version I. For the second model of quadratic and cubic non-linearities, it has been shown that results are identical. In all solutions, the direct-perturbation method is used, since this method produces more accurate results compared to the discretization-perturbation method. General solution algorithms are presented for the proposed non-linear models. These algorithms are then applied to specific problems.

As mentioned earlier in reference [8], when complex amplitudes A depend on the first slow time scale T_1, for higher order perturbation schemes, the MMS version I produces extra non-physical results (Section 2). On the contrary, when the complex amplitudes A

are independent of T_1, both methods produce identical results for steady state solutions (Section 3).

In conclusion, for partial differential systems, MMS version II combined with the direct-perturbation method is recommended for finding steady state solutions.

ACKNOWLEDGMENT

This study is supported by the Scientific and Technical Research Council of Turkey (TUBITAK) under project number TBAG-1346.

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