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IMPROVING LOW ORDER EIGENFREQUENCY ESTIMATES DERIVED FROM THE WAVE PROPAGATION METHOD FOR AN EULER–BERNOULLI BEAM

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1. Introduction and review of the wave propagation method

The Euler–Bernoulli beam equation

$$m \frac{\partial^2}{\partial t^2} w(x, t) + EI \frac{\partial^4}{\partial x^4} w(x, t) = f(x, t), \qquad 0 < x < 1, \quad t > 0, \tag{1}$$

is a mathematical model of basic importance in structural engineering because machinery, bridges and buildings often have components which can be simply modelled as beams. In equation (1), the beam length has been normalized to 1. The external force $f(x, t)$ in equation (1) will excite resonant vibrations which are of major concern in structural engineering design. Resonance can be determined by an eigenvalue problem corresponding to the time-reduced form of equation (1) subject to properly given boundary conditions. To fix ideas, assume throughout that the Euler–Bernoulli beam is clamped at the left end:

$$w(0, t) = w_x(0, t) = 0, \qquad t \geqslant 0. \tag{2}$$

At the right end $x = 1$, one of four types of boundary conditions normally occurs:

$$\begin{aligned}
&\text{clamped (C)}, && w(1, t) = w_x(1, t) = 0, && t \geqslant 0;\\
&\text{simply supported (S)}, && w(1, t) = w_{xx}(1, t) = 0, && t \geqslant 0;\\
&\text{roller-supported (R)}, && w_x(1, t) = w_{xxx}(1, t) = 0, && t \geqslant 0;\\
&\text{free (F)}, && w_{xx}(1, t) = w_{xxx}(1, t) = 0, && t \geqslant 0.
\end{aligned} \tag{3}$$

The time-reduced form of equation (1), under the assumption that there is no external force, is obtained by letting $w(x, t) = \phi(x)\, \mathrm{e}^{-ik^2 t}$ (and $f(x, t) \equiv 0$) in equation (1), giving

$$\phi^{(4)}(x) - \tilde{k}^4 \phi(x) = 0, \qquad 0 < x < 1, \tag{4}$$

where

$$\tilde{k} = \alpha k, \qquad \alpha \equiv (m/EI)^{1/4}. \tag{5}$$

As stipulated in equations (2) and (3), the boundary conditions for ϕ consist of one of the following four sets:

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$$\text{(C–C)} \quad \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0; \qquad \text{(C–S)} \quad \phi(0) = \phi'(0) = \phi(1) = \phi''(1) = 0;$$

$$\text{(C–R)} \quad \phi(0) = \phi'(0) = \phi'(1) = \phi'''(1) = 0; \ \text{(C–F)} \quad \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0. \tag{6}$$

The *eigenvalue problem*, therefore, is to determine all values of $\tilde{k}^4$ which satisfy equation (4), subject to the appropriate set of boundary conditions given in equation (6).

It is well known that the eigenvalue problem (4) and (6) does not have any closed form solutions. We first recall the standard, straightforward approach to determine $\tilde{k}$ that is familiar: write

$$\phi(x) = A\,\mathrm{e}^{-\mathrm{i}\tilde{k}x} + B\,\mathrm{e}^{\mathrm{i}\tilde{k}x} + C\,\mathrm{e}^{-\tilde{k}x} + D\,\mathrm{e}^{\tilde{k}x}, \tag{7}$$

and substitute equation (7) into, say, the fourth set (C–F) of boundary conditions in equation (6), yielding

$$\begin{gathered} A + B + C + D = 0, \qquad -\mathrm{i}\tilde{k}A + \mathrm{i}\tilde{k}B - \tilde{k}C + \tilde{k}D = 0, \\ -\tilde{k}^2\,\mathrm{e}^{-\mathrm{i}\tilde{k}}A - \tilde{k}^2\,\mathrm{e}^{\mathrm{i}\tilde{k}}B + \tilde{k}^2\,\mathrm{e}^{-\tilde{k}}C + \tilde{k}^2\,\mathrm{e}^{\tilde{k}}D = 0, \\ \mathrm{i}\tilde{k}^3\,\mathrm{e}^{-\mathrm{i}\tilde{k}}A - \mathrm{i}\tilde{k}^3\,\mathrm{e}^{\mathrm{i}\tilde{k}}B - \tilde{k}^3\,\mathrm{e}^{-\tilde{k}}C + \tilde{k}^3\,\mathrm{e}^{\tilde{k}}D = 0. \end{gathered} \tag{8}$$

Therefore (after a slight simplification of equation (8)) $\tilde{k}$ satisfies the transcendental equation determined by the zero determinant condition

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -\mathrm{i} & \mathrm{i} & -1 & 1 \\ \mathrm{e}^{-\mathrm{i}\tilde{k}} & \mathrm{e}^{\mathrm{i}\tilde{k}} & -\mathrm{e}^{-\tilde{k}} & -\mathrm{e}^{\tilde{k}} \\ -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\tilde{k}} & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\tilde{k}} & \mathrm{e}^{-\tilde{k}} & -\mathrm{e}^{\tilde{k}} \end{vmatrix} = 0. \tag{9}$$

Actual estimation of $\tilde{k}$ from equation (9) is rather tedious, because the expansion of the determinant in equation (9) leads to 24 terms, each of which must be carefully scrutinized before any are discarded in order to do the asymptotic analysis (see, e.g., reference [1]).

Traditionally, engineers have been using Ritz and/or Galerkin finite element methods (FEM) to compute such eigenvalues. Although the computed data get more accurate when more and more basis (shape) functions are used, such eigenvalue data are of lower order. One *does not obtain analytical insights* from FEM as to what happens in the high frequency range, since FEM are purely numerical in nature. To obtain analytical insights, one needs to resort to a different, subtler way: the *wave propagation method* (WPM). It was first developed by Keller and Rubinow [2] for estimating the eigenvalues of the Laplacian Δ, a second order multi-dimensional partial differential operator. In Chen and Zhou [3], and Chen, Coleman and Zhou [4], the method was first adapted for the fourth order problem (4) in one space dimension, and later for the fourth order biharmonic operator Δ^2 in two space dimensions. To make this note sufficiently self-contained, let us provide a quick review of the WPM for equation (1) as in reference [3]. Consider $f(x, t) \equiv 0$ in equation (1) subject to the boundary conditions (C–F), i.e., the cantilever case:

$$\begin{gathered} \frac{\partial^2}{\partial t^2} w(x,t) + \alpha^{-4} \frac{\partial^4}{\partial x^4} w(x,t) = 0, \quad 0 < x < 1,\ t > 0, \qquad \text{(cf., equation (5) for } \alpha), \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = w_{xxx}(1,t) = 0, \qquad t > 0. \end{gathered} \tag{10}$$

We write the solution ω of equation (10) as a linear combination of four waves:

$$w(x,t) = \underbrace{A\,\mathrm{e}^{-\mathrm{i}k(\alpha x + kt)}}_{\text{Wave I}} + \underbrace{B\,\mathrm{e}^{\mathrm{i}k(\alpha x - kt)}}_{\text{Wave II}} + \underbrace{C\,\mathrm{e}^{\mathrm{i}k(\mathrm{i}\alpha x - kt)}}_{\text{Wave III}} + \underbrace{D\,\mathrm{e}^{-\mathrm{i}k[\mathrm{i}\alpha(x-1)+kt]}}_{\text{Wave IV}}, \tag{11}$$

where k is stipulated to be *positive*. Each of these waves is dispersive. Wave I travels leftward and Wave II travels rightward, while Waves III and IV are, respectively, *evanescent waves* near the left endpoint $x = 0$ and the right endpoint $x = 1$. Now, we first focus our attention on the boundary conditions at the left endpoint $w(0, t) = w_x(0, t) = 0$. Wave II impinges toward $x = 0$ and is reflected. After reflection, it becomes Wave I plus the evanescent Wave III. The total field is thus

$$A\,\mathrm{e}^{-\mathrm{i}k(\alpha x - kt)} + B\,\mathrm{e}^{\mathrm{i}k(\alpha x - kt)} + C\,\mathrm{e}^{\mathrm{i}k(\mathrm{i}\alpha x - kt)}. \tag{12}$$

In comparing equation (12) with equation (11), we note that Wave IV has been dropped in equation (12) because it is evanescent near $x = 1$ and is thus negligible near $x = 0$. Substituting equation (12) into the clamped boundary conditions at $x = 0$, we obtain

$$A\,\mathrm{e}^{-\mathrm{i}k^2 t} + B\,\mathrm{e}^{-\mathrm{i}k^2 t} + C\,\mathrm{e}^{-\mathrm{i}k^2 t} = 0, \qquad -\mathrm{i}\tilde{k}A\,\mathrm{e}^{-\mathrm{i}k^2 t} + \mathrm{i}\tilde{k}B\,\mathrm{e}^{-\mathrm{i}k^2 t} - \tilde{k}C\,\mathrm{e}^{-\mathrm{i}k^2 t} = 0, \tag{13}$$

i.e.,

$$A + B + C = 0, \qquad -\mathrm{i}A + \mathrm{i}B - C = 0, \tag{14}$$

which gives the amplitudes of the reflected waves

$$A = -\mathrm{i}B, \qquad C = \mathrm{i}B, \tag{15}$$

in terms of the incoming waves. We now repeat the same argument at the right endpoint $x = 1$. The incoming wave is Wave I, $A\,\mathrm{e}^{-\mathrm{i}k(\alpha x + kt)}$, while the reflected waves are Waves II and IV. Therefore, the total field is

$$A\,\mathrm{e}^{-\mathrm{i}k(\alpha x + kt)} + B\,\mathrm{e}^{-\mathrm{i}k(\alpha x - kt)} + D\,\mathrm{e}^{-\mathrm{i}k[\mathrm{i}\alpha(x-1)+kt)]}. \tag{16}$$

Substituting equation (16) (as w) into $w_{xx}(1, t) = w_{xxx}(1, t) = 0$, we obtain

$$-\tilde{k}^2\,\mathrm{e}^{-\mathrm{i}\tilde{k}}A\,\mathrm{e}^{-\mathrm{i}k^2 t} - \tilde{k}^2\,\mathrm{e}^{\mathrm{i}\tilde{k}}B\,\mathrm{e}^{-\mathrm{i}k^2 t} + \tilde{k}^2 D\,\mathrm{e}^{-\mathrm{i}k^2 t} = 0,$$
$$\mathrm{i}\tilde{k}^3\,\mathrm{e}^{-\mathrm{i}\tilde{k}}A\,\mathrm{e}^{-\mathrm{i}k^2 t} - \mathrm{i}\tilde{k}^3\,\mathrm{e}^{\mathrm{i}\tilde{k}}B\,\mathrm{e}^{-\mathrm{i}k^2 t} + \tilde{k}^3 D\,\mathrm{e}^{-\mathrm{i}k^2 t} = 0, \tag{17}$$

i.e.,

$$\mathrm{e}^{-\mathrm{i}\tilde{k}}A + \mathrm{e}^{\mathrm{i}\tilde{k}}B - D = 0, \qquad -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\tilde{k}}A + \mathrm{i}\,\mathrm{e}^{\mathrm{i}\tilde{k}}B - D = 0, \tag{18}$$

which gives the amplitudes of the reflected waves

$$B = -\mathrm{i}\,\mathrm{e}^{-2\mathrm{i}\tilde{k}}A, \qquad D = (1 - \mathrm{i})\,\mathrm{e}^{-\mathrm{i}\tilde{k}}A, \tag{19}$$

in terms of the amplitude A of the incoming wave.

Since Waves I and II are the *dominant waves* propagating back and forth on the beam, an eigenmode shape forms when the reflected wave does not have any phase difference with the incoming wave; i.e., when *resonance* occurs. We need only pay attention to the relationships between A and B in equations (15) and (19); we obtain

$$B = -\mathrm{i}\,\mathrm{e}^{-2\mathrm{i}\tilde{k}}A = -\mathrm{i}\,\mathrm{e}^{-2\mathrm{i}\tilde{k}}(-\mathrm{i}B) = -\mathrm{e}^{-2\mathrm{i}\tilde{k}}B.$$

This is possible if and only if

$$-2\mathrm{i}\tilde{k} = -\mathrm{i}(2n + 1)\pi, \qquad n = \text{a positive integer}.$$

Therefore

$$\tilde{k} = \frac{(2n+1)\pi}{2}, \qquad n = \text{a positive integer}, \tag{20}$$

and the eigenvalue problem

$$\phi^{(4)}(x) - \lambda\phi(x) = 0, \quad 0 < x < 1, \qquad \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0, \tag{21}$$

has non-trivial solutions ϕ when

$$\lambda = \tilde{k}^4 \approx \left[\left(\frac{2n+1}{2}\right)\pi\right]^4, \qquad n = 0, 1, 2, \ldots,$$

or

$$\sqrt{\lambda} = \tilde{k}^2 \approx \left[\left(\frac{2n+1}{2}\right)\pi\right]^2, \qquad n = 0, 1, 2, \ldots. \tag{22}$$

As mentioned earlier, exact values of λ in equation (21) are not known. However, they can be computed by a high accuracy Legendre-tau spectral method (see Gottlieb and Orszag [5], and Chen and Zhou [6, §3.3]); see the left column of Table 1, where $\sqrt{\lambda}$ are listed. The values of $\sqrt{\lambda}$ obtained from WPM in equation (22) are listed in the second column of Table 1. These values agree closely with data published elsewhere in the literature; cf., e.g., references [7] and [8]. The reader can easily see that WPM gives highly accurate eigenvalues for λ in equation (21) even at medium to low frequencies. However, for the lowest few eigenvalues, the deviations are larger. This is the nature of using an asymptotic method such as WPM. From an applications point of view, *the lowest few eigenvalues are actually the most important ones*, because the corresponding low order eigenmodes manifest prominently in the vibration of a beam. The crux of this note, therefore, is to develop a formal perturbation procedure which can estimate the accuracy of these lower order eigenvalues to any degree of desired accuracy. This procedure is given in the following section.

TABLE 1

The eigenfrequencies $k^2 = \sqrt{\lambda}$ of the cantilever beam, equation (21). *The left column lists the nearly exact values of $\sqrt{\lambda}$ computed by the Legendre-tau* (L-T) *spectral method, agreeing with those values found elsewhere in the literature, while the right column lists $\sqrt{\lambda}$ obtained by WPM in equation* (22)

$n\backslash\sqrt{\lambda_n}$	L-T	WPM
1	3·516015	2·467401
2	22·034492	22·206610
3	61·697214	61·685028
4	120·901916	120·902654
5	199·859530	199·859489
6	298·555531	298·555533
7	416·990786	416·990786
8	555·165248	555·165248

2. A FORMAL PERTURBATION PROCEDURE TO OBTAIN HIGH ACCURACY FOR LOW ORDER EIGENVALUES

We consider the equation (4) subject to the general combination of any set of boundary conditions (6). Without loss of generality, we let $\alpha = 1$ in equation (5) so that $\tilde{k} = k$, and k^2 is the *eigenfrequency* because of the separation of variables $w(x, t) = \phi(x)\,\mathrm{e}^{-\mathrm{i}k^2 t}$. Write

$$\phi(x) = A\,\mathrm{e}^{-\mathrm{i}kx} + B\,\mathrm{e}^{\mathrm{i}kx} + C\,\mathrm{e}^{-kx} + D\,\mathrm{e}^{k(x-1)}, \qquad 0 < x < 1. \tag{23}$$

Note that equation (23) is the time-reduced form of equation (11). Substituting equation (23) into each set of the boundary conditions (6) with slight simplification, we obtain a general matrix equation

$$\begin{bmatrix} 1 & 1 & 1 & \mathrm{e}^{-k} \\ -\mathrm{i} & \mathrm{i} & -1 & \mathrm{e}^{-k} \\ a_1\,\mathrm{e}^{-\mathrm{i}k} & a_2\,\mathrm{e}^{\mathrm{i}k} & \mathrm{e}^{-k} & a_3 \\ b_1\,\mathrm{e}^{-\mathrm{i}k} & b_2\,\mathrm{e}^{\mathrm{i}k} & \mathrm{e}^{-k} & b_3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \equiv M \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{24}$$

where a_j and b_j are constants depending on which set of the boundary conditions is under consideration:

$$\begin{aligned} &(\text{C–C}), && a_1 = a_2 = a_3 = 1, \quad b_1 = \mathrm{i}, \quad b_2 = -\mathrm{i}, \quad b_3 = -1; \\ &(\text{C–S}), && a_1 = a_2 = a_3 = 1, \quad b_1 = b_2 = -1, \quad b_3 = 1; \\ &(\text{C–R}), && a_1 = \mathrm{i}, \quad a_2 = -\mathrm{i}, \quad a_3 = -1, \quad b_1 = -\mathrm{i}, \quad b_2 = \mathrm{i}, \quad b_3 = -1; \\ &(\text{C–F}), && a_1 = a_2 = -1, \quad a_3 = 1, \quad b_1 = -\mathrm{i}, \quad b_2 = \mathrm{i}, \quad b_3 = -1. \end{aligned} \tag{25}$$

Therefore k is determined by

$$\det M = 0.$$

Let us define

$$M_\varepsilon = \begin{bmatrix} 1 & 1 & 1 & \varepsilon \\ -\mathrm{i} & \mathrm{i} & -1 & \varepsilon \\ a_1\,\mathrm{e}^{-\mathrm{i}k} & a_2\,\mathrm{e}^{\mathrm{i}k} & \varepsilon & a_3 \\ b_1\,\mathrm{e}^{-\mathrm{i}k} & b_2\,\mathrm{e}^{\mathrm{i}k} & \varepsilon & b_3 \end{bmatrix}. \tag{26}$$

Note that when $\varepsilon = 0$, $\det M_0 = 0$ corresponds to the WPM, as can be easily seen from equations (14) and (18), for the (C–F) set of the boundary conditions (25).

We again use the cantilever case (C–F) to illustrate our perturbation approach; the remaining cases proceed similarly. Thus, using the fourth set of data in equation (25) for a_j and b_j, and writing

$$k = k_0 + k_1\varepsilon + k_2\varepsilon^2 + O(\varepsilon^3), \tag{27}$$

we substitute equation (27) into equation (26), obtaining

$$\begin{aligned} \det M_\varepsilon = &-2\mathrm{i}(\mathrm{e}^{\mathrm{i}k_0} + \mathrm{e}^{-\mathrm{i}k_0}) + 2[-4\mathrm{i} + k_1(\mathrm{e}^{\mathrm{i}k_0} - \mathrm{e}^{-\mathrm{i}k_0})]\varepsilon \\ &+ [\mathrm{i}(k_1^2 - 2)(\mathrm{e}^{\mathrm{i}k_0} + \mathrm{e}^{-\mathrm{i}k_0}) + 2k_2(\mathrm{e}^{\mathrm{i}k_0} - \mathrm{e}^{-\mathrm{i}k_0})]\varepsilon^2 + O(\varepsilon^3), \end{aligned} \tag{28}$$

TABLE 2

A comparison of the lowest six eigenfrequencies k^2 for the clamped–clamped beam, as estimated by the Legendre-tau method (L-T), *the wave propagation method* (WPM), *and the iterated approximants* (36). *In this and the subsequent tables, "*" denotes numerical values which do not show further improvement of accuracy, and thus are omitted*

	WPM				
L-T	k_0^2	$[k^{(1)}]^2$	$[k^{(2)}]^2$	$[k^{(3)}]^2$	$[k^{(4)}]^2$
22·373285	22·206610	22·376264	22·373237	22·373291	22·373290
61·672823	61·685028	61·672832	61·672823	61·672823	*
120·903392	120·902654	120·903392	120·903392	*	*
199·859448	199·859489	199·859448	199·859448	*	*
298·555535	298·555533	298·555535	298·555535	*	*
416·990786	416·990786	416·990786	*	*	*

TABLE 3

A comparison of the lowest five eigenfrequencies k^2 for the clamped–simply supported beam, as estimated by the Legendre-tau method (L-T), *the wave propagation method* (WPM) *and the iterated approximants* (37)

	WPM			
L-T	k_0^2	$[k^{(1)}]^2$	$[k^{(3)}]^2$	$[k^{(3)}]^2$
15·418206	15·421257	15·418208	15·418206	15·418206
49·964862	49·964872	49·964862	49·964862	*
104·247696	104·247696	104·247696	*	*
178·269729	178·269729	178·269729	*	*
272·030971	272·030971	272·030971	*	*

where we have used the approximation that

$$e^{\pm i(k_0 + k_1\varepsilon + k_2\varepsilon^2)} = e^{\pm ik_0}\left[1 \pm ik_1\varepsilon + \left(-\frac{k_1^2}{2} \pm k_2\right)\varepsilon^2 + O(\varepsilon^3)\right].$$

Setting the coefficients of different powers of ε to zero in equation (28), we obtain

$$\varepsilon^0\!: e^{ik_0} + e^{-ik_0} = 0; \tag{29}$$

$$\varepsilon^1\!: -4i + k_1(e^{ik_0} - e^{-ik_0}) = 0; \tag{30}$$

$$\varepsilon^2\!: i(k_1^2 - 2)(e^{ik_0} + e^{-ik_0}) + 2k_2(e^{ik_0} - e^{-ik_0}) = 0; \tag{31}$$

$$\vdots$$

Equation (29) leads to

$$k_0 = \frac{(2n+1)\pi}{2}, \qquad n = 0, 1, 2, \ldots. \tag{32}$$

This result is consistent with equation (20) obtained from WPM. Using k_0 from equation

(32) in equation (30), we obtain

$$k_1 = -2\mathrm{i}\,\mathrm{e}^{\mathrm{i}k_0} = 2 \times (-1)^n, \qquad n = 0, 1, 2, \ldots .$$

We can obtain k_2 similarly from equation (31). However, as it turns out, in this case we do not need to actually find k_2.

Repeating the above argument for the remaining cases, we obtain

$$\text{(C–C)}, \qquad k_0 = \frac{(2n+1)\pi}{2}, \quad k_1 = 2 \times (-1)^{n+1}, \qquad n = 0, 1, 2, \ldots ;$$

$$\text{(C–S)}, \qquad k_0 = \frac{(4n+1)\pi}{4}, \quad k_1 = 0, \quad k_2 = -1, \qquad n = 0, 1, 2, \ldots ;$$

$$\text{(C–R)}, \qquad k_0 = \frac{(4n-1)\pi}{4}, \quad k_1 = 0, \quad k_2 = 1, \qquad n = 1, 2, 3, \ldots .$$

Note that in the cases (C–S) and (C–R), the correction for k is of the second order in ε:

$$k = k_0 + k_2\varepsilon^2 + O(\varepsilon^3). \tag{33}$$

TABLE 4

A comparison of the lowest five eigenfrequencies k^2 for the clamped–roller-supported beam, as estimated by the Legendre-tau method (L-T), *the wave propagation method* (WPM) *and the iterated approximants* (37)

	WPM				
L-T	k_0^2	$[k^{(1)}]^2$	$[k^{(2)}]^2$	$[k^{(3)}]^2$	$[k^{(4)}]^2$
5·593321	5·551652	5·594066	5·593309	5·593322	5·593322
30·225848	30·225663	30·225848	30·225848	*	*
74·638884	74·638883	74·638884	74·638884	*	*
138·791312	138·791312	138·791312	*	*	*
222·682949	222·682949	222·682949	*	*	*

TABLE 5

A comparison of the lowest eight eigenfrequencies for the clamped–free beam, as estimated by the Legendre-tau method (L-T), *the wave propagation method* (WPM) *and the iterated approximants* (36). *The first two columns here are the same as those in Table* 1

	WPM				
L-T	k_0^2	$[k^{(1)}]^2$	$[k^{(2)}]^2$	$[k^{(3)}]^2$	$[k^{(4)}]^2$
3·516015	2·467401	3·946403	3·404507	3·560036	3·511526
22·034492	22·206610	22·037602	22·034544	22·034488	22·034487
61·697214	61·685028	61·697224	61·697214	61·697214	*
120·901916	120·902654	120·901916	120·901916	*	*
199·859530	199·859489	199·859530	199·859530	*	*
298·555531	298·555533	298·555531	298·555531	*	*
416·990786	416·990786	416·990786	*	*	*
555·165248	555·165248	555·165248	*	*	*

Now, we return and continue with the cantilever case (C–F) at hand. To improve the estimate of k, we first set

$$\varepsilon = \varepsilon_0 = e^{-k_0}, \tag{34}$$

the choice of which is obvious by comparing the matrices M and M_ε in equations (24) and (26). This gives us a first improved value of k:

$$k^{(1)} \equiv k_0 + k_1\varepsilon_0 = \frac{(2n+1)\pi}{2} + 2 \times (-1)^n \exp\left(-\frac{(2n+1)\pi}{2}\right), \qquad n = 0, 1, 2, \ldots . \tag{35}$$

Next, we can again improve $k^{(1)}$ by updating our choice of ε, from equation (34) to

$$\varepsilon = \varepsilon_1 = \exp(-k_0 - k_1\varepsilon_0),$$

and let

$$k^{(2)} \equiv k_0 + k_1\varepsilon_1.$$

This process can be continued recursively and indefinitely:

$$\varepsilon_0 = e^{-k_0}, \qquad \varepsilon = \varepsilon_j = \exp(-k_0 - k_1\varepsilon_{j-1}), \qquad k^{(j+1)} = k_0 + k_1\varepsilon_j, \quad j = 1, 2, 3, \ldots . \tag{36}$$

(For the (C–S) and (C–R) cases when $k_1 = 0$ and equation (33) holds, the updating procedure becomes

$$\varepsilon_0 = e^{-k_0}, \qquad \varepsilon_j = \exp(-k_0 - k_2\varepsilon_{j-1}^2), \qquad k^{(j+1)} = k_0 + k_2\varepsilon_j^2, \quad j = 1, 2, 3, \ldots . \tag{37}$$

The numerical results of applying the above to the four cases C–C, C–S, C–R and C–F, along with those (nearly exact values) obtained using a Legendre-tau spectral approximation, are given in Tables 2–5, respectively. We see that, in all four cases, we get strong agreement for the lowest few eigenfrequencies after only a few updates. (Of course, if the eigenfrequencies are high, then WPM already supplies very accurate estimates, and no improvements are needed at all.) We again mention that for each case in Tables 2–5, our Legendre-tau calculated data agree closely with the values obtained elsewhere; cf., references [7, 8].

As a final conclusion, we state that WPM, when combined with the perturbation procedure as developed in this work, enables us to capture the entire (low, medium and high) range of the spectrum of vibration of an Euler–Bernoulli beam, both asymptotically and numerically. The amount of numerical work required is minimal. The methodology also applies to other more complicated situations such as a flexible single-link robotic manipulator with a payload attached at an endpoint; see a new paper by the second author [9].

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