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BIFURCATIONS AND CHAOS IN A FOUR-DIMENSIONAL MECHANICAL SYSTEM WITH DRY FRICTION

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1. introduction

Figure 1(a) shows a mechanical model which displays friction induced self-sustained oscillations. Energy is continuously introduced into the system by the moving belt which slowly drives the two blocks with constant velocity V_{dr}. Starting from rest, both blocks stick on the belt while the energy is initially accumulated by the two linear springs which connect each block with a fixed support, then, when the elastic forces acting on a block exceed the maximum static friction force (*slip condition*), the block starts slipping due to the decreasing magnitude of the dynamic friction (Figure 1(b)). Such a motion changes the length of the linearly elastic coupling spring and influences the forces acting on the other block which, eventually, initiates a slipping motion. Finally, if the driving velocity is not too high, after the dissipation of a certain amount of kinetic energy due to friction, both blocks stop slipping and stick again on the belt until a new slip condition occurs. The sequence of stick–slip motions can generate a periodic or a non-periodic motion as was shown elsewhere [1].

In this letter the interest is in the classification of the bifurcations of the system. The system is not smooth, with a phase space dimension varying between two, when both blocks ride on the belt, and four, when both blocks slip, so that the usual analytical or numerical methods are not easily applicable. In the next section a one-dimensional map is defined (introduced in [1]) which allows a straightforward identification of the bifurcational behaviour of the system and the computation of the most significant Lyapunov exponent. The non-dimensionalized equations of motions [1] are given by:

$$\ddot{X}_1 + X_1 + \alpha(X_1 - X_2) = \pm 1/(1 + \gamma|V_1 - V_{dr}|), \tag{1a}$$

$$\ddot{X}_2 + X_2 + \alpha(X_2 - X_1) = \pm \beta/(1 + \gamma|V_2 - V_{dr}|). \tag{1b}$$

The conditions which indicate the passage between stick and slip motions are given by:

$$X_1 + \alpha(X_1 - X_2) = \pm 1, \qquad X_2 + \alpha(X_2 - X_1) = \pm \beta. \tag{2a, b}$$

Equations (1) and (2) were derived in [1] for the case of blocks with the same mass and external springs with the same stiffness $k_1 = k_2 = k$ (see Figure 1). α denotes the ratio between the coupling spring stiffness and the stiffness of the other two springs k_c/k and can vary between 0 and $+\infty$; β corresponds to the ratio between the maximum static friction force acting on the second mass and the same force acting on the first mass, and can also vary between 0 and $+\infty$; γ, the shape coefficient of the dynamic friction law, can vary between $-\infty$ and $+\infty$. The parameters assume the following values: $\alpha = 1{\cdot}2$, $\beta = 1{\cdot}3$, $\gamma = 3{\cdot}0$, while V_{dr} is kept in the range of small driving velocity, see [1] for more details.

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2. ONE DIMENSIONAL MAP

If the driving velocity V_{dr} is not too high the motions of the system are composed of a sequence of phases in which both blocks stick on the belt (stick phase) or at least one block slips (slip phase). A stick phase is characterised by a constant value of the variable

$$d = X_2 - X_1, \tag{3}$$

therefore a generic motion of the system generates in a natural way a sequence of values $d_0, d_1, \ldots, d_n$, which can be thought of as a one-dimensional map of the type:

$$d_{n+1} = f(d_n), \tag{4}$$

where f is implicitly defined and it is computed via the numerical integration of the system dynamics. A similar one-dimensional map, describing a three-dimensional non-smooth mechanical system with dry friction, was derived and analysed in reference [2].

In [1] it was shown that the system possesses different attractors. In this paper one concentrates on one of them, the bifurcation path which is entirely contained in the small driving velocity range. In this way the one-dimensional map is always well defined. Figure 2 shows the bifurcation path of such an attractor. The figure is generated by plotting the X_1 co-ordinate of a three-dimensional Poincaré map. Following the definition given in [1], a three-dimensional Poincaré section of the four-dimensional phase space is detected by the condition $V_1 = 0$ and the Poincaré map is constituted by the successive values of the variables (X_1, X_2, V_2) when V_1 is equal to zero and passes from negative values to positive ones. The attractor exists approximately in the range $0{\cdot}08058 < V_{dr} < 0{\cdot}170$. The introduction of the above defined one-dimensional map allows the bifurcations which appear in that diagram to be understood.

2.1. *Remark*

It is worth noting that for the above defined parameter set, the variable d is contained in the range $-0{\cdot}6764705882 < d < 0{\cdot}6764705882$. Therefore in the following Figures 3, 4 and 5, only the portions of the one-dimensional map corresponding to the attractor of

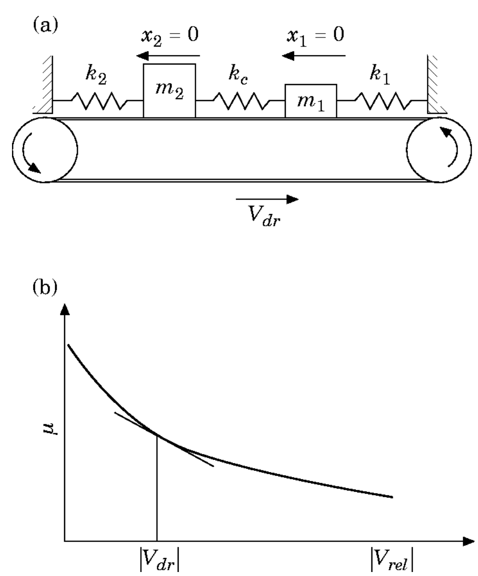

Figure 1. (a) Mechanical system; (b) friction characteristics.

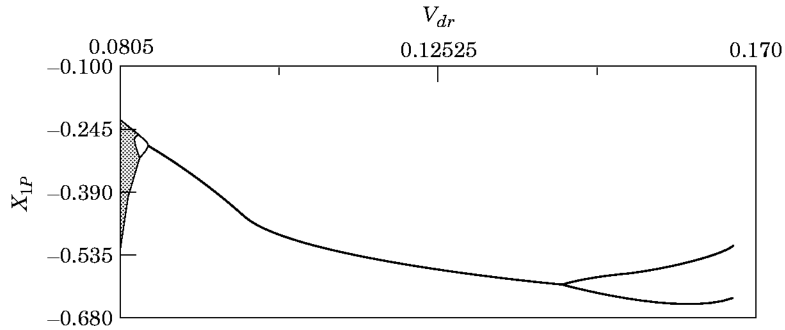

Figure 2. Bifurcation diagram of an attractor. The X_1 co-ordinate of the Poincaré section is plotted versus the driving velocity.

Figure 2 are shown. Clearly other branches belong to the map but they lie out of the zones shown in the figures and correspond to other attractors.

3. BIFURCATIONS

3.1. *Collision of the attractor with the basin boundaries*

The introduction of the one-dimensional map allows one to identify the sudden crisis of the chaotic attractor. Figure 3 shows the evolution of the one-dimensional map as the parameter V_{dr} decreases. When V_{dr} decreases the permanent branches of the map spread towards the boundaries of the basin. The attractor disappears for the value $V_{dr} \approx 0{\cdot}08058$ associated with the collision of the attractor with its basin boundaries. Consequently it is believed that a similar collision takes place in the four-dimensional phase space, between the attractor and a stable manifold [3].

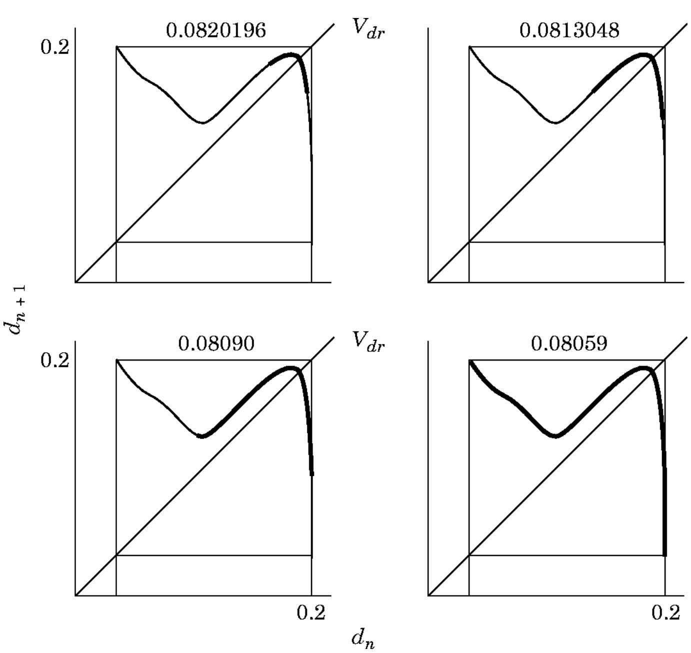

Figure 3. Collision between chaotic attractor and its basin boundaries. Thick lines indicate the attractor whereas thin lines indicate transient motions.

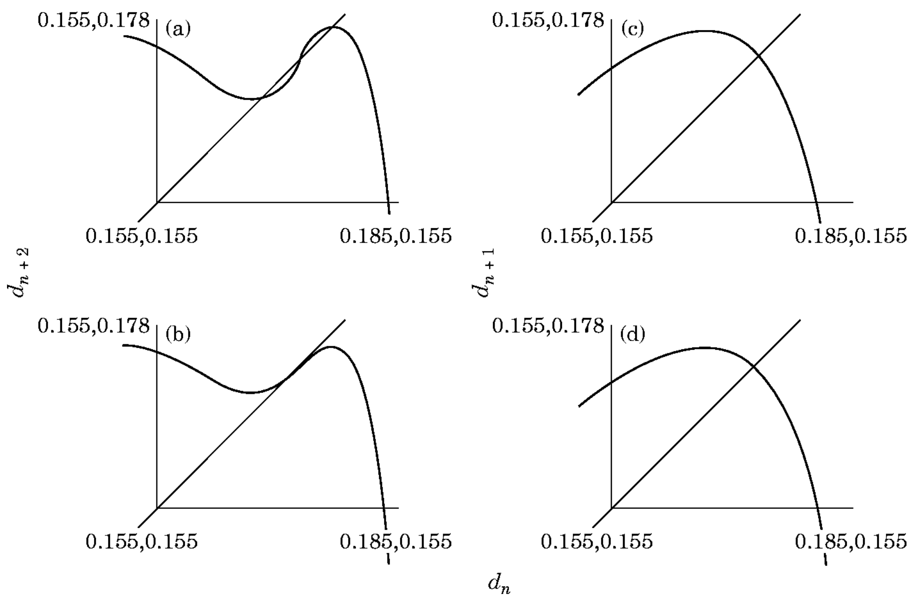

Figure 4. Period doubling bifurcation. V_{dr} values (a) 0·0838; (b) 0·0842; (c) 0·0838; (d) 0·0842.

3.2. *Flip bifurcations*

In the range of driving velocities $0{\cdot}0820 < V_{dr} < 0{\cdot}0840$ the system dynamics seem to undergo a reverse period-doubling cascade which is clearly described and confirmed by the use of the one-dimensional map. In Figure 4(a) the second iterated map is shown to have two stable attractors separated by an unstable fixed point. As the driving velocity increases the shape of the map changes in such a way that the three fixed points collapse into a unique stable fixed point shown in Figure 4(b). In the same way it is possible to show that the bifurcation close to the value $V_{dr} \approx 0{\cdot}1432$ is a flip bifurcation.

3.3. *Fold bifurcation*

The attractor disappears at a fold bifurcation of the second iterated map for $V_{dr} \approx 0{\cdot}170$. Figure 5(a) shows f^2 for $V_{dr} = 0{\cdot}165$. The second iterated map has two stable fixed points,

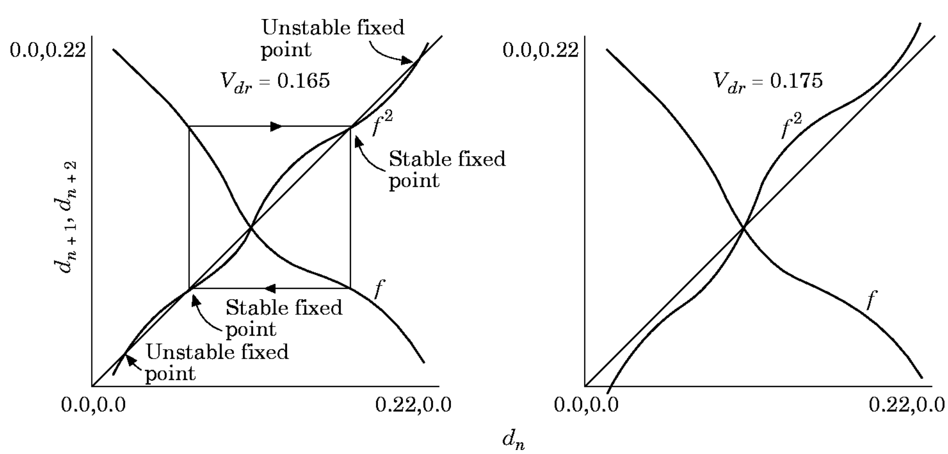

Figure 5. Fold bifurcation.

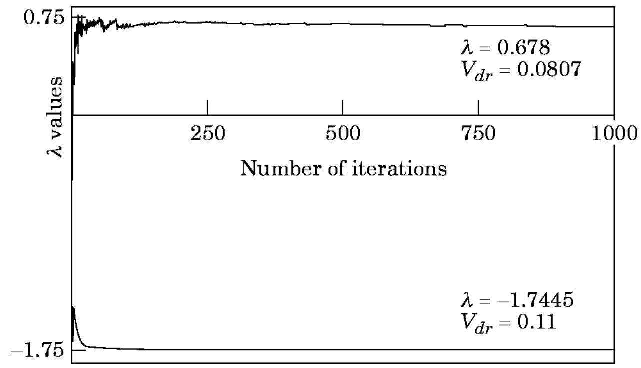

Figure 6. Convergence of some calculations of Lyapunov exponents.

separated by a central unstable fixed point, and two other external unstable fixed points that are the boundaries of the basin of attraction of the stable solutions. As V_{dr} increases the shape of the map changes and the stable intersections of the map with the bisection line coincide with the two external unstable ones and then disappear as can be seen in Figure 5(b) for $V_{dr} = 0{\cdot}175$. The unstable fixed point exists also for larger values of V_{dr}.

4. LYAPUNOV EXPONENTS

As described in the previous section the non-smooth four-dimensional system can be reduced to a one-dimensional map. By means of this reduction three Lyapunov exponents have been lost. Loosely speaking, since the phase dimension of the system varies between two and four, two Lyapunov exponents can be considered as $\lambda_3 \to -\infty$, $\lambda_4 \to -\infty$ because of the degeneration of the motion along the stick phases. A third exponent equals zero,

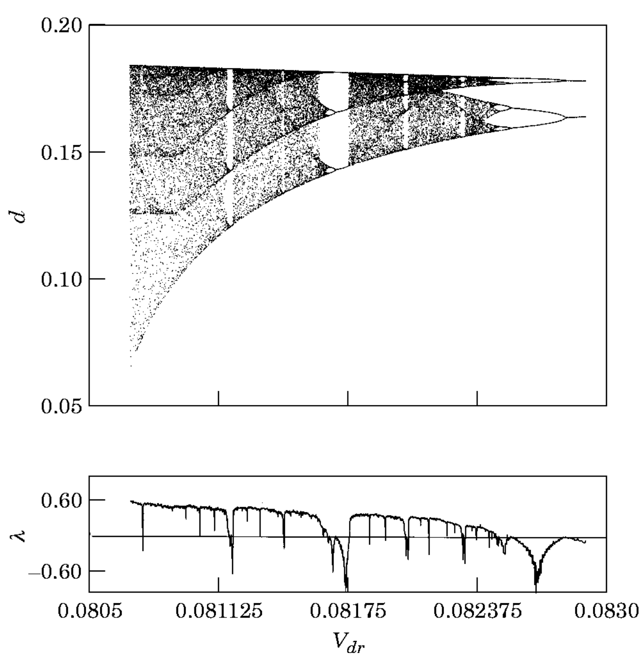

Figure 7. Bifurcation diagram and corresponding Lyapunov exponent in the range $0{\cdot}0807 < V_{dr} < 0{\cdot}0829$.

corresponding to a perturbation in the tangential direction. The fourth exponent can be evaluated by computing the one-dimensional map. In the case of one-dimensional maps, the Lyapunov exponent can be given as [2, 4]:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln |f'(d_n)|. \tag{5}$$

Figure 6 shows the computed Lyapunov exponent for two different values of the driving velocity and Figure 7 shows the good agreement existing between the bifurcation diagram and λ values in the range $0{\cdot}0806 < V_{dr} < 0{\cdot}0829$. One observes that where period doubling occurs the slope of the map is $f' = -1$ and therefore the Lyapunov exponent is $\lambda = 0$ as can be clearly seen in the figure. The Lyapunov exponents of non-smooth systems can be computed in a different way [5], but the approach presented here is more suitable for the system under investigation.

5. CONCLUSIONS

In this letter a one-dimensional map has been introduced in order to classify the bifurcations of a non-smooth dynamical system with four-dimensional phase space. The same map reduction can be utilised to evaluate the unique unknown Lyapunov exponent of the system, so that it is possible to clearly diagnose the chaotic nature of some motions.

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