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A NEW DISCRETE APPROACH TO EIGENVALUE SENSITIVITY WITH RESPECT TO CONSTRAINT LOCATIONS

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The sensitivity of eigenvalues to the location of bracing and/or boundary support is formulated by using a generalized variational principle. The constraints are incorporated through the use of Lagrange multipliers. A numerical example is provided to illustrate the proposed method and its accuracy in application.

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1. INTRODUCTION

Slender beams, thin plates and some built-up structures, such as ships, are susceptible to lateral–torsional buckling and/or vibration. It is often necessary to specify intermediate supports for such structures to reduce the possibility of design failure. The locations of the braces and intermediate supports affect significantly the buckling load and natural vibration frequencies [1, 2]. Therefore, it is desirable to guide selection of the optimal locations of these constraints. Eigenvalue sensitivity analysis with respect to the locations of these constraints reveals the relative importance of design parameters to specified performance measures like buckling load or natural frequency magnitude. Sensitivity analysis is particularly useful in large and complex analytical models where optimal bracing and intermediate support locations are not obvious and can be expensive to locate by repeated analysis. In addition, sensitivity approximates eigenvalues following perturbation of bracing and/or intermediate support locations, and is used for shaping vibration modes to reduce dynamic displacements at particular locations.

In recent years, optimal specification of constraints for beams to prohibit buckling and constrain vibration has received substantial attention [1–9]. Liu *et al*. [3] presented a generalized variational approach to the derivation of eigenvalue sensitivity with respect to support location for continuous systems. Wang [4] derived a sensitivity formula from a normal mode method. Chuang and Hou [5, 6] derived eigenvalue sensitivity with respect to the support location in a beam through a material derivative. Hsieh and Arora [10] investigated the dynamic response of structural systems when general boundary conditions are imposed during the analysis phase. Pierre [11] studied eigenvalue sensitivity

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formulation with respect to the natural boundary conditions. Very recently, Liu and Hu [12] presented a method to formulate eigenpair sensitivity with respect to boundary shape. In the continuum methods the eigenvalue sensitivity is derived from a weak variational form. In discrete methods the sensitivity is obtained following spatial discretization of the governing differential equations.

The continuum approach to eigenvalue sensitivity with respect to bracing and support locations has been evaluated in references [3–6]. However, the discrete derivation for eigenvalue sensitivity with respect to bracing and support locations appears to have not been brought to closure. Lund and Olhoff [9] noted that erroneous results for eigenvalue sensitivity can be obtained when the characteristic equation is differentiated at the locations of boundary supports. In this case the essential boundary equations, necessary for specification of the space of admissible functions, can be violated. A derivation of eigenvalue sensitivity that avoids this problem and a numerical algorithm for prediction of sensitivity are presented in this paper.

The sensitivity of eigenvalues with respect to the locations of constraints is formulated here through a generalized Rayleigh Quotient. The stationarity of the Rayleigh Quotient subject to constraints is transformed into a free stationary problem without constraints by means of Lagrange multipliers. The eigenvalue sensitivity is then obtained by differentiating the generalized Rayleigh Quotient.

2. PROBLEM DESCRIPTION

Assume that a continuous structure has been spatially discretized into a N-degree of freedom (e.g., finite element) model so that its undamped free linear vibration and/or buckling analysis leads to a finite eigenvalue problem,

$$KZ_i - \lambda_i M Z_i = 0, \tag{1}$$

where $K \in R^{N \times N}$, $M \in R^{N \times N}$ and $Z_i \in R^N$. In a vibration problem, λ_i is the square of the ith natural frequency and in a buckling analysis λ_i is the buckling load factor. Z_i is the ith mode associated with λ_i. K and M are symmetric, K is positive semidefinite, and M is positive definite in vibration. The constraints due to supports and/or braces are represented in the form

$$X_r = 0, \tag{2}$$

The partition of equation (1) in the form $Z^{\mathrm{T}} = [X_r^{\mathrm{T}}, X_f^{\mathrm{T}}]$, following introduction of equation (2), yields

$$\left(\begin{bmatrix} K_{rr} & K_{rf} \\ K_{fr} & K_{ff} \end{bmatrix} - \lambda_i \begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix}\right)\begin{bmatrix} X_{ri} \\ X_{fi} \end{bmatrix} = 0. \tag{3}$$

From equations (2) and (3), one obtains the reduced eigenproblem

$$(K_{ff} - \sigma_i M_{ff}) Y_i = 0, \tag{4}$$

where σ_i and Y_i are respectively the ith eigenvalue and the corresponding eigenvector of the constrained system.

Let the locations of the constrained co-ordinates be perturbed to a new co-ordinate vector $U(h) = U + hZ$, where Z is the direction of perturbation, U is the co-ordinate vector of the unperturbed constrained nodes and h is a scaling parameter. The eigenproblem associated with the perturbed locations of the constraints is then

$$(K_{ff}^* - \sigma_i^* M_{ff}^*) Y_i^* = 0, \tag{5}$$

where K_{ff}^* and M_{ff}^* reflect the perturbed locations of constraints, σ_i^* is the ith eigenvalue and Y_i^* is the associated eigenvector. When $h = 0$, one has $\sigma_i^* = \sigma_i$ and $Y_i^* = Y_i$.

The eigenvalue sensitivity becomes

$$\frac{\mathrm{d}\sigma_i}{\mathrm{d}h} = \lim_{h \to 0} \frac{\sigma_i^* - \sigma_i}{h} \tag{6}$$

and the problem is to formulate $\mathrm{d}\sigma_i/\mathrm{d}h$ to facilitate its application in optimization.

3. RAYLEIGH QUOTIENT $R(x, y)$

Two Rayleigh Quotients are used to establish the relation between σ_i^* and h, one including and one excluding the constraint.

The Rayleigh Quotient associated with equation (3) is

$$R_3 = \frac{[X_r^{\mathrm{T}}, X_f^{\mathrm{T}}] \begin{bmatrix} K_{rr} & K_{rf} \\ K_{fr} & K_{ff} \end{bmatrix} \begin{bmatrix} X_r \\ X_f \end{bmatrix}}{[X_r^{\mathrm{T}}, X_f^{\mathrm{T}}] \begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix} \begin{bmatrix} X_r \\ X_f \end{bmatrix}} \tag{7}$$

and the eigenvalue σ_i is the ith stationarity of R_3, reference [8],

$$\lambda_i = s.t. R_3 = s.t. \frac{[X_r^{\mathrm{T}}, X_f^{\mathrm{T}}] \begin{bmatrix} K_{rr} & K_{rf} \\ K_{fr} & K_{ff} \end{bmatrix} \begin{bmatrix} X_r \\ X_f \end{bmatrix}}{[X_r^{\mathrm{T}}, X_f^{\mathrm{T}}] \begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix} \begin{bmatrix} X_r \\ X_f \end{bmatrix}} \tag{8}$$

where $s.t.$ denotes the stationary value with respect to X_f and X_r. This equation can be further expressed as

$$R_3(X_{ri}, X_{fi}) = \lambda_i \tag{9}$$

and when $[X_r, X_f]^{\mathrm{T}} = [X_{ri}, X_{fi}]^{\mathrm{T}}$

$$\partial R_3/\partial X_r = 0, \qquad \partial R_3/\partial X_f = 0, \tag{10, 11}$$

The Rayleigh Quotient for equation (4) is

$$R_4(Y) = Y^{\mathrm{T}} K_{ff} Y / Y^{\mathrm{T}} M_{ff} Y \tag{12}$$

and

$$R_4(Y_i) = Y_i^{\mathrm{T}} K_{ff} Y_i / Y_i^{\mathrm{T}} M_{ff} Y_i = \sigma_i \qquad \partial R_4(Y)/\partial Y = 0 \qquad \text{when} \qquad Y = Y_i. \tag{13, 14}$$

Define the generalized Rayleigh Quotient as

$$R_G(X_r, X_f, \rho) = \frac{[X_r^{\mathrm{T}}, X_f^{\mathrm{T}}]\begin{bmatrix} K_{rr} & K_{rf} \\ K_{fr} & K_{ff} \end{bmatrix}\begin{bmatrix} X_r \\ X_f \end{bmatrix} - 2[\rho^{\mathrm{T}}, 0]\begin{bmatrix} X_r \\ 0 \end{bmatrix}}{[X_r^{\mathrm{T}}, X_f^{\mathrm{T}}]\begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix}\begin{bmatrix} X_r \\ X_f \end{bmatrix}} \tag{15}$$

where ρ is an N_r order column vector of Lagrange multipliers to be determined.

Stationarity of $R_G(X_r, X_f, \rho)$ requires

$$\frac{\partial R_G}{\partial \rho} = \frac{-2X_r}{[X_r^{\mathrm{T}}, X_f^{\mathrm{T}}]\begin{bmatrix} M_{aa} & M_{ab} \\ M_{ba} & M_{bb} \end{bmatrix}\begin{bmatrix} X_r \\ X_f \end{bmatrix}}, \tag{16}$$

$$\frac{\partial R_G}{\partial X_f} = \frac{2([K_{fr}, K_{ff}] - R_G[M_{fr}, M_{ff}])\begin{bmatrix} X_r \\ X_f \end{bmatrix}}{[X_r, X_f]\begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix}\begin{bmatrix} X_r \\ X_f \end{bmatrix}}, \tag{17}$$

$$\frac{\partial R_G}{\partial X_r} = \frac{2([K_{rr}, K_{rf}] - R_G[M_{rr}, M_{rf}])\begin{bmatrix} X_r \\ X_f \end{bmatrix} - 2[\rho^{\mathrm{T}}, 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{[X_r, X_f]\begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix}\begin{bmatrix} X_r \\ X_f \end{bmatrix}}, \tag{18}$$

The stationary of R_G with respect to ρ, equation (16), provides

$$X_r = 0 \tag{19}$$

The stationary of R_G with respect to X_f, equations (17) and (19), leads to (see equation (4))

$$[K_{ff} - R_G M_{ff}]X_f = 0. \tag{20}$$

The stationary of R_G with respect to X_r, equations (18) and (19), gives

$$[K_{rf} - R_G M_{rf}]X_f = \rho. \tag{21}$$

Equations (20) and (21) yield

$$\left[\begin{bmatrix} K_{rr} & K_{rf} \\ K_{fr} & K_{ff} \end{bmatrix} - R_G\begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix}\right]\begin{bmatrix} 0 \\ X_f \end{bmatrix} = \begin{bmatrix} \rho \\ 0 \end{bmatrix}, \tag{22}$$

illustrating the force of constraint provided by the Lagrange multipliers.

When R_G is stationary,

$$\delta R_G = 0. \tag{23}$$

Then

$$R_G = \frac{[X_r^{\mathrm{T}}, X_f^{\mathrm{T}}]\begin{bmatrix} K_{rr} & K_{rf} \\ K_{fr} & K_{ff} \end{bmatrix}\begin{bmatrix} X_r \\ X_f \end{bmatrix} - 2[\rho^{\mathrm{T}}, 0]\begin{bmatrix} X_r \\ X_f \end{bmatrix}}{[X_r^{\mathrm{T}}, X_f^{\mathrm{T}}]\begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix}\begin{bmatrix} X_r \\ X_f \end{bmatrix}} = \sigma_i \tag{24}$$

and

$$X_r = 0, \qquad X_f = X_{fi}, \qquad \rho = \rho_i = K_{rf}X_{fi} - \sigma_i M_{rf} X_{fi}.$$

4. EIGENVALUE SENSITIVITY

Because X_{ri}, X_{fi}, ρ_i, K and M all depend on h for the given perturbation direction Z, σ_i depends on h. Variation in h, δh, results in a variation in σ_i, denoted as $\delta\sigma_i$,

$$\delta\sigma_i = \delta_h \sigma_i + \delta_h^0 \sigma_i, \tag{25}$$

where $\delta_h^0\sigma_i$ denotes the variation of σ_i induced by X_{ri}, X_{fi} and ρ_i alone with $h = 0$. Because σ_i is a stationary value of R_G with respect to X_{ri}, X_{fi} and ρ_i,

$$\delta_h^0 \sigma_i = 0. \tag{26}$$

Therefore,

$$\delta\sigma_i = \delta_h^0 \sigma_i, \tag{27}$$

where $\delta_h\sigma_i$ is the variation of σ_i induced by h with X_{ri}, X_{fi} and ρ_i remaining unchanged. For a distinct σ_i, differentiation of equation (24) with respect to h yields

$$\mathrm{d}\sigma_i/\mathrm{d}h = [\partial V/\partial h - \sigma_i \partial L/\partial h - 2\rho_i^{\mathrm{T}} X'_{ri}] X_{fi}^{\mathrm{T}} M_{ff} X_{fi}, \tag{28}$$

where

$$\frac{\partial V}{\partial h} = \frac{\partial}{\partial h}\left([\tilde{X}_r^{\mathrm{T}}, \tilde{X}_f^{\mathrm{T}}]\begin{bmatrix} K_{rr} & K_{rf} \\ K_{fr} & K_{ff} \end{bmatrix}\begin{bmatrix} \tilde{X}_r \\ \tilde{X}_f \end{bmatrix}\right), \tag{29}$$

$$\frac{\partial L}{\partial h} = \frac{\partial}{\partial h}\left([\tilde{X}_r^{\mathrm{T}}, \tilde{X}_f^{\mathrm{T}}]\begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix}\begin{bmatrix} \tilde{X}_r \\ \tilde{X}_f \end{bmatrix}\right), \tag{30}$$

with a tilde ($\tilde{\ }$) over a variable denoting that the variable is kept fixed during the partial differentiation; and X' is the spatial slope of X defined by

$$X' = \lim_{h \to 0} (X(h) - X(0))/h. \tag{31}$$

The first two terms on the right side of equation (28) represent changes of potential energy and kinetic energy due to the variation of the constraint location; the last term reflects the product of the reaction forces of the constraint with the slope of the eigenvector at the point of constraint.

For repeated σ_0 with multiplicity m and a corresponding set of independent eigenvectors $x_{01}, x_{02}, \ldots, x_{0m}$, X_0 will also be an eigenvector associated with an σ_0,

$$X_0 = [x_{01}, x_{02}, \ldots, x_{0m}]\alpha, \tag{32}$$

where α is an arbitrary m order column vector. Partition of X_0 with respect to constraints gives

$$X_0 = \begin{bmatrix} X_{r0} \\ X_{f0} \end{bmatrix}\alpha. \tag{33}$$

Introduction of equation (33) and equation (15) yields

$$\sigma_0 = \frac{\alpha^{\mathrm{T}}[X_{r0}^{\mathrm{T}}, X_{f0}^{\mathrm{T}}]\begin{bmatrix} K_{rr} & K_{rf} \\ K_{fr} & K_{ff} \end{bmatrix}\begin{bmatrix} X_{r0} \\ X_{f0} \end{bmatrix}\alpha - 2[\rho_0^{\mathrm{T}}, 0]\begin{bmatrix} X_{r0} \\ X_{f0} \end{bmatrix}\alpha}{\alpha^{\mathrm{T}}[X_{r0}^{\mathrm{T}}, X_{f0}^{\mathrm{T}}]\begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix}\begin{bmatrix} X_{r0} \\ X_{f0} \end{bmatrix}\alpha}, \tag{34}$$

where ρ_0 is given by

$$\rho_0 = K_{rf}X_{fo}\alpha - \sigma_0 M_{rf}X_{f0}\alpha. \tag{35}$$

Recalling that $\delta_h\sigma_0$ is the variation induced by h alone with X_{r0}, X_{f0} and ρ_0 unchanged, one has

$$\delta_h\sigma_0 = \frac{\alpha^{\mathrm{T}}[\partial V/\partial h - \sigma_0\partial L/\partial h - 2\rho_0^{\mathrm{T}}X_{r0}']\alpha}{\alpha^{\mathrm{T}}X_{f0}^{\mathrm{T}}M_{ff}X_{f0}\alpha}\,\delta h, \tag{36}$$

where

$$\frac{\partial V}{\partial h} = \frac{\partial}{\partial h}\left([\tilde{X}_{r0}^{\mathrm{T}}, \tilde{X}_{f0}^{\mathrm{T}}]\begin{bmatrix} K_{rr} & K_{rf} \\ K_{fr} & K_{ff} \end{bmatrix}\begin{bmatrix} \tilde{X}_{r0} \\ \tilde{X}_{f0} \end{bmatrix}\right), \tag{37}$$

$$\frac{\partial L}{\partial h} = \frac{\partial}{\partial h}\left([\tilde{X}_{r0}^{\mathrm{T}}, \tilde{X}_{f0}^{\mathrm{T}}]\begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix}\begin{bmatrix} \tilde{X}_{r0} \\ \tilde{X}_{f0} \end{bmatrix}\right), \tag{38}$$

(39)

As shown in reference [8], $\delta_h\sigma_0$ is stationary with respect to α. After substitution of equation (35) into equation (36) and noting that $\delta\sigma_0 = \delta_h\sigma_0$, equation (36) becomes

$$\delta\sigma_0 = s.t.\,\frac{\alpha^{\mathrm{T}}[\partial V/\partial h - \sigma_0\partial L/\partial h - 2[K_{rf}X_{f0} - \sigma_0 M_{rf}X_{f0}]^{\mathrm{T}}X_{r0}']\alpha}{\sigma^{\mathrm{T}}X_{f0}^{\mathrm{T}}M_{ff}X_{f0}\alpha}\,\delta h, \tag{40}$$

where $s.t.$ denotes a stationary value with respect to α. Stationarity of equation (40) yields the eigenproblem,

$$\left(A - \frac{\mathrm{d}\sigma_0}{\mathrm{d}h}\,B\right)\alpha = 0, \tag{41}$$

where

$$A = \partial V/\partial h - \sigma_0\partial L/\partial h - 2[K_{rf}X_{f0} - \sigma_0 M_{rf}X_{f0}]^{\mathrm{T}}X_{r0}', \tag{42}$$

$$\mathrm{B} = \mathrm{X}_{f0}^{\mathrm{T}}M_{ff}X_{f0}, \tag{43}$$

whose m eigenvalues have the sensitivities of σ_0.

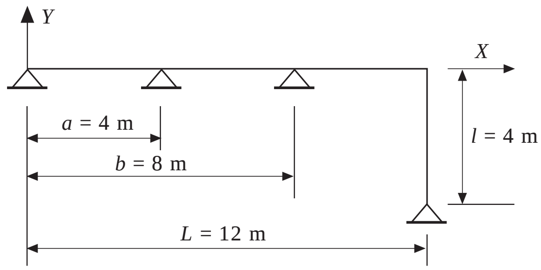

Figure 1. A plane frame with two intermediate supports.

5. NUMERICAL EXAMPLE

For illustration, the eigenvalue sensitivity of the first four eigenvalues of the plane frame shown in Figure 1 with respect to the locations of the intermediate supports are calculated. The plane frame is described by Young's modulus $= 2{\cdot}1 \times 10^{11}$ N/m^2, mass density $= 7{\cdot}8 \times 10^3$ kg/m^3, cross-sectional area $= 0{\cdot}02 \times 0{\cdot}02 = 0{\cdot}0004$ m^2, Poisson ratio $= 0{\cdot}3$, $L = 12$ m, $l = 4$ m, $a = 4{\cdot}00$ m and $b = 8{\cdot}00$ m.

The first four eigensolutions have been computed for a plane frame modelled by 32 uniform beam finite elements. The two supports are located at node 9 $(X, Y, Z) = (4{\cdot}000, 0{\cdot}000, 0{\cdot}000)$ and node 17 $(X, Y, Z) = (8{\cdot}000, 0{\cdot}000, 0{\cdot}000)$. The support locations are perturbed to new positions given by $(X, Y, Z) = (4{\cdot}000, 0{\cdot}000, 0{\cdot}000) + h(1{\cdot}000, 0{\cdot}000, 0{\cdot}000)$ and $(X, Y, Z) = (8{\cdot}000, 0{\cdot}000, 0{\cdot}000) + h(1{\cdot}000, 0{\cdot}000, 0{\cdot}000)$. Because the spatial slope of the mode, X'_{ri}, at a nodal point is the rotation in Z at that nodal point, the eigenvalue derivatives with respect to h can be evaluated using (28). The results are summarized in Table 1. For comparison, the first four eigenvalue sensitivities were also approximated by using a forward difference method (FD) with $h = 0{\cdot}001$ and those results are included in Table 1. As can be seen in the last two rows of Table 1, the results obtained by using equation (28) and the forward difference method agree well.

6. CONCLUSION

A generalized Rayleigh Quotient is defined so that the equation for the constraints due to the bracing and/or supports can be included in the functional R_G. The Lagrange

TABLE 1

Eigenvalue sensitivity with respect to support locations

	Mode number			
	1	2	3	4
Unperturbed eigenvalues ($h = 0{\cdot}000$)	78·3156	149·8498	402·1168	639·9145
Perturbed eigenvalues ($h = 0{\cdot}001$)	78·2565	149·9337	402·8429	640·0390
Eign-rate by FD σ_i/h	− 59·1	89·9	174·8	124·5
Eign-rate by Eq. (28) $d\sigma_i/dh$	− 54·0	84·3	179·7	122·4

multiplier terms, $\rho_r^{\mathrm{T}} X_r$, give the work done by the reaction forces at the bracing and/or supports of interest [3, 8]. Equation (28) shows that eigenvalue sensitivity with respect to location of the constraint depends not only on the changes in potential energy and kinematic energy due to the variation of constraint location, but also on the product of the spatial slope of the corresponding eigenvector at the point of constraint with the constraint force.

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