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A GENERALIZED TREATMENT OF THE ENERGETICS OF TRANSLATING CONTINUA, PART I: STRINGS AND SECOND ORDER TENSIONED PIPES

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The energetics of translating one-dimensional uniform strings and highly tensioned pipes with vanishing bending stiffness and flowing fluid are analyzed for fixed, free and damped boundary conditions. The interaction between the translating continua and the boundary supports causes energy transfer. At a fixed boundary, the transverse component of tension does work, and the Coriolis forces at a free-end cause energy flux into the second-order continuum. Under a symmetric boundary configuration, the total energy of free oscillation varies periodically at the fundamental natural frequency. Asymmetric boundary supports in the pipe-fluid system lead to damped or self-excited motions. At a viscously damped boundary, the condition for maximal energy dissipation, the destabilizing effect of dissipation and the stabilizing effect of negative damping are examined analytically using travelling wave solutions. The energies transferred at the different boundary supports are quantified by energy reflection coefficients which are determined completely by the boundary conditions. Numerical simulations verify the analytically predicted energy variations.

1. INTRODUCTION

Understanding vibration of translating continua is important for the design of high speed magnetic tapes, band saws, power transmission chains and belts, textile and composite fibers, aerial cable tramways, pipes transporting fluid and other similar systems. Recent developments are reviewed by Wickert and Mote [1], Wang and Liu [2], and Paidoussis and Li [3].

A translating uniform string is the simplest model of axially moving continua. Earlier studies concern the dependence of the frequency spectrum on the transport speed, complex eigenfunctions, and the existence of a divergence instability [4–8]. The natural frequencies decrease with increasing transport speed, and the translating continua experience divergence instability at a critical speed. The eigenfunctions are complex and speed-dependent due to a convective acceleration component in the equations of motion. The phases of the natural oscillations are not constant and propagate upstream at the phase propagation velocity. Wickert and Mote [9] derived exact, closed-form, expressions for the response of translating continua to arbitrary excitation and initial conditions by using a complex modal analysis and a Green's function.

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Another interesting feature of the translating continuum is the periodic variation of total mechanical energy. For an undamped non-translating string, the total energy is constant. However under translation, the energy of each natural mode periodically transfers into and out of the span. Chubachi [6] and Miranker [10] discussed periodicity of the energy transfer in a translating string. Wickert and Mote [11] showed that the energy flux at a fixed support is the product of the string tension and the convective component of a velocity. An analogous problem, with similar governing equation, is the problem of a highly tensioned pipe conveying fluids under the assumptions that the contribution of bending to the stiffness of the pipe and the effect of fluid pressure are negligible.

In this paper the energetics of translating uniform strings and highly tensioned pipes conveying fluids are investigated using traveling waves. By addressing the energetics of both translating strings and tensioned pipes under a fixed, free or damped boundary support in a generalized manner, this paper extends the previous work by Wickert and Mote [11] on the energy variation of a single-mode wave in a translating string with a fixed support. This paper, firstly, identifies the generalized forces and the convective velocities, that result in energy flux at the boundary, by the one-dimensional transport theorem. Then, the magnitude of the energy transfer is quantified completely by the boundary condition specified through traveling wave solutions. The dynamic stability of the second-order continua is discussed based on the energy transfer mechanism at the boundaries.

2. THE EQUATIONS OF MOTION

2.1. TRANSLATING STRING

Consider a uniform string translating at constant speed and tension between two supports separated by distance L (Figure 1(a)). The linear equation of the transverse motion of the string is [4, 5]

$$\rho(w_{tt} + 2vw_{xt} + v^2 w_{xx}) - Pw_{xx} = 0, \qquad x \in (0, L) \tag{1}$$

where ρ is the linear density of the string, v is the constant transport speed, and P is the constant tension. The linear model (1) is restricted by the assumptions that the transverse displacement is small compared to the span length L and the initial tension is sufficiently large that its variation due to extension of the string is negligible. The contributions of the non-linear terms in the equation of motion increase with transport speed [12]. When the stiffness operator in (1) is no longer positive definite, divergence instability occurs at a critical speed

$$v_c = \sqrt{P/\rho}. \tag{2}$$

The critical speed is actually the phase velocity of a wave with $v = 0$. In this study, v is subcritical ($v < v_c$).

2.2. TENSIONED PIPE CONVEYING FLUIDS

Consider an idealized, highly tensioned pipe with negligible bending stiffness conveying fluids with a steady flow velocity u between two supports, as illustrated in Figure 1 (b–c). If gravity, pressurization effects and flexural restoring forces are negligible, then the pipe–fluid system is modelled by a *string conveying fluids*. The linear equation of transverse motion $w(x, t)$ of the tensioned pipe is

$$(m_f + m_p)w_{tt} + 2m_f u w_{xt} + m_f u^2 w_{xx} - Pw_{xx} = 0, \qquad x \in (0, L) \tag{3}$$

where m_f and m_p are the mass densities of the fluid and pipe, and P is the tension on the pipe. Similar to a beam conveying fluids [13], fluid-frictional effects acting on this pipe are absent in equation (3). The centrifugal force $m_f u^2 w_{xx}$ is analogous to a compressive force to the pipe, and the pipe system experiences divergence instability at a critical flow velocity

$$u_c = \sqrt{P/m_f} \tag{4}$$

which is obtained from the time-independent terms in equation (3).

3. TRAVELING WAVE CHARACTERISTICS

3.1. TRAVELING WAVES

A one-dimensional traveling wave has the form

$$w(x, t) = A\, \mathrm{e}^{\mathrm{i}(\omega t - kx)}, \tag{5}$$

where ω and k are the frequency and wavenumber. The traveling wave solution of an infinite, translating second-order continuum is represented by two independent traveling waves

$$w(x, t) = A_d\, \mathrm{e}^{\mathrm{i}(\omega t - k_d x)} + A_u\, \mathrm{e}^{\mathrm{i}(\omega t + k_u x)}, \tag{6}$$

where k_d and k_u are the wavenumbers for downstream (forward) and upstream (backward) traveling waves. Substitution of equation (5) into equation (1) gives the dispersion relation of the translating string

$$(v_c^2 - v^2)k^2 + 2v\omega k - \omega^2 = 0, \tag{7}$$

and it leads to the wavenumbers

$$k_d = \frac{\omega}{v_c + v}, \qquad k_u = \frac{\omega}{v_c - v}. \tag{8}$$

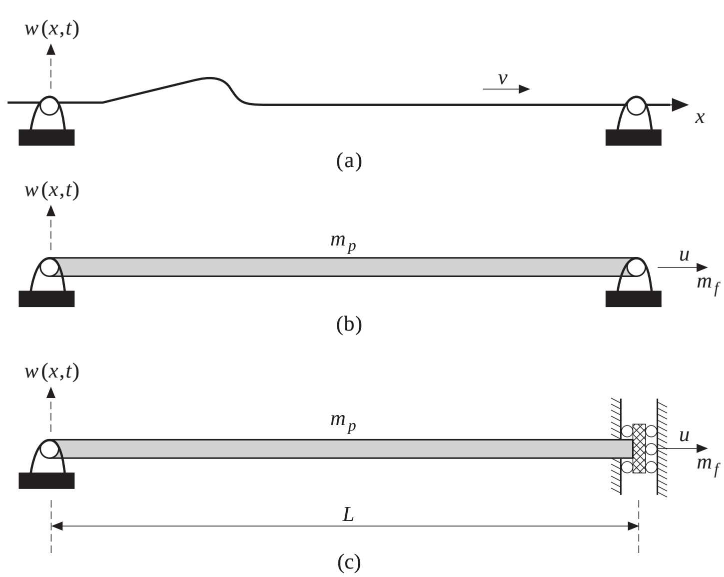

Figure 1. Schematics of the second order translating continua; (a) an axially moving string, (b) a tensioned pipe conveying fluid with fixed supports, and (c) a tensioned pipe with a free-end at $x = L$.

The phase velocities of the downstream and upstream waves are, respectively, $c_d = v_c + v$, and $c_u = v_c - v$. For a tensioned pipe transporting fluids, the dispersion relation is

$$\beta(u_c^2 - u^2)k^2 + 2u\beta\omega k - \omega^2 = 0, \tag{9}$$

where $\beta = m_f/(m_f + m_p)$. The downstream and upstream wavenumbers are

$$k_d = \frac{\omega}{\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}, \qquad k_u = \frac{\omega}{-\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}. \tag{10}$$

The phase velocities, $c_d = \beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}$ and $c_u = -\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}$, are independent of wavenumber. In the non-dispersive medium, the energy propagation velocity (group velocity) equals the phase velocity. For a stationary fluid ($u = 0$), $k_d = k_u = \omega/(u_c\sqrt{\beta})$. For the limiting case $\beta = 1$, the wavenumbers become $k_d = \omega/(u_c + u)$ and $k_u = \omega/(u_c - u)$, which are similar to equation (8). Accordingly, the translating string (1) can be considered a special case of the second order fluid–pipe system (3).

When a wave, propagating along an elastic medium, is incident on a discontinuity in the medium, a wave is reflected with its amplitude and phase determined by the reflection coefficient

$$r = A_r/A_i = |r|\,\mathrm{e}^{\mathrm{i}\phi}, \tag{11}$$

where A_i and A_r are amplitudes of the incident and reflected waves. When the boundary has no external energy source (or sink), such as a fixed support, energy conservation requires that $|r| = 1$.

3.2. PHASE CLOSURE PRINCIPLE: NATURAL FREQUENCY

In the infinite, translating continua without constraints, all wavenumbers are permissible. The natural modes of vibration (standing waves) are representable by the superposition of equal but opposite traveling waves. As a wave propagates, the phase difference between two points in the continuum is characterized by the wavenumber k, which is the phase change per unit length. In the translating continua the total phase change is $k_d L$, as a wave propagates from the upstream boundary to the downstream one. Similarly the phase change of a wave travelling the upstream boundary to the downstream one is $k_u L$. The total phase change, as the wave travels the domain, becomes

$$L(k_d + k_u) + \phi_d + \phi_u, \tag{12}$$

where ϕ_d and ϕ_u are phase changes produced at the downstream and upstream boundaries. The phase–closure principle [14, 15] states that, if this total phase change is an integer multiple of 2π, equation (12) identifies a natural frequency of the system. For the translating string with fixed supports, the total phase difference

$$\omega L(1/(v_c + v) + 1/(v_c - v)) + \pi + \pi = 2\pi n \tag{13}$$

gives the natural frequencies of the classical moving threadline [7],

$$\omega_n = \frac{n\pi(v_c^2 - v^2)}{v_c L}, \tag{14}$$

where $n = 1, 2, 3, \cdots$. The natural frequencies of the fluid–pipe system (3) are determined in a similar manner (Table 1). For the case of a fixed-free boundary configuration, ω_n represents the real part of the natural frequencies of the tensioned pipe.

TABLE 1

Natural frequencies

Boundary	String fixed–fixed	Tensioned pipe fixed–fixed	Tensioned pipe fixed–free	Tensioned pipe free–free
ω_n	$\dfrac{n\pi(v_c^2 - v^2)}{v_c L}$	$\dfrac{n\pi\beta(u_c^2 - u^2)}{L\sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}$	$\dfrac{(n-1/2)\pi\beta(u_c^2 - u^2)}{L\sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}$	$\dfrac{(n-1)\pi\beta(u_c^2 - u^2)}{L\sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}$

4. ENERGETICS OF SECOND ORDER CONTINUA

Consider a continum moving axially at a constant speed v. The instantaneous transverse velocity of the material particle in inertial co-ordinates is $w_t + vw_x$. The total energy per unit length of a translating string is the sum of the kinetic and potential energy densities:

$$\hat{E} = \tfrac{1}{2}\,\rho(w_t + vw_x)^2 + \tfrac{1}{2}Pw_x^2. \tag{15}$$

The energy density of a tensioned pipe conveying fluid becomes

$$\hat{E} = \tfrac{1}{2}m_f(w_t + uw_x)^2 + \tfrac{1}{2}m_p w_t^2 + \tfrac{1}{2}Pw_x^2. \tag{16}$$

The total mechanical energy $E(t)$ contained in the material particles belonging to the fixed region $0 \leqslant x \leqslant L$ is

$$E(t) = \int_0^L \hat{E}\,\mathrm{d}x. \tag{17}$$

When the continuum transports mass at speed v, the time-rate of change of the total energy is expressed by the one-dimensional transport theorem including the effect by nonconservative forces acting on two boundaries:

$$\dot{E}(t) = E_t + v\hat{E}_m|_0^L + \mathscr{F}|_0^L, \tag{18}$$

where $(\dot{\ }) = \mathrm{d}/\mathrm{d}t$, $()_t = \partial/\partial t$, and $\hat{E}_m$ is the energy density of the continuum crossing the boundary supports. The first term on the right side of equation (18) describes the local rate of change of energy within the domain, while the second term represents the net rate of outward energy flux at any instant of time [16]. For a translating string, $\hat{E}_m = \hat{E}$ in equation (15), and for a tensioned pipe conveying fluids,

$$\hat{E}_m = \tfrac{1}{2}m_f(m_t + uw_x)^2. \tag{19}$$

The last term $\mathscr{F}$ of equation (18) denotes energy flux into the continuum by non-conservative forces at the boundaries. The non-conservative flux term vanishes at fixed boundaries because of the zero displacements. Substitution of equations (15) and (17) into equation (18), and use of equation (1) yield the time-rate of change of the total energy of the translating string:

$$\dot{E}(t) = Pw_x(w_t + vw_x)|_0^L + \mathscr{F}|_0^L. \tag{20}$$

Similarly, the expression for total energy flux in the fluid-pipe system is obtained by substituting equation (17) and (19) into equation (18) and using equation (3):

$$\dot{E}(t) = Pw_x(w_t + uw_x)|_0^L - \int_0^L m_p w_{tt}(uw_x)\,\mathrm{d}x + \mathscr{F}|_0^L. \tag{21}$$

4.1. FIXED SUPPORT

From equation (20), the time-rate of change of energy in a translating string with fixed boundary conditions $w(0, t) = w(L, t) = 0$ and $\mathscr{F} = 0$ is

$$\dot{E}(t) = Pw_x(vw_x)|_0^L. \tag{22}$$

At the fixed boundary, the instantaneous velocity of the material particle, vw_x, mutiplied by the transverse component of the string tension, Pw_x, gives energy flux into the string [11]. The string segment passing through a downstream fixed boundary gains energy, and it loses energy at an upstream one.

For tensioned pipe with fixed supports, the rate of change of energy is, from equation (21),

$$\dot{E}(t) = Pw_x(uw_x)|_0^L - \int_0^L m_p w_{tt}(uw_x)\,\mathrm{d}x. \tag{23}$$

The first term represents energy flux into the pipe by the boundary force Pw_x and the convective velocity uw_x of the fluid to the pipe. The second term of equation (21) represents energy flux resulted from inertial force by the transverse motion in the domain. For a traveling wave $w(x, t) = A\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}$, the inertia force and the convective velocity at a material particle on the pipe are given by

$$m_p w_{tt} = -m_p\omega^2 A\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}, \qquad uw_x = -iukA\,\mathrm{e}^{\mathrm{i}(\omega k - kx)}, \tag{24}$$

showing that the phase difference between the local inertia force and the convective velocity is always $\pi/2$. Thus, the energy flux term vanishes for both downstream and upstream traveling waves and it is represented only by the pipe tension Pw_x and the relative velocity uw_x at the boundaries.

Equations (22) and (23) show that total energy of free oscillation in both systems varies with time due to energy transfer at both boundaries. All waves recover their original forms after two consecutive reflections at the end supports. For the translating string, the period of the energy variation, with zero net energy flux by both boundary supports, is

$$T_e = L/(v_c + v) + L/(v_c - v) = 2v_c L/(v_c^2 - v^2), \tag{25}$$

which equals the fundamental period of the string. For the pipe, the energy of free vibration varies with a period

$$T_e = \frac{L}{c_d} + \frac{L}{c_u} = \frac{2L\sqrt{\beta^2 u^2 + \beta(u_c^2 - v^2)}}{\beta(u_c^2 - u^2)}, \tag{26}$$

which is also the period of the first mode of the fluid-pipe system. In both cases, the resultant energy variation of each system vanishes over T_e:

$$\Delta W = \int_0^{T_e} \dot{E}(t)\,\mathrm{d}t = 0. \tag{27}$$

4.2. FREE SUPPORT

When the exit end of a tensioned pipe is unconstrained, the momentum flux of the fluid acts as a non-conservative force. The fluid–dynamic force does work on the boundary, while energy flux induced by the boundary force Pw_x at the end vanishes. The non-conservative force is obtained from Hamilton's principle [17]:

$$\delta \int_{t_1}^{t_2} \mathscr{L}\, \mathrm{d}t + \int_{t_1}^{t_2} \delta W\, \mathrm{d}t = 0, \tag{28}$$

where $\mathscr{L} = T - V$ is the Lagrangian function made up of the kinetic energy (T) and potential energy (V) of the pipe and fluid and δW is the virtual work due to the non-conservative force not included in the Lagrangian. The substitution of the Lagrangian for the second order fluid–pipe system,

$$\mathscr{L} = \frac{1}{2} \int_0^L (m_f(w_t^2 + 2uw_x w_t + u^2 w_x^2) + m_p w_t^2 - P w_x^2)\, \mathrm{d}x,$$

and the virtual work by the unknown non-conservative force $F_L(t)$,

$$\delta W = F_L(t) \delta w(L, t),$$

into equation (28) gives the boundary condition at $x = L$:

$$\{-Pw_x + m_f u(w_t + uw_x) + F_L\}\delta w = 0. \tag{29}$$

With the boundary condition $w_x(L, t) = 0$ at the free-end, equation (29) gives the non-conservative force

$$F_L(t) = -m_f u w_t(L, t). \tag{30}$$

In this case, the Coriolis force induced by the fluid transport does work at the free end. However, the centrifugal force $-m_f u^2 w_x(L, t)$, which is not zero at a free boundary of a beam conveying fluids, vanishes in the second order continuum. The fluid–dynamic force at an upstream free end is $F_0(t) = m_f uwt(0, t)$. Accordingly, with energy flux $\mathscr{F} = Fw_t = m_f u w_t^2$, the rate of change of energy in the pipe system with free supports both downstream and upstream is

$$\dot{E}(t) = \mathscr{F}|_0^L = m_f u w_t^2(0, t) - m_f u w_t^2(L, t). \tag{31}$$

The pipe always loses energy at the downstream free-end and gains energy at the upstream free-end. The total energy varies with period

$$T_e = \frac{2L\sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}{\beta(u_c^2 - u^2)} \tag{32}$$

which equals equation (26) for the case with fixed supports, because the fundamental frequencies of both cases are the same. For any initial condition, the total energy flux by the nonconservative forces vanishes over T_e:

$$\Delta W = \int_0^{T_e} \dot{E}(t)\, \mathrm{d}t = m_f u \int_0^{T_e} \{w_t^2(0, t) - w_t^2(L, t)\}\, \mathrm{d}t = 0. \tag{33}$$

4.3. ASYMMETRIC BOUNDARY CONFIGURATION

Consider a translating continuum subjected to an asymmetric boundary configuration. In a tensioned cantilevered pipe with fluid exiting from the downstream free end, the total energy flux becomes, from equations (23) and (31),

$$\dot{E}(t) = -Puw_x^2(0, t) - m_f u w_t^2(L, t) \leqslant 0. \tag{34}$$

Energy flux into the pipe is always negative and free motion is damped. However a tensioned cantilevered pipe with an upstream free end loses stability by flutter, because the total energy flux into the pipe,

$$\dot{E}(t) = m_f u w_t^2(0, t) + P u w_x^2(L, t), \tag{35}$$

is always positive. In either case, the period of energy fluctuation in damped or self-excited free oscillation becomes

$$T_e = 4L \frac{\sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}{\beta(u_c^2 - u^2)}, \tag{36}$$

which is twice the period (26) for the case with a symmetric boundary configuration as shown in Table 1.

The forces and the corresponding transverse velocity terms resulting in energy flux at fixed and free supports are summarized in Table 2. A positive sign indicates that energy is transferred into the system at the boundary.

5. GENERALIZED EXPRESSION FOR ENERGY TRANSFER

The explicit expression for energy transfer, between translating continua and different types of boundary supports, is examined by considering the energies of incident and reflected travelling waves on a boundary. The energy contained in one wavelength $\lambda = 2\pi/k$ of a single harmonic wave $A\,\mathrm{e}^{\mathrm{i}(\omega t - kx)}$ in a translating continuum is

$$E_\lambda = \int_x^{x+\lambda} \hat{E}\,\mathrm{d}x = \pi P k A^2 = \pi\omega \mathscr{Z} A^2, \tag{37}$$

where $\mathscr{Z} = P/c$ is the mechanical impedance of the continuum. The energy ΔW, transferred into the continuum span over one period by wave reflection at a boundary, equals the difference between the energies of the reflected and incident waves:

$$\Delta W = E_r - E_i = \pi\omega(\mathscr{Z}_r A_r^2 - \mathscr{Z}_i A_i^2), \tag{38}$$

where $\mathscr{Z}_r$ and $\mathscr{Z}_i$ are the mechanical impedances of the continuum, and A_r and A_i are the amplitudes of reflected and incident waves. The different impedances, $\mathscr{Z}_r \neq \mathscr{Z}_i$, in the translating medium lead to energy flux into the continuum at the boundary even when the incident and reflected waves have the same amplitude ($r = 1$). An energy reflection coefficient is then introduced to quantify the energy transfer at the boundary. The energy reflection coefficient equals the ratio of the reflected wave energy to the incident wave energy,

$$R = E_r/E_i = \mathscr{Z}_r/\mathscr{Z}_i (A_r/A_i)^2 = (k_r/k_i) r^2. \tag{39}$$

TABLE 2

Energy flux at fixed and free boundaries

Boundary	String fixed	Tensioned pipe fixed	Tensioned pipe free
Force	Pw_x	Pw_x	$-m_f u w_t$
Convective velocity	vw_x	uw_x	w_t
Energy flux at $x = L$	+	+	−
Energy flux at $x = 0$	−	−	+

The simple equation (39) determines the energy transfer without identifying the conservative or non-conservative forces acting on the boundary. The energy transferred into the continuum over a cycle of the wave is then

$$\Delta W = (1 - R)E_i. \tag{40}$$

For the translating string and the fluid-conveying pipe, the impedance (wavenumber) ratio of downstream waves to upstream waves is, from equations (8) and (10),

$$\frac{\mathscr{Z}_d}{\mathscr{Z}_u} = \frac{k_d}{k_u} = \frac{v_c - v}{v_c + v}, \qquad \frac{\mathscr{Z}_d}{\mathscr{Z}_u} = \frac{k_d}{k_u} = \frac{-\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}{\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}. \tag{41}$$

5.1. ENERGY TRANSFER AT A DOWNSTREAM BOUNDARY

For a harmonic wave solution

$$w(x, t) = A_d \, e^{i(\omega t - k_d x)} + A_u \, e^{i(\omega t + k_u x)}, \tag{42}$$

any travelling wave propagating towards a boundary can be considered as the incident wave. At a downstream boundary, A_d is the amplitude of the incident wave, and A_u is the amplitude of that reflected wave. Substitution of equation (42) into the boundary condition gives $r = A_u/A_d$ at the boundary, and R_d is determined completely from equation (39). For a fixed boundary support ($w(L, t) = 0$),

$$r = -1, \qquad R_d = (k_u/k_d)r^2 = k_u/k_d. \tag{43}$$

For the translating string and pipe, the energy reflection coefficients are, from equations (41) and (43),

$$R_d = \frac{v_c + v}{v_c - v}, \qquad R_d = \frac{\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}{-\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}. \tag{44}$$

Note that $R_d \geqslant 1$ in both cases, and the incident wave energy increases with a factor k_u/k_d at the fixed end. $R_d = 1$ at zero transport speed ($v = 0$ or $u = 0$) and R_d increases with the speed.

For a fluid-pipe system with a free end at $x = L$ ($Pw_x(L, t) = 0$), the coefficients are

$$r = R_d = \frac{k_d}{k_u} = \frac{-\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}{\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}} \leqslant 1. \tag{45}$$

The wave reflected from the downstream free end decreases its amplitude and energy by k_d/k_u. Both r and R_d asymptotically approach zero as u is increased to u_c.

When viscous damping is attached to the free end of the tensioned pipe system, the boundary condition becomes $Pw_x(L, t) = -\mathrm{d}w_t(L, t)$ and

$$r = \frac{Pk_d - \mathrm{d}\omega}{Pk_u + \mathrm{d}\omega} = \frac{\mathscr{Z}_d - d}{\mathscr{Z}_u + d}, \qquad R_d = \frac{k_u(\mathscr{Z}_d - d)^2}{k_d(\mathscr{Z}_u + d)^2}. \tag{46}$$

If the damping coefficient is tuned to the impedance,

$$d_{opt} = \mathscr{Z}_d = P/\{\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}\}, \tag{47}$$

then r and R_d both vanish at the downstream boundary regardless of frequency. For a translating string with a viscous damper at $x = L$, the expressions for r and R_d in equations (46) are valid with the assumption that the string particles exiting the boundary have no contribution to the force balance ($Pw_x(L^+, t) = 0$). Under this assumption, the impedance of a downstream wave in the axially moving string,

$$d_{opt} = \mathscr{Z}_d = P/(v_c + v), \tag{48}$$

TABLE 3

Critical damping coefficient and energy flux at a viscous damped support

Boundary	Downstream ($x = L$)			Upstream ($x = 0$)		
String	$d_{1,2} = \frac{P}{v}\left(1 \mp \frac{v_c}{\sqrt{v_c^2 - v^2}}\right)$			$d_{1,2} = \frac{P}{v}\left(-1 \mp \frac{v_c}{\sqrt{v_c^2 - v^2}}\right)$		
Tensioned pipe	$d_{1,2} = \frac{P}{\beta u}\left(1 \mp \sqrt{\frac{\beta^2 u^2 + \beta(u_c^2 - u^2)}{\beta(u_c^2 - u^2)}}\right)$			$d_{1,2} = \frac{P}{\beta u}\left(-1 \mp \sqrt{\frac{\beta^2 u^2 + \beta(u_c^2 - u^2)}{\beta(u_c^2 - u^2)}}\right)$		
Damping coefficient	$d < d_1$	$d_1 < d < d_2$	$d > d_2$	$d < d_1$	$d_1 < d < d_2$	$d > d_2$
Energy flux	+	−	+	−	+	−

becomes the optimal coefficient for complete wave dissipation. Lee and Mote [18] have recently used this result to stabilize vibration of an axially moving string through boundary control. The damping coefficients leading to $R_d = 1$, where energy dissipated by the damping component equals energy transferred into the pipe are

$$d_{1,2} = \frac{P}{\beta u}\left(1 \mp \sqrt{\frac{\beta^2 u^2 + \beta(u_c^2 - u^2)}{\beta(u_c^2 - u^2)}}\right). \tag{49}$$

When $d > d_2$, $R_d > 1$ and energy transferred into the pipe at the damped support exceeds energy dissipation by the damping component. This interesting phenomenon, the destabilizing effect of dissipation, is also found in a cantilevered, fourth order beam conveying fluids [19, 20].

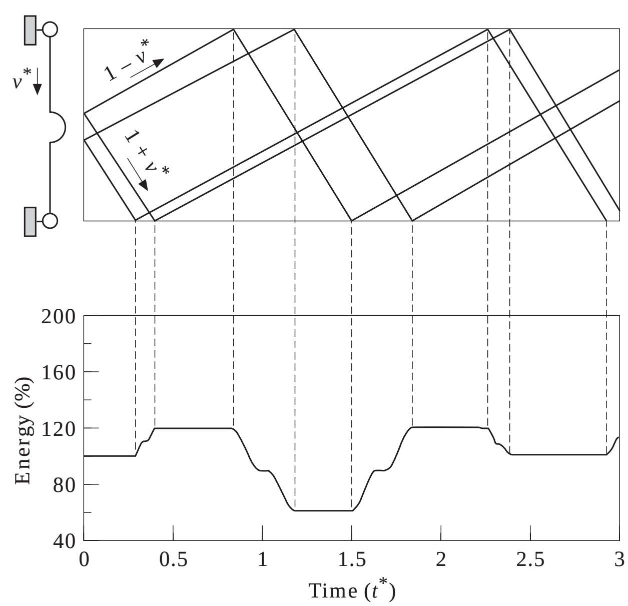

Figure 2. Characteristic lines and energy $E(t)$ of a string translating at $v = 0{\cdot}5$ with fixed supports and a midspan disturbance at $0{\cdot}4 < x^* < 0{\cdot}6$. The period of the energy fluctuation is $T_e = 2/(1 - v^{*2}) = 2{\cdot}667$.

5.2. ENERGY TRANSFER AT AN UPSTREAM BOUNDARY

When an upstream travelling wave of amplitude A_u impinges on a boundary at $x = 0$, the reflection and energy reflection coefficients at the upstream boundary are obtained in a similar manner. For

fixed support: $w(0, t) = 0$,

$$r = A_d/A_u = -1, \qquad R_u = (k_d/k_u)r^2 = k_d/k_u; \tag{50}$$

free support: $w_x(0, t) = 0$,

$$r = k_u/k_d, \qquad R_u = k_u/k_d; \tag{51}$$

viscously damped support: $Pw_x(0, t) = dw_t(0, t)$,

$$r = (\mathscr{L}_u - d)/(\mathscr{L}_d + d) \qquad R_u = (k_d/k_u)(\mathscr{L}_u - d)^2/(\mathscr{L}_d + d)^2. \tag{52}$$

From equation (50), the energy reflection coefficients for the translating string and the tensioned pipe at the fixed-end are always less than unity:

$$R_u = \frac{v_c - v}{v_c + v}, \qquad R_u = \frac{-\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}{\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}\,. \tag{53}$$

At an upstream free support in the fluid-pipe system, the Coriolis force, $m_f u w_t(0, t)$, is induced by mass transport through the inlet boundary and does positive work on the pipe. From equation (51), both r and R_u are larger than unity,

$$r = R_u = \frac{\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}}{-\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}} \geqslant 1, \tag{54}$$

and increase with transport speed. Thus the free boundary, where energy flux into the system is positive, can cause large increases in energy as the fluid speed approaches u_c.

If damping is attached on the upstream free end of the fluid–pipe system, the damping force dissipates the incident wave energy but the non-conservative force due to fluid particles does positive work on the end. The critical damping coefficients leading to $R_u = 1$, (energy dissipated due to the damping equals energy transferred from inlet fluid particles), are

$$d_{1,2} = \frac{P}{\beta u}\left(-1 \mp \sqrt{\frac{\beta^2 u^2 + \beta(u_c^2 - u^2)}{\beta(u_c^2 - u^2)}}\right). \tag{55}$$

It is noted that, firstly, the upstream wave energy can increase at the damped boundary when the damping coefficient satisfies

$$d_1 < d < d_2. \tag{56}$$

Secondly, a negatively damped support can dissipate the incident wave energy, because $R_u < 1$ for the range $d < d_1 < 0$. The damping coefficient of the boundary damping for perfect wave dissipation,

$$d_{opt} = \mathscr{L}_u = P/\{-\beta u + \sqrt{\beta^2 u^2 + \beta(u_c^2 - u^2)}\}, \tag{57}$$

is easily obtained from R_u. The results are summarized in Table 3.

6. NUMERICAL COMPARISON

Introduction of the variables

$$x^* = x/L, \qquad w^* = w/L, \qquad t^* = t/L\sqrt{P/\rho}, \qquad v^* = v\sqrt{\rho/P}$$

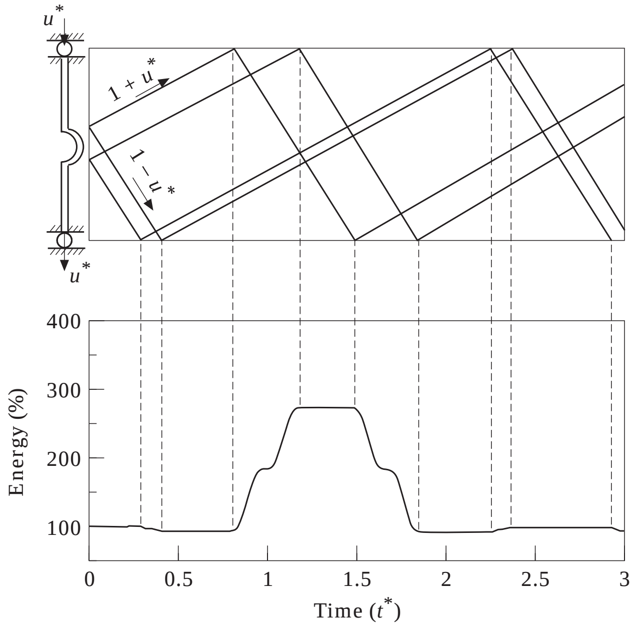

Figure 3. Characteristic lines and energy variation with time of the fluid–pipe system with free ends. Fluid speed is $u = 0{\cdot}5$ and $\beta = 1$. The initial midspan wave repeats its motion with period $T_e = 2{\cdot}667$.

into equation (1) gives the normalized equation of motion for the translating string:

$$w^*_{t^*t^*} + 2v^* w^*_{x^*t^*} - (1 - v^{*2}) u^*_{x^*x^*} = 0, \qquad x^* \in (0, 1). \tag{58}$$

For the tensioned pipe (3), the equation has the form

$$w^*_{t^*t^*} + 2\sqrt{\beta} u^* w^*_{x^*t^*} - (1 - u^{*2}) w^*_{x^*x^*} = 0, \qquad x^* \in (0, 1), \tag{59}$$

using the dimensionless parameters

$$x^* = \frac{x}{L}, \qquad w^* = \frac{w}{L}, \qquad t^* = \frac{t}{L}\sqrt{\frac{P}{m_f + m_p}}, \qquad u^* = u\sqrt{\frac{m_f}{P}}, \qquad \beta = \frac{m_f}{m_f + m_p}.$$

The corresponding normalized period of free oscillation for the string and the tensioned pipe are

$$T_e = 2/(1 - v^{*2}), \qquad T_e = 2\sqrt{(1 + (\beta - 1)u^{*2})}/(1 - u^{*2}), \tag{60}$$

respectively.

Numerical simulations by an explicit finite difference method are compared to the analytical predictions. The unit length of the string is divided into n equal intervals of length $h = 1/n$, and the time increment is κ. The displacement and velocity at each point are then

$$u_i^j = w^*(ih, j\kappa), \qquad v_i^j = w^*_{t^*}(ih, j\kappa). \tag{61}$$

The explicit difference equations for u_i^{j+1} and v_i^{j+1} from the normalized equation (58) are

$$v_i^{j+1} = v_i^j - v\kappa \frac{v_{i+1}^j - v_{i-1}^j}{h} + (1 - v^2)\kappa \frac{u_{i+1}^j - 2u_i^j - u_{i-1}^j}{h^2},$$

$$u_i^{j+1} = u_i^j + (\kappa/2)(v_i^j + v_i^{j+1}). \tag{62}$$

A mesh number $n = 400$ and a time step $\kappa = 2 \times 10^{-6}$ gave stable difference solutions.

The energy fluctuation and the characteristic lines of a fixed boundary string translating at $v^* = 0{\cdot}5$ are shown in Figure 2. Consider two waves, one travelling upstream and the other downstream, starting from an initial midspan disturbance ($0{\cdot}4 < x^* < 0{\cdot}6$) with zero transverse velocity and a total energy E_I. The energies contained in the upstream and downstream waves are calculated by integrating equation (15):

$$\frac{E_u}{E_I} = \frac{(1+v^*)^2}{2(1+v^{*2})}, \qquad \frac{E_d}{E_I} = \frac{(1-v^*)^2}{2(1+v^{*2})}. \tag{63}$$

The upstream wave energy, E_u, always exceeds the downstream one, E_d. For $v^* = 0{\cdot}5$, an upstream wave contains $(1+v^*)^2/(2(1+v^{*2})) = 0{\cdot}9E_1$, while a downstream wave has $0{\cdot}1E_1$. The downstream travelling wave impinging on the downstream boundary increases its energy by $(1+v^*)/(1-v^*) = 3$, as predicted by equation (43). For an initial energy $E_I = 100$, the total energy becomes $10 \times 3 + 90 = 120$ at $t^* = 0{\cdot}6/(1+v^*) = 0{\cdot}4$. The energy of backward travelling wave decreases to $(1-v^*)/(1+v^*) \times 90 = 30$ when it is reflected from the upstream boundary, and the total energy decreases to $30 + 30 = 60$ at $t^* = 0{\cdot}6/(1-v^*) = 1{\cdot}2$. The total energy returns to 100 at $t^* = 0{\cdot}6/(1+v^*) + 1/(1-v^*) = 2{\cdot}4$. The energy fluctuation has the period $T_e = 2/(1-v^{*2}) = 2{\cdot}667$. As predicted earlier, it equals the time required for a disturbance to propagate the length of the string downstream and upstream and return to the initial position.

The total energy of a tensioned pipe with free–free supports is shown in Figure 3. In this case, the mass parameter is $\beta = 1$ and fluid speed is $u^* = 0{\cdot}5$ for the comparison with the previous string case. A traveling wave incident on the downstream free support decreases its energy to one-third during reflection. An upstream wave energy, which is

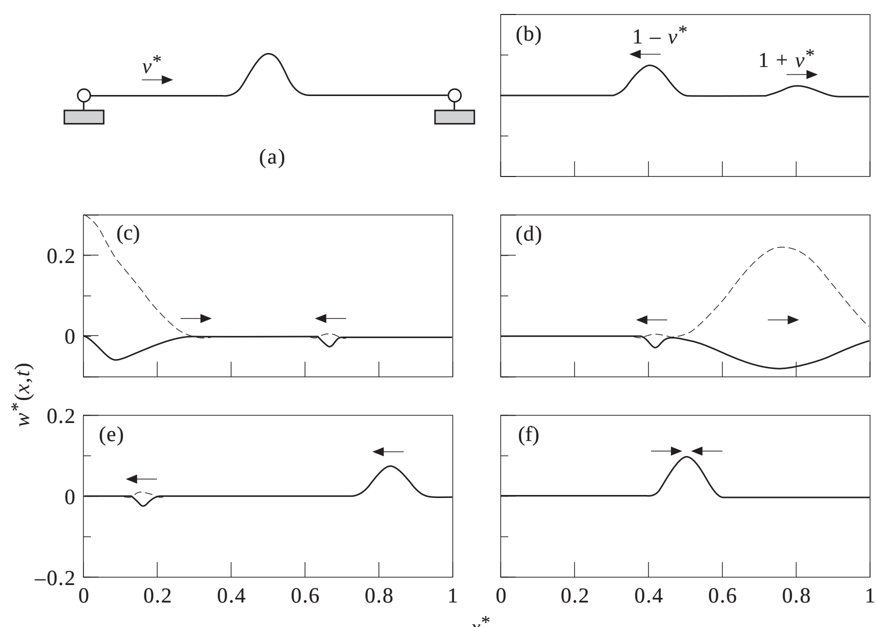

Figure 4. Transverse response $w^*(x^*, t^*)$ of a translating string with a midspan initial disturbance at intermediate times: (a) $t^* = 0$, (b) $t^* = 0{\cdot}2$, (c) $t^* = 1$, (d) $t^* = 1{\cdot}5$, (e) $t^* = 2$, (f) $t^* = T_e = 2{\cdot}667$. Transport speed is $v^* = 0{\cdot}5$. Fixed supports (solid line) and free supports (dashed line).

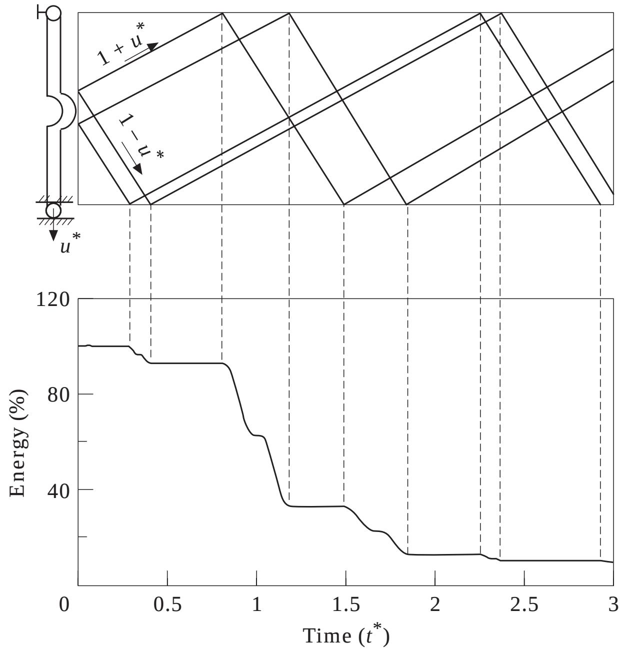

Figure 5. Characteristic lines and energy variation with time of the fluid-pipe system with fixed upstream and free downstream supports. Fluid speed is $u^* = 0{\cdot}5$ and $\beta = 1$. The wave energy decreases at both boundaries.

initially 90, increases three times ($(1 + u^*)/(1 - u^*) = 3$) at the upstream free support. At that time the total energy becomes $10 \times 1/3 + 90 \times 3 = 273{\cdot}3$. The forward travelling wave, reflected from the backward wave at $x = 0$, propagates to $x = 1$ where the total energy becomes $10 \times 1/3 + 270 \times 1/3 = 93{\cdot}3$ at $t^* = 0{\cdot}6/(1 - u^*) + 1/(1 + u^*) = 1{\cdot}87$. The period of energy fluctuation is $T_e = 2{\cdot}667$. The corresponding displacements of the fixed string and the pipe with free ends at intermediate times are plotted in Figure 4. When a travelling wave is incident on each boundary, the wave reflected from a fixed support has the amplitude of the incident wave and a phase of π. At a free support, both the amplitude and the energy of the reflected wave decrease by a factor of 2/3 at $x = 1$ and increase by a factor 2 at $x = 0$. The incident wave is reflected from the free end without phase change.

The energy fluctuation and the characteristic lines of a tensioned pipe with fixed–free boundary configuration are shown in Figure 5 when $\beta = 1$ and $u^* = 0{\cdot}5$. Free oscillation is damped by the interaction between the fluid flow and the pipe at the boundaries. A downstream wave, resulting from the initial disturbance, propagates to $x = 1$ which decreases its energy to a factor of $(1 - u^*)/(1 + u^*) = 1/3$. The energy of an upstream wave decreases by $90 \times 2/3 = 60$ at $x = 0$. Then the total energy becomes $10 \times 1/3 + 90 \times 1/3 = 33{\cdot}3$ at $t^* = 0{\cdot}6/0{\cdot}5 = 1{\cdot}2$. At $t^* = 2{\cdot}4$, the total wave energy decreases to $33{\cdot}3/3 = 11{\cdot}1$.

The energy and free response of the first and second modes of a translating string are shown in Figures 6(a) and (b). The total wave energy varies periodically with a period $T_e = 2{\cdot}669$ in each case. The displacements at intermediate times $t^*/T_e = 0, 0{\cdot}25, 0{\cdot}5, 0{\cdot}75$ and 1 are plotted together. The total energy of the fundamental mode of a tensioned pipe system is simulated in Figure 7 when $\beta = 0{\cdot}7$ and $u^* = 0{\cdot}2$. Initial condition is $w(0, t^*) = 0{\cdot}1 \sin \pi x^*$ for fixed–fixed, $w^*(0, t^*) = 0{\cdot}1 \sin (\pi x^*/2)$ for fixed–free and $w^*(0, t^*) = 0{\cdot}1 \cos (\pi x^*/2)$ for free–fixed supports. As predicted from equation (60), the

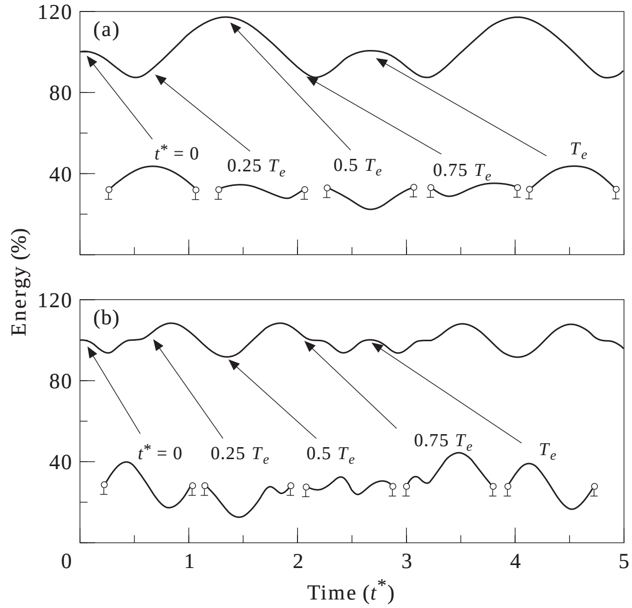

Figure 6. Total energy variation of a translating string: (a) free response of the first mode ($w^*(0, t^*) = 0{\cdot}1 \sin(\pi x^*)$); (b) the second mode ($w^*(0, t^*) = 0{\cdot}1 \sin(2\pi x^*)$). The displacement of the string is shown at $t^*/T_e = 0$, 0·25, 0·5, 0·75 and 1. Transport speed is $v^* = 0{\cdot}5$ and the period of free motion is $T_e = 2{\cdot}667$.

vibration energy of the fixed–fixed pipe varies with a period $T_e = 2\sqrt{(1 + (\beta - 1)u^{*2})/(1 - u^{*2})} = 2{\cdot}058$. For the fixed–free configuration, the free vibration is damped with time because energy is always transferred out of the pipe at both boundaries. However, in the free–fixed case, energy flux into the pipe is positive at both boundaries and the flowing fluid becomes an energy source. The amplitude of the oscillation increases with time.

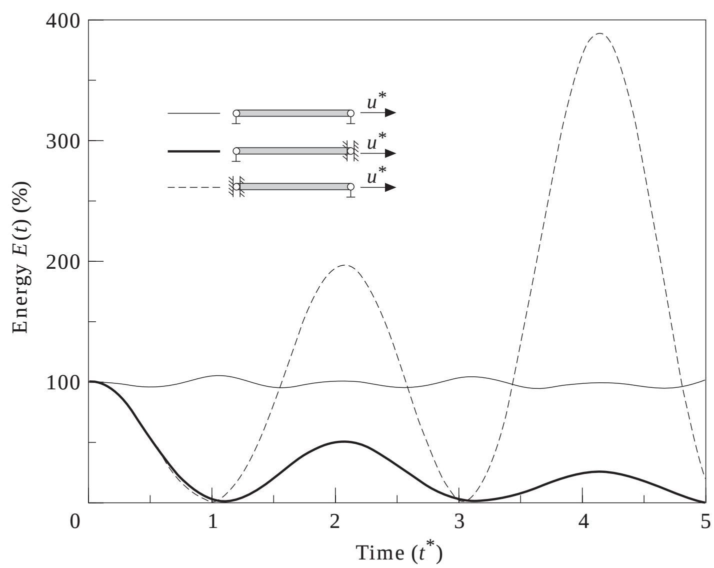

Figure 7. Total energy variation of the first mode in a tensioned pipe for fixed–fixed, fixed–free and free–fixed supports. Fluid speed is $u^* = 0{\cdot}2$ and $\beta = 0{\cdot}7$. The period of free oscillation in the pipe with fixed supports is $T_e = 2{\cdot}058$.

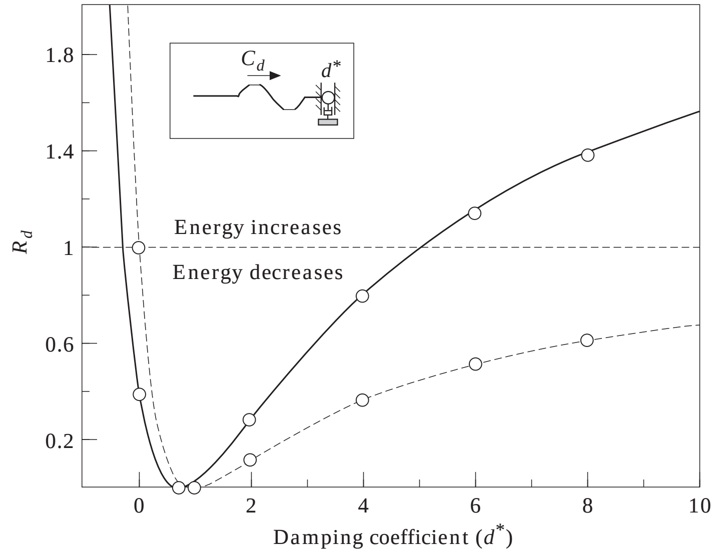

Figure 8. The energy reflection coefficient R_d at a viscously damped, downstream boundary when $\beta = 0{\cdot}7$: ——, analytical, $u^* = 0{\cdot}5$; - - - -, analytical, $u^* = 0$; ○, numerical.

When viscous damping is attached to a downstream free end of the tensioned pipe, analytical solutions (46) and finite difference calculations for the energy reflection coefficient R_d are compared in Figure 8. The coefficient R_d is numerically calculated using energies of a single harmonic, incident and reflected waves when $\beta = 0{\cdot}7$ and $u^* = 0$ or $0{\cdot}5$. The agreement is satisfactory over the ranges examined. Under $u^* = 0{\cdot}5$, the viscously damped support leads to an energy increase when

$$d^* > d_2 = 1/(\sqrt{\beta}u^*)(1 + \sqrt{(1 + (\beta - 1)u^{*2})/(1 - u^{*2})}) = 5{\cdot}0452.$$

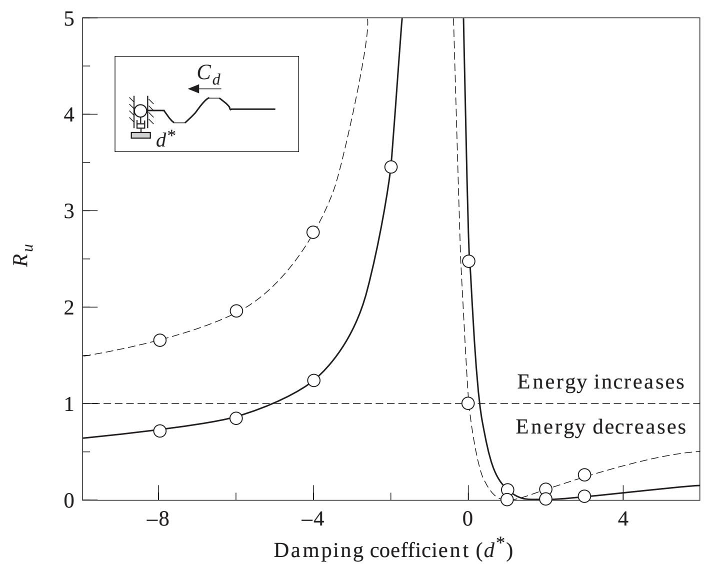

Figure 9. The energy reflection coefficient R_u at a viscously damped, upstream boundary when $\beta = 0{\cdot}7$: ——, analytical, $u^* = 0{\cdot}5$; - - - -, analytical, $u^* = 0$; ○, numerical.

The optimal damping coefficient required to dissipate all the incident wave energy is $d^* = Z_d = 1/[\sqrt{\beta u^*} + \sqrt{1 + (\beta - 1)u^{*2}}] = 0{\cdot}7246$ for $u^* = 0{\cdot}5$ and 1 for $u^* = 0$.

The energy reflection coefficient R_u at a viscous damped support attached to $x = 0$ is shown in Figure 9 for $u^* = 0$ and 0·5. For complete wave dissipation ($R_u = 0$), $d^* = \mathscr{Z}_u = 1{\cdot}8451$ at $u^* = 0{\cdot}5$. The damping coefficients leading to $R_u = 1$ are

$$d_1 = -5{\cdot}042 \qquad d_2 = 0{\cdot}2643 \tag{64}$$

for $\beta = 0{\cdot}7$ and $u^* = 0{\cdot}5$. At $u^* = 0$ (dot line), energy flux into the pipe is positive for $d^* < 0$ and negative for $d^* > 0$. However the case of $u^* = 0{\cdot}5$ (solid line) shows that the pipe loses energy at negative damping ($d^* < d_1$). In this case, energy transferred out of the pipe is larger than the energy increase by the negative damping. Accordingly, unstable motions in a pipe with free–fixed boundary conditions can be avoided by attaching a viscous damper whose damping coefficient satisfies $d^* < d_1$ or $d^* > d_2$.

7. CONCLUSIONS

(1) The energy transferred into the translating continua, due to the conservative or non conservative forces at the various boundary supports is determined completely by the energy reflection coefficient that depends on the mechanical impedance and the ratio of the amplitudes of the reflected to incident waves.

(2) At a fixed boundary, the boundary force (tension) leads to energy flux into a translating string or tensioned pipe. The energy flux into the second-order continuum over one cycle of a travelling wave is always positive at a fixed downstream boundary and negative at an upstream one. The Coriolis force at a free end, resulting from mass transport through the boundary, causes energy flux into the continuum. The sign of the energy flux is opposite to the case at the fixed boundary.

(3) The total energy of free oscillations in the string or tensioned pipe system with a symmetric boundary configuration (fixed–fixed and free–free) varies periodically at the fundamental frequency.

(4) Asymmetric boundary configurations lead to damped (fixed–free) or self-excited (free–fixed) motions in free vibration of the tensioned pipe.

(5) A boundary viscous damper, whose damping coefficient is tuned to the mechanical impedance of the translating string or tensioned pipe completely dissipates all the wave energy incident on the boundary.

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