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CONTROL OF SLEW MANEUVER OF A FLEXIBLE BEAM MOUNTED NON-RADIALLY ON A RIGID HUB: A GEOMETRICALLY EXACT MODELLING APPROACH

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The positioning control using a joint torque is studied for a hub–beam system with a tip payload. In order to make the axial vibration of the beam accessible to the joint torque, the flexible beam mounted non-radially on the rigid hub is considered. Neither model truncation nor small deflection assumption is employed throughout the process of dynamic modelling and controller design. A joint PD controller with additional feedback of joint acceleration and root strain of the beam is derived using a Lyapunov-type method. Global asymptotic stability of the desired equilibrium position is proved rigorously. The geometrically exact formulation presented in this work can be viewed as a generalization of the related work based on the small deflection assumption investigated in the existing literature.

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1. INTRODUCTION

The control of a slewing elastic beam has been extensively studied in connection with flexible robot arms and spacecrafts with flexible appendages. A vast majority of these studies have designed controllers based on truncated finite-dimensional models (see, e.g., references [1–10]). Only in a few related papers have there been construction of controllers making direct use of infinite-dimensional models (see, e.g., references [11–19]). Linear and non-linear dynamic models with varying degrees of complexity have been considered in these papers. However, the small-deflection assumption has been invoked in all these studies, using either truncated finite-dimensional models or linear infinite-dimensional models.

For the problem of a rapidly driven flexible beam, the occurrence of large elastic deflections is inevitable. Indeed, it has been observed experimentally [20] that the tip deflection can exceed one-tenth of the length of the beam when large rotational speed and acceleration are involved. Numerical simulation of a hub–beam system employing a PD joint control has also demonstrated that large tip deflections of the beam can occur even for a realistically fast positioning operation [21]. The rest-to-rest maneuver of a horizontally slewing torque-driven beam (without a tip payload) undergoing geometrically exact elastic deflections has recently been studied in reference [22]. It was found that a joint-based PD type controller stabilizes the highly non-linear coupled system described by a set of integro-partial differential equations with coupled non-linear boundary conditions. However, the proof of global asymptotic stability requires the assumption of inextensibility at the root of the beam. This difficulty arises because the flexible beam was

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assumed to be mounted radially on a rigid hub [22]. Thus if the axial vibration of the beam is excited, it cannot be suppressed by the joint torque. To overcome this difficulty, it is suggested in the present work that the flexible beam be mounted non-radially on the rigid hub.

In this paper, the rest-to-rest horizontal maneuver of a torque-driven hub–beam system with a tip payload is considered. The objective of this investigation is to show that a joint-based PD controller with additional feedback of either the joint angular acceleration of the hub or the root strain of the beam can accomplish the maneuver with all the vibrations suppressed asymptotically. In deriving the feedback law, neither model truncation nor the small deflection assumption is imposed. Although feedback laws of similar type have previously been established by many investigators (see e.g., references [2, 3, 5, 8, 13, 15, 16, 18, 19]) using either truncated finite-dimensional models or linear distributed parameter models, the issues of truncated error and ignored dynamics have never been resolved. The primary contribution of this paper is thus to provide further theoretical understanding to earlier work in this area.

2. EQUATIONS OF MOTION

The horizontal slewing hub-beam system with a tip payload driven by an externally applied torque $\tau(t)$ is considered, as shown in Figure 1. The undeformed elastic beam of length L, area moment of inertia I, cross-sectional area A, mass per unit length A_ρ, mass moment of inertia per unit length I_ρ, shear coefficient k_s, shear modulus G, and Young's modulus E is mounted non-radially on the rigid hub of radius a and mass moment of inertia I_h. The rigid payload has a mass m_p with an inertia I_p with respect to its own center of mass Q'. The point Q' is specified by the vector $q_1\mathbf{n} + q_2\mathbf{t}$ relative to the centroid of the tip cross-section of the beam, where q_1 and q_2 are linear dimensions, and $\mathbf{n}$ and $\mathbf{t}$ denote the unit vectors normal and tangential to the tip cross-section, respectively. It is

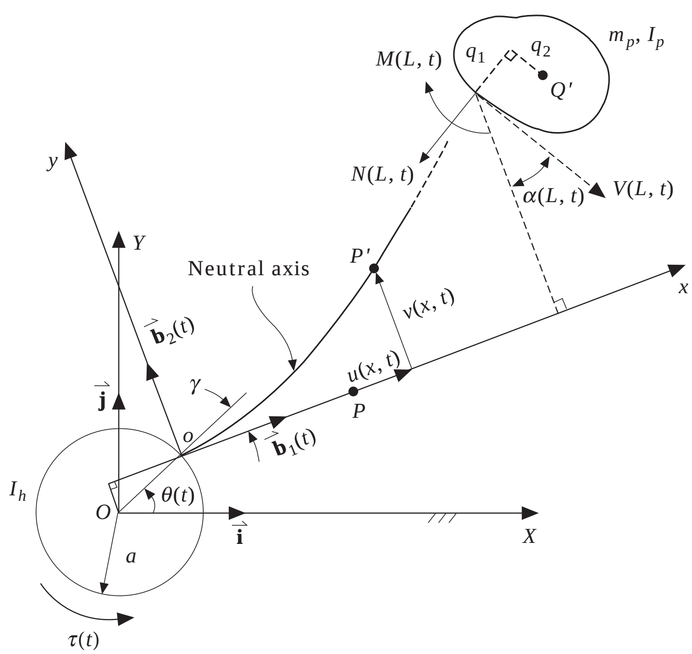

Figure 1. A schematic of a horizontally slewing flexible beam.

assumed that the beam possesses a line of symmetry in the plane of bending, plane sections orthogonal to the neutral axis remain plane during deformation, and that the line of centroid coincides with the neutral axis.

Let $o(x, y)$ be a floating frame fixed on the hub with x-axis coincident with the neutral axis of the undeformed beam and let $O(X, Y)$ be a fixed frame. The basis vectors of the two co-ordinate frames are related by

$$\mathbf{b}_1 = \mathbf{i} \cos(\theta - \gamma) + \mathbf{j} \sin(\theta - \gamma), \tag{1}$$

$$\mathbf{b}_2 = -\mathbf{i} \sin(\theta - \gamma) + \mathbf{j} \cos(\theta - \gamma), \tag{2}$$

where θ is the angular displacement of the hub and γ is a fixed inclined angle of the undeformed neutral axis of the beam (see Figure 1). It is evident from equations (1) and (2) that $\dot{\mathbf{b}}_1 = \dot{\theta}\mathbf{b}_2$ and $\dot{\mathbf{b}}_2 = -\theta\dot{\mathbf{b}}_1$. The position vector of an arbitrary point P' on the deformed neutral axis is given by

$$\mathbf{R}_{P'} = (a \cos\gamma + x + u)\mathbf{b}_1 + (a \sin\gamma + v)\mathbf{b}_2, \tag{3}$$

where $u(x, t)$ and $v(x, t)$ are the axial and transverse displacements of the point P on the undeformed neutral axis of the beam. The velocity and acceleration of the point P' can be written, respectively, as

$$\dot{\mathbf{R}}_{P'} = [\dot{u} - (a \sin\gamma + v)\dot{\theta}]\mathbf{b}_1 + [\dot{v} + (a \cos\gamma + x + u)\dot{\theta}]\mathbf{b}_2, \tag{4}$$

$$\begin{aligned}\ddot{\mathbf{R}}_{P'} = &[\ddot{u} - (a \sin\gamma + v)\ddot{\theta} - 2\dot{v}\dot{\theta} - (a \cos\gamma + x + u)\dot{\theta}^2]\mathbf{b}_1 \\ &+ [\ddot{v} + (a \cos\gamma + x + u)\ddot{\theta} + 2\dot{u}\dot{\theta} - (a \sin\gamma + v)\dot{\theta}^2]\mathbf{b}_2.\end{aligned} \tag{5}$$

Let $\alpha(x, t)$ be the angle of rotation of a beam cross-section from its undeformed configuration. Define $u_L = u(L, t)$, $v_L = v(L, t)$, and $\alpha_L = \alpha(L, t)$. The position vector, velocity and acceleration of the point Q' of the tip-payload can be expressed, respectively, as

$$\begin{aligned}\mathbf{R}_{Q'} = &(a \cos\gamma + L + u_L + q_1 \cos\alpha_L + q_2 \sin\alpha_L)\mathbf{b}_1 \\ &+ (\mathrm{a} \sin\gamma + v_L + q_1 \sin\alpha_L - q_2 \cos\alpha_L)\mathbf{b}_2,\end{aligned} \tag{6}$$

$$\begin{aligned}\dot{\mathbf{R}}_{Q'} = &[(\dot{u}_L - q_1\dot{\alpha}_L \sin\alpha_L + q_2\dot{\alpha}_L \cos\alpha_L) \\ &- (a \sin\gamma + v_L + q_1 \sin\alpha_L - q_2 \cos\alpha_L)\dot{\theta}]\mathbf{b}_1 \\ &+ [(\dot{v}_L + q_1\dot{\alpha}_L \cos\alpha_L + q_2\dot{\alpha}_L \sin\alpha_L) \\ &+ (a \cos\gamma + L + u_L + q_1 \cos\alpha_L + q_2 \sin\alpha_L)\dot{\theta}]\mathbf{b}_2,\end{aligned} \tag{7}$$

$$\ddot{\mathbf{R}}_{Q'} = \pi_1\mathbf{b}_1 + \pi_2\mathbf{b}_2, \tag{8}$$

where

$$\begin{aligned}\pi_1 = &\ddot{u}_L - q_1(\dot{\alpha}_L^2 \cos\alpha_L + \ddot{\alpha}_L \sin\alpha_L) + q_2(\ddot{\alpha}_L \cos\alpha_L - \dot{\alpha}_L^2 \sin\alpha_L) \\ &- (a \sin\gamma + v_L + q_1 \sin\alpha_L - q_2 \cos\alpha_L)\ddot{\theta} \\ &- 2(\dot{v}_L + q_1\dot{\alpha}_L \cos\alpha_L + q_2\dot{\alpha}_L \sin\alpha_L)\dot{\theta} \\ &- (a \cos\gamma + L + u_L + q_1 \cos\alpha_L + q_2 \sin\alpha_L)\dot{\theta}^2,\end{aligned} \tag{9}$$

$$\pi_2 = \ddot{v}_L + q_1(\ddot{\alpha}_L \cos\alpha_L - \dot{\alpha}_L^2 \sin\alpha_L) + q_2(\ddot{\alpha}_L \sin\alpha_L + \dot{\alpha}_L^2 \cos\alpha_L)$$
$$+ (a\cos\gamma + L + u_L + q_1\cos\alpha_L + q_2\sin\alpha_L)\ddot{\theta}$$
$$+ 2(\dot{u}_L - q_1\dot{\alpha}_L \sin\alpha_L + q_2\dot{\alpha}_L\cos\alpha_L)\dot{\theta}$$
$$- (a\sin\gamma + v_L + q_1\sin\alpha_L - q_2\cos\alpha_L)\dot{\theta}^2. \quad (10)$$

The moment of inertia forces of the elastic beam and the tip payload relative to the point O can be computed as

$$M_0(t) = -\int_0^L A_\rho(\mathbf{R}_{P'} \times \ddot{\mathbf{R}}_{P'}) \cdot \mathbf{b}_3 \, \mathrm{d}x - \int_0^L I_\rho(\ddot{\theta} + \ddot{\alpha}) \, \mathrm{d}x$$
$$- m_p(\mathbf{R}_{Q'} \times \ddot{\mathbf{R}}_{Q'}) \cdot \mathbf{b}_3 - I_p(\ddot{\theta} + \ddot{\alpha}_L), \quad (11)$$

where $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2$ is used and $a \ll L$ is assumed. The rotational equation of motion of the hub-beam system can now be written as

$$\tau + M_0 = I_h\ddot{\theta}. \quad (12)$$

After substituting equations (3), (5), (6), (8) and (11) into equation (12), one obtains

$$\left\{I_h + \int_0^L A_\rho[(a\cos\gamma + x + u)^2 + (a\sin\gamma + v)^2] \, \mathrm{d}x\right\}\ddot{\theta}$$
$$+ 2\left\{\int_0^L A_\rho[(a\cos\gamma + x + u)\dot{u} + (a\sin\gamma + v)\dot{v}] \, \mathrm{d}x\right\}\dot{\theta}$$
$$+ \int_0^L A_\rho[(a\cos\gamma + x + u)\ddot{v} - (a\sin\gamma + v)\ddot{u}] \, \mathrm{d}x$$
$$+ \int_0^L I_\rho(\ddot{\theta} + \ddot{\alpha}) \, \mathrm{d}x + m_p[-(a\sin\gamma + v_L + q_1\sin\alpha_L - q_2\cos\alpha_L)\pi_1$$
$$+ (a\cos\gamma + L + u_L + q_1\cos\alpha_L + q_2\sin\alpha_L)\pi_2] + I_p(\ddot{\theta} + \alpha_L) = \tau(t). \quad (13)$$

In order to establish the equations of motion for the elastic deflections, a differential element of the beam is considered, as shown in Figure 2. One can easily show from $|d\mathbf{R}_{P'}| = \mathrm{d}l$ that

$$(\mathrm{d}l)^2 = [(1 + u_x)^2 + v_x^2](\mathrm{d}x)^2, \quad (14)$$

$$\sin\beta = v_x[(1 + u_x)^2 + v_x^2]^{-1/2}, \quad (15)$$

$$\cos\beta = (1 + u_x)[(1 + u_x)^2 + v_x^2]^{-1/2}, \quad (16)$$

where $\beta(x, t)$ is the angle between the tangents of deformed and undeformed neutral axes and the subscript x denotes a partial derivative with respect to x. Let the longitudinal force, the shear force and the bending moment acting on the element be denoted by N, V and

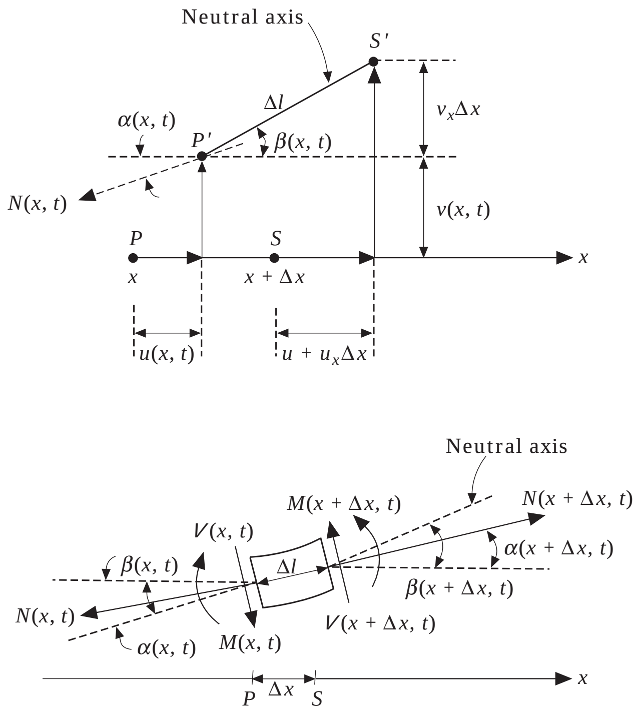

Figure 2. A deformed beam element.

M, respectively. Summing the forces in the x and y directions and the moments of forces acting on the element, one obtains

$$A_\rho[\ddot{u} - (a \sin \gamma + v)\ddot{\theta} - 2v\dot{\theta} - (a \cos \gamma + x + u)\dot{\theta}^2] = (N \cos \alpha - V \sin \alpha)_x, \quad (17)$$

$$A_\rho[\ddot{v} + (a \cos \gamma + x + u)\ddot{\theta} + 2\dot{u}\dot{\theta} - (a \sin \gamma + v)\dot{\theta}^2] = (N \sin \alpha + V \cos \alpha)_x, \quad (18)$$

$$I_\rho(\ddot{\theta} + \ddot{\alpha}) = M_x - v_x(N \cos \alpha - V \sin \alpha) + (1 + u_x)(N \sin \alpha + V \cos \alpha), \quad (19)$$

where equation (5) has been used in arriving at equations (17) and (18). Note that equations (17)–(19) reduce to those given in [23] for the case $\gamma = 0$, wherein the vibration of the beam caused by a prescribed $\dot{\theta}(t)$ (the so-called spin-up problem) was studied. Although equation (13) is not relevant to the spin-up beam dynamics, it is essential for torque-driven beam dynamics. Now the clamped-mass boundary conditions of the beam can be written as

$$u(0, t) = v(0, t) = \alpha(0, t) = 0, \quad (20)$$

$$N(L, t) = -m_p \ddot{\mathbf{R}}_{Q'} \cdot \mathbf{n} = -m_p(\pi_1 \cos \alpha_L + \pi_2 \sin \alpha_L), \quad (21)$$

$$V(L, t) = -m_p \ddot{\mathbf{R}}_{Q'} \cdot \mathbf{t} = m_p(\pi_1 \sin \alpha_L - \pi_2 \cos \alpha_L), \quad (22)$$

$$M(L, t) = q_1 V(L, t) + q_2 N(L, t) - I_p(\ddot{\theta} + \ddot{\alpha}_L)$$

$$= m_p[\pi_1(q_1 \sin \alpha_L - q_2 \cos \alpha_L) - \pi_2(q_1 \cos \alpha_L + q_2 \sin \alpha_L)] - I_p(\ddot{\theta} + \ddot{\alpha}_L). \quad (23)$$

Substituting $\ddot{u}$, $\ddot{v}$ and $\ddot{\alpha}$ from equations (17)–(19) into equation (13), performing integration by parts and using the boundary conditions (20)–(23), one obtains an alternative but more appealing form for equation (13):

$$I_h\ddot{\theta} - V(0, t)a\cos\gamma + N(0, t)a\sin\gamma - M(0, t) = \tau(t). \tag{24}$$

At this point, it may be worthwhile to note the following:

(i) When $\gamma = 0$, $N(0, t)$ is not accessible to the input torque $\tau(t)$. This implies that the longitudinal vibration $u(x, t)$ of the beam, if it is ever being excited, will persist even if $\dot{\theta}(t)$, $v(x, t)$ and $\alpha(x, t)$ are driven to zero by $\tau(t)$. This is true because no internal damping of the beam is assumed in this work.
(ii) The equations derived thus far depend neither on the strain (small or large) nor on the constitutive equation (linear or non-linear) of the beam.

To complete the formulation, the stress–strain relations for the homogeneous, isotropic linearly elastic beam can be taken as

$$N = EA\Gamma_1, \qquad V = k_sGA\Gamma_2, \qquad M = EI\kappa, \tag{25–27}$$

where the normal strain Γ_1, the shear strain Γ_2, and the bending curvature κ are described by

$$\Gamma_1 = (1 + u_x)\cos\alpha + v_x\sin\alpha - 1, \qquad \Gamma_2 = -(1 + u_x)\sin\alpha + v_x\cos\alpha, \qquad \kappa = \alpha_x, \tag{28–30}$$

as a special case of the fully nonlinear three-dimensional strain measures given in [24]. Without prescribing equations (28)–(30) as *a priori*, a new derivation of these equations is given in Appendix A.

The principle of conservation of energy can now be verified for the hub–beam system as follows. Let the kinetic energy T and the potential energy U be written as

$$T = T_h + T_b + T_p, \tag{31}$$

where

$$T_h = \tfrac{1}{2}I_h\dot{\theta}^2, \tag{32}$$

$$T_b = \tfrac{1}{2}\int_0^L A_p|\dot{\mathbf{R}}_{P'}|^2\,\mathrm{d}x + \tfrac{1}{2}\int_0^L I_p(\dot{\theta} + \dot{\alpha})^2\,\mathrm{d}x, \tag{33}$$

$$T_p = \tfrac{1}{2}m_p|\dot{\mathbf{R}}_{Q'}|^2 + \tfrac{1}{2}I_p(\dot{\theta} + \dot{\alpha}_L)^2 \tag{34}$$

and

$$U = \tfrac{1}{2}\int_0^L (EA\Gamma_1^2 + k_sGA\Gamma_2^2 + EI\kappa^2)\,\mathrm{d}x. \tag{35}$$

The time rate change of the total energy ($\varepsilon = T + U$) of the hub–beam system can be computed by substituting equations (4), (7) and (25)–(30) into equations (31) and (35), making use of equations (13) and (17)–(23), integration by parts and tedious algebraic manipulations to yield

$$\dot{\varepsilon} = \tau\dot{\theta}. \tag{36}$$

It may appear that equations (24) and (36) can be obtained merely by a physical argument. This is indeed true for the foregoing geometrically exact formulation. However, an inadequate approximation or a premature linearization of the exact equations of motion can destroy either equation (24) or equation (36) or both. For example, the second order beam theory given in reference [23], which provides an appropriate account for the centrifugal stiffening effect for the beam dynamics driven by a prescribed $\dot{\theta}(t)$, fails to satisfy equation (24) for the torque-driven beam dynamics. This is illustrated in Appendix B. Also as illustrated in Appendix C, the effective applied force approach [25] to account for the centrifugal stiffening effect does not satisfy equation (36) for the torque-driven beam dynamics. The significance of using equations (24) and (36) in establishing the stability of the torque-driven hub–beam system will be demonstrated in the next section.

3. CONTROLLER DESIGN AND STABILITY ANALYSIS

Consider the system given by equations (13), (17)–(23) and (25)–(30). It can be shown, with some algebra, that the equilibrium states of the system without input torque are given by $\dot{\theta}(t) = u(x, t) = v(x, t) = \alpha(x, t) = 0$ for $0 \leqslant x \leqslant L$ and $t \geqslant t_0$, where t_0 is a real non-negative constant. The problem is to find an appropriate control τ that drives the state of the system $(\theta, \dot{\theta}, u, \dot{u}, v, \dot{v}, \alpha, \dot{\alpha})$ from the initial state $(0, 0, \dots 0)$ to the target state $(\theta_d, 0, \dots, 0)$, where θ_d is a constant.

Let a weighted error function with respect to the target state be the Lyapunov functional candidate

$$\phi = \varepsilon + K_a(T_b + T_p + U) + \tfrac{1}{2}K_p(\theta - \theta_d)^2, \tag{37}$$

where $\varepsilon = T_h + T_b + T_p + U$, and the real constants $K_a > -1$ and $K_p > 0$ are design parameters. It is obvious that ϕ is positive definite and has a global minimum zero at the target state. Using equation (36), the time rate change of ϕ can be computed as

$$\dot{\phi} = \dot{\theta}[\tau + K_a(\tau - I_h\ddot{\theta}) + K_p(\theta - \theta_d)]. \tag{38}$$

If τ is chosen such that

$$\tau = \frac{1}{1 + K_a}[K_a I_h\ddot{\theta} - K_d\dot{\theta} - K_p(\theta - \theta_d)], \tag{39}$$

where the real constant $K_d > 0$ is also a design constant, then the target state becomes the unique equilibrium point of the closed loop system, and equation (38) reduces to

$$\dot{\phi} = -K_d\dot{\theta}^2. \tag{40}$$

Note that $\dot{\phi}$ is negative semi-definite. Thus only stability (but not asymptotic stability) of the target state can be concluded. However, equation (40) implies that $\dot{\phi}(t) = 0$ if and only if $\dot{\theta}(t) = 0$. Thus, if one can show that $\dot{\theta}(t) = 0$ for $t \geqslant t_0$ implies $\phi(t) = 0$ for $t \geqslant t_0$, then global asymptotic stability of the target state is established. Toward this end, substituting equation (39) into equation (24) gives the closed loop dynamics for the rigid hub:

$$I_h\ddot{\theta} + K_d\dot{\theta} + K_p\theta = (1 + K_a)[V(0, t)a\cos\gamma - N(0, t)a\sin\gamma + M(0, t)] + K_p\theta_d. \tag{41}$$

Now suppose that $\dot{\theta}(t) = 0$ for $t \geqslant t_0$. Thus $\ddot{\theta}(t) = 0$ for $t \geqslant t_0$. In view of equation (41), one obtains, for $t \geqslant t_0$,

$$V(0, t)a\cos\gamma - N(0, t)a\sin\gamma + M(0, t) = C, \tag{42}$$

where C is a constant. Furthermore, equations (13) and (24) can be combined to yield

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^L A_\rho[(a\cos\gamma + x + u)\dot{v} - (a\sin\gamma + v)\dot{u}]\,\mathrm{d}x\right.\\
&\left.+\int_0^L I_\rho\dot{\alpha}\,\mathrm{d}x + m_p f_L + I_p\dot{\alpha}_L\right\},\\
&= V(0,t)a\cos\gamma - N(0,t)a\sin\gamma + M(0,t)\\
&= C,
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
f_L &= (a\cos\gamma + L + u_L + q_1\cos\alpha_L + q_2\sin\alpha_L)(\dot{v}_L + q_1\dot{\alpha}_L\cos\alpha_L + q_2\dot{\alpha}_L\sin\alpha_L)\\
&\quad - (a\sin\gamma + v_L + q_1\sin\alpha_L - q_2\cos\alpha_L)(\dot{u}_L - q_1\dot{\alpha}_L\sin\alpha_L + q_2\dot{\alpha}_L\cos\alpha_L).
\end{aligned}
\tag{44}
$$

Because $\phi(t)$ is non-increasing due to equation (40), every term in equation (43) must be bounded. Consequently, one must have $C = 0$ for $t \geqslant t_0$. It then follows from equations (41) and (24) that $\theta(t) = \theta_d$ and $\tau(t) = 0$ for $t \geqslant t_0$. Since the remaining free vibration problem (with the fixed hub) is independent of the hub radius a and the inclined angle γ of the beam, $V(0,t)a\cos\gamma - N(0,t)a\sin\gamma + M(0,t) = 0$ for $t \geqslant t_0$ implies $V(0,t) = N(0,t) = M(0,t) = 0$ for $t \geqslant t_0$. It then leads to $u_x(0,t) = v_x(0,t) = \alpha_x(0,t) = 0$ for $t \geqslant t_0$ by virtue of equations (28)–(30) and (20). From equations (17)–(19) and (25)–(30), one further obtains $u_{xx}(0,t) = v_{xx}(0,t) = \alpha_{xx}(0,t) = 0$ for $t \geqslant t_0$. Now it is straightforward to show, by taking repeated x-differentiation of equations (17)–(19), that all higher order partial derivatives of u, v and α with respect to x are zero at $x = 0$. As a result, it must be true that $u(x,t) = v(x,t) = \alpha(x,t) = 0$ for $0 \leqslant x \leqslant L$, $t \geqslant t_0$. At this point, we have shown that $\dot{\theta}(t) = 0$ for $t \geqslant t_0$ implies $\phi(t) = 0$ for $t \geqslant t_0$. Note that the foregoing results were reached without specifying t_0. Referring now to equation (41), it is clear that $\dot{\theta}(t)$ approaches zero asymptotically as $t \to \infty$. Therefore, $\phi(t)$ and $\dot{\phi}(t) = -K_d\dot{\theta}^2$ tend to zero as $t \to \infty$. Global convergence follows from equation (37). The proof of global asymptotic stability is completed.

Note that the control law given by equation (39) involves the feedback of angular position, angular velocity and angular acceleration of the joint. However, by the substitution of $I_h\ddot{\theta}$ from equation (24) into equation (39), one obtains the PD plus strain feedback controller as

$$
\tau = K_a[V(0,t)a\cos\gamma - N(0,t)a\sin\gamma + M(0,t)] - K_d\dot{\theta} - K_p(\theta - \theta_d).
\tag{45}
$$

An important observation can be made at this point: if a simplified dynamic model were adopted at the very begining, the stability result would remain unaffected so long as the validity of equations (24) and (36) is preserved. It may be remarked that the proof of asymptotic stability even for a non-linear distributed parameter model, based on the small deflection assumption, of the hub–beam system is not yet available in the existing literature.

4. CONCLUDING REMARKS

The positioning control of a hub–beam system with a tip payload was considered, for the first time, under the framework of geometrically exact formulation. The non-radially mounted flexible beam makes the axial vibration of the beam controllable to the joint torque. The result presented is a more general theory as compared with the related work relying on the small deflection assumption existing in the current literature. An important implication of this study is that the stability of the joint-based controller designed based on truncated finite-dimensional models or simplified linear distributed parameter models is guaranteed as long as the model simplifications of the open loop system does not destroy the conservation equations for the angular momentum and energy. Although the stability is guaranteed, the performance (transient responses, etc.) may degrade due to truncation error and ignored dynamics. Since the methods of numerical simulation for a slewing beam using the geometrically exact beam model are well-documented in the literature, they are not pursued in this work.

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APPENDIX A: A DERIVATION OF THE NON-LINEAR STRAIN MEASURES

Instead of using Newton's second law, the equations of motion can also be derived via the extended Hamilton's principle. Without loss of generality, the hub–beam system without a tip payload is considered. The strains, which depend on the deformation gradient, can be written as

$$\Gamma_1 = \Gamma_1(u_x, v_x, \alpha), \qquad \Gamma_2 = \Gamma_2(u_x, v_x, \alpha), \qquad \kappa = \alpha_x, \tag{A1–A3}$$

where the dependence of α in equations (A1) and (A2) is due to equations (17) and (18), and equation (A3) is a direct consequence of the definition of bending curvature. As a result of the application of Hamilton's principle with the kinetic and potential energies given by equations (31)–(35) and the stress–strain relations given by equations (25)–(27), the equations of motion related to the deformation variables are given by

$$A_\rho[\ddot{u} - (a \sin\gamma + v)\ddot{\theta} - 2\dot{v}\dot{\theta} - (a\cos\gamma + x + u)\dot{\theta}^2] = \left(N\frac{\partial \Gamma_1}{\partial u_x} + V\frac{\partial \Gamma_2}{\partial u_x}\right)_x, \tag{A4}$$

$$A_\rho[\ddot{v} + (a\cos\gamma + x + u)\ddot{\theta} - 2\dot{u}\dot{\theta} - (a\sin\gamma + v)\dot{\theta}^2] = \left(N\frac{\partial \Gamma_1}{\partial v_x} + V\frac{\partial \Gamma_2}{\partial v_x}\right)_x, \tag{A5}$$

$$I_\rho(\ddot{\theta} + \ddot{\alpha}) = M_x - \left(N\frac{\partial \Gamma_1}{\partial \alpha} + V\frac{\partial \Gamma_2}{\partial \alpha}\right). \tag{A6}$$

After comparing equations (A4)–(A6) with equations (17)–(19) and making use of clamped–free boundary conditions, one can easily obtain

$$N\left(\frac{\partial \Gamma_1}{\partial u_x} - \cos\alpha\right) + V\left(\frac{\partial \Gamma_2}{\partial u_x} + \sin\alpha\right) = 0, \tag{A7}$$

$$N\left(\frac{\partial \Gamma_1}{\partial v_x} - \sin\alpha\right) + V\left(\frac{\partial \Gamma_2}{\partial v_x} - \cos\alpha\right) = 0, \tag{A8}$$

$$N\left[\frac{\partial \Gamma_1}{\partial \alpha} + (1+u_x)\sin\alpha - v_x\cos\alpha\right] + V\left[\frac{\partial \Gamma_2}{\partial \alpha} + (1+u_x)\cos\alpha + v_x\sin\alpha\right] = 0. \tag{A9}$$

Equations (A7)–(A9) are satisfied if Γ_1 and Γ_2 are solutions of

$$\frac{\partial \Gamma_1}{\partial u_x} = \cos\alpha, \qquad \frac{\partial \Gamma_1}{\partial v_x} = \sin\alpha, \qquad \frac{\partial \Gamma_1}{\partial \alpha} = v_x\cos\alpha - (1+u_x)\sin\alpha. \tag{A10}$$

$$\frac{\partial \Gamma_2}{\partial u_x} = -\sin\alpha, \qquad \frac{\partial \Gamma_2}{\partial v_x} = \cos\alpha, \qquad \frac{\partial \Gamma_2}{\partial \alpha} = -(1+u_x)\cos\alpha - v_x\sin\alpha. \tag{A11}$$

Hence equations (28) and (29) can be obtained provided that $\Gamma_1(0, t) = u_x(0, t)$ and $\Gamma_2(0, t) = v_x(0, t)$.

APPENDIX B: THE INCONSISTENCY OF THE SECOND ORDER BEAM THEORY FOR TORQUE-DRIVEN BEAM DYNAMICS

For simplicity, the hub–beam system without a payload is considered. According to the second order beam theory given in reference [23], one may write

$$\bar{N} = EA\bar{\Gamma}_1 = EA(u_x + \alpha v_x - \tfrac{1}{2}\alpha^2), \tag{B1}$$

$$\bar{V} = k_s GA\bar{\Gamma}_2 = k_s GA(v_x - \alpha - \alpha u_x), \tag{B2}$$

$$\bar{M} = EI\alpha_x. \tag{B3}$$

The equations of motion for the deformation variables are given by [23]:

$$A_\rho[\ddot{u} - (a\sin\gamma + v)\ddot{\theta} - 2\dot{v}\dot{\theta} - (a\cos\gamma + x + u)\dot{\theta}^2] = (\bar{N} - \alpha\bar{V})_x, \tag{B4}$$

$$A_\rho[\ddot{v} + (a\cos\gamma + x + u)\ddot{\theta} + 2\dot{u}\dot{\theta} - (a\sin\gamma + v)\dot{\theta}^2] = (\bar{V} + \alpha\bar{N})_x, \tag{B5}$$

$$I_\rho(\ddot{\theta} + \ddot{\alpha}) = \bar{M}_x + (1 + \bar{\Gamma}_1)\bar{V} - \bar{\Gamma}_2\bar{N}. \tag{B6}$$

After substituting $\ddot{u}$, $\ddot{v}$ and $\ddot{\alpha}$ from equations (B4)–(B6) into equation (13), one obtains

$$I_h\ddot{\theta} - \bar{V}(0, t)a\cos\gamma + \bar{N}(0, t)a\sin\gamma - \bar{M}(0, t) = \tau(t) + \tfrac{1}{2}\int_0^L \alpha^2\bar{V}\,\mathrm{d}x, \tag{B7}$$

which is not consistent with equation (24). One might be tempted to neglect all terms of third degree in the elastic variables in equations (B4)–(B6), so that the last term in equation (B7) can be dropped. However, accompanying this procedure is the omission of some strain energy terms computed from equations (B1)–(B2). As a result, the principle of conservation of energy is destroyed. Hence the second order beam theory is not an adequate approximation for torque-driven beam dynamics from an angular momentum viewpoint.

APPENDIX C: THE INCONSISTENCY OF THE EFFECTIVE APPLIED FORCE APPROACH FOR TORQUE-DRIVEN BEAM DYNAMICS

According to reference [25], the kinetic and potential energies of the hub–beam system with $\gamma = m_p = I_p = I_\rho = 0$ can be respectively, written as

$$T = \tfrac{1}{2} I_h \dot{\theta}^2 + \tfrac{1}{2} \int_0^L A_\rho \{[(a+x)\dot{\theta} + \dot{v}]^2 + (v\dot{\theta})^2\} \, \mathrm{d}x, \tag{C1}$$

$$U = \tfrac{1}{2} \int_0^L EI v_{xx}^2 \, \mathrm{d}x + \tfrac{1}{2} \dot{\theta}^2 \int_0^L A_\rho S v_x^2 \, \mathrm{d}x, \tag{C2}$$

where

$$S = \int_x^L (a+x) \, \mathrm{d}x. \tag{C3}$$

Note that a fictitious work term caused by the centrifugal force has been included in U in an *ad hoc* manner [26]. As a result of the application of Hamilton's principle, one obtains the following equations of motion [25]:

$$\left\{ I_h + \int_0^L A_\rho [(a+x)^2 + v^2 - S v_x^2] \, \mathrm{d}x \right\} \ddot{\theta} + 2\dot{\theta} \int_0^L A_\rho (v\dot{v} - S v_x \dot{v}_x) \, \mathrm{d}x + \int_0^L A_\rho (a+x) \ddot{v} \, \mathrm{d}x = \tau, \tag{C4}$$

$$A_\rho \{\ddot{v} + (a+x)\ddot{\theta} - \dot{\theta}^2 [v + (S v_x)_x]\} + EI v_{xxxx} = 0. \tag{C5}$$

Using equations (C1)–(C5), it can be easily shown that

$$\frac{\mathrm{d}}{\mathrm{d}t}(T+U) = \tau\dot{\theta} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\theta}^2 \int_0^L A_\rho S v_x^2 \, \mathrm{d}x \right). \tag{C6}$$

This result is not consistent with equation (36). The reason for this flaw is that the centrifugal force is not an externally applied force but, rather, an inertia effect. Hence the effective applied force approach is not adequate for torque-driven beam dynamics from an energy viewpoint.