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SUPPRESSION OF VIBRATION IN STRETCHED STRINGS BY THE BOUNDARY CONTROL

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1. INTRODUCTION

In this note, an elastic string of unit length and unit mass density, represented by the following non-linear partial differential equation, is considered

$$y_{tt}(x, t) = [1 + \tfrac{3}{2}by_x^2(x, t)]y_{xx}(x, t), \tag{1a}$$

for all $x\in(0, 1)$ and $t \geqslant 0$, with the boundary conditions

$$y(0, t) = 0, \qquad T(1, t)y_x(1, t) = u(t), \tag{1b, c}$$

for all $t \geqslant 0$, and the initial conditions

$$y(x, 0) = f(x), \qquad y_t(x, 0) = g(x), \tag{1d}$$

for all $x\in(0, 1)$. In equations (1), $y(\cdot, \cdot)\in\mathbb{R}$ denotes the transversal displacement of the string, $T(1, t) > 0$ denotes the tension in the string at $x = 1$ for all $t \geqslant 0$, $u(\cdot)\in\mathbb{R}$ is a control input, $y_x:=\partial y/\partial x$, $y_{xx}:=\partial^2 y/\partial x^2$, $y_t:=\partial y/\partial t$, $y_{tt}:=\partial^2 y/\partial t^2$, and $b > 0$ is a constant real number.

There are several non-linear mathematical models that describe the transversal vibration of stretched strings. One such model is presented in equation (1a). This model was derived in reference [1] and has been studied by researchers from the physical and mathematical points of view; see, e.g., references [2–6] and the references therein.

The boundary condition in equation (1b) implies that the string is fixed at $x = 0$. The boundary condition in equation (1c) represents the balance of the transversal component of the tension in the string and the control input u, which is applied transversally at $x = 1$. The tension in the string represented by equation (1a) is *not* constant and is given by

$$T(x, t) = 1 + \tfrac{1}{2}by_x^2(x, t), \tag{2}$$

for all $x\in[0, 1]$ and $t \geqslant 0$ (see references [1, 2]). Therefore, the boundary condition in equation (1c) can be written as

$$[1 + \tfrac{1}{2}by_x^2(1, t)]y_x(1, t) = u(t), \tag{3}$$

for all $t \geqslant 0$.

In equation (1d), the initial displacement and velocity of the string are, respectively, denoted by $f(x)$ and $g(x)$ for all $x\in(0, 1)$. One assumes that $f\in C^1[0, 1]$, and that at least one of the functions f or g is not identically zero over [0, 1].

The control input u in equation (3) is commonly known as the *boundary control*. In this note, the stabilization of the string in equaion (1a) by u is studied. More precisely, a u that results in $y(x, t)\to 0$ as $t\to\infty$ for all $x\in[0, 1]$, is studied. As a stabilizing control input, one proposes

$$u(t) = -ky_t(1, t), \tag{4}$$

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for all $t \geqslant 0$, where $k > 0$ is a constant real number. With this choice of u, the boundary control is the negative feedback of the transversal velocity of the string at $x = 1$, with the gain k. It is known that *linear* strings represented by equations (1)–(3), for which $b = 0$, can be stabilized by the control law in equation (4), see, e.g., references [7–12]. Roughly speaking, the boundary control in equation (4) provides a dissipative effect in linear strings, because it is of the form of negative velocity feedback. This is in accordance with the well known fact that the negative velocity feedback increases damping in most finite dimensional inertial systems, such as large flexible systems and robotic manipulators.

The authors' goal in this note is to show that the boundary control u in equation (4) stabilizes the non-linear string in equations (1)–(3), i.e., u results in $y(x, t) \to 0$ as $t \to \infty$ for all $x \in [0, 1]$. The approach taken in this note to show the stabilization of the string in equations (1)–(4) is similar to that in reference [13], where the stabilization of the Kirchhoff's non-linear string by the boundary control was presented.

2. STABILIZATION BY BOUNDARY CONTROL

The authors' plan to establish the stability of the non-linear string represented by equations (1)–(4) is as follows. One defines an energy like (Lyapunov) function of time for the string and denote it by $t \mapsto V(t)$. One shows that V tends to zero exponentially.

The scalar-valued function V is defined as

$$V(t) := E(t) + \gamma \int_0^1 x y_t(x, t) y_x(x, t) \, \mathrm{d}x, \tag{5}$$

for all $t \geqslant 0$, where $\gamma > 0$ is a constant real number,

$$E(t) := \tfrac{1}{2} \int_0^1 [y_t^2(x, t) + y_x^2(x, t)] \, \mathrm{d}x + \frac{b}{8} \int_0^1 y_x^4(x, t) \, \mathrm{d}x, \tag{6}$$

and $y(\cdot, \cdot)$ satisfies equations (1)–(4). From equations (5), (6), and (1d), one obtains

$$E(0) = \tfrac{1}{2} \int_0^1 [g^2(x) + f_x^2(x)] \, \mathrm{d}x + \frac{b}{8} \int_0^1 f_x^4(x) \, \mathrm{d}x, \tag{7a}$$

$$V(0) = E(0) + \gamma \int_0^1 x g(x) f_x(x) \, \mathrm{d}x, \tag{7b}$$

where $f_x(x) := \mathrm{d}f(x)/\mathrm{d}x$. Since at least one of the functions f or g is not identically zero over $[0, 1]$, one has $E(0) > 0$.

Now, a property of V is proved.

Lemma 2.1. Let γ in equation (5) satisfy

$$\gamma < 1. \tag{8}$$

Then, the function V satisfies

$$0 \leqslant K_1 E(t) \leqslant V(t) \leqslant K_2 E(t), \tag{9}$$

for all $t \geqslant 0$, where $K_1 > 0$ and $K_2 > 0$ are constant real numbers given by

$$K_1 = 1 - \gamma, \qquad K_2 = 1 + \gamma. \tag{10a, b}$$

Proof. For the integral term in equation (5), whose coefficient is γ, one has (the argument (x, t) of the functions is deleted)

$$\int_0^1 xy_t y_x \, \mathrm{d}x \leqslant \int_0^1 x|y_t||y_x| \, \mathrm{d}x \leqslant \tfrac{1}{2}\int_0^1 y_t^2 \, \mathrm{d}x + \tfrac{1}{2}\int_0^1 y_x^2 \, \mathrm{d}x \leqslant E(t), \tag{11}$$

for all $t \geqslant 0$. Similarly, one obtains

$$\int_0^1 xy_t y_x \, \mathrm{d}x \geqslant -E(t), \tag{12}$$

for all $t \geqslant 0$. Using equations (11) and (12) in equation (5), one obtains inequality (9). □

Remark. Let γ satisfy inequality (8). Then, by inequality (9) and the fact that $E(0) > 0$, it is concluded that $V(0) > 0$. □

Next, equation (4) is used in equation (3) and the boundary conditions are rewritten as

$$y(0, t) = 0, \qquad y_x(1, t) = -ky_t(1, t)/(1 + by_x^2(1, t)/2), \tag{13a, b}$$

for all $t \geqslant 0$. One now proves some identities for the functions satisfying equations (13).

Lemma 2.2. Let $y(\cdot, \cdot)$ satisfy the boundary conditions in equations (13). Then,

$$\int_0^1 (y_{xx}y_t + y_{xt}y_x) \, \mathrm{d}x = -\frac{ky_t^2(1, t)}{1 + by_x^2(1, t)/2}, \tag{14a}$$

$$\int_0^1 (3y_{xx}y_x^2 y_t + y_x^3 y_{xt}) \, \mathrm{d}x = -\frac{k^3 y_t^4(1, t)}{[1 + by_x^2(1, t)/2]^3}, \tag{14b}$$

$$\int_0^1 xy_{xt}y_t \, \mathrm{d}x = \tfrac{1}{2}y_t^2(1, t) - \tfrac{1}{2}\int_0^1 y_t^2 \, \mathrm{d}x, \tag{14c}$$

$$\int_0^1 xy_{xx}y_x \, \mathrm{d}x = \frac{k^2 y_t^2(1, t)}{2[1 + by_x^2(1, t)/2]^2} - \tfrac{1}{2}\int_0^1 y_x^2 \, \mathrm{d}x, \tag{14d}$$

$$\int_0^1 xy_{xx}y_x^3 \, \mathrm{d}x = \frac{k^4 y_t^4(1, t)}{4[1 + by_x^2(1, t)/2]^4} - \tfrac{1}{4}\int_0^1 y_x^4 \, \mathrm{d}x, \tag{14e}$$

for all $t \geqslant 0$.

Proof. From equation (13a), one has $y_t(0, t) = 0$ for all $t \geqslant 0$. Thus, one obtains

$$\int_0^1 (y_{xx}y_t + y_{xt}y_x) \, \mathrm{d}x = \int_0^1 (y_x y_t)_x \, \mathrm{d}x = y_x(1, t)y_t(1, t), \tag{15}$$

for all $t \geqslant 0$. Using equation (13b) in equation (15), one obtains equation (14a).

Having $y_t(0, t) = 0$ for all $t \geqslant 0$, one next obtains

$$\int_0^1 (3y_{xx}y_x^2 y_t + y_x^3 y_{xt})\,\mathrm{d}x = \int_0^1 (y_x^3 y_t)_x\,\mathrm{d}x = y_x^3(1, t)y_t(1, t), \tag{16}$$

for all $t \geqslant 0$. Using equation (13b) in equation (16), one obtains equation (14b).

Next one writes

$$\int_0^1 x y_{xt} y_t\,\mathrm{d}x = \tfrac{1}{2}\int_0^1 (x y_t^2)_x\,\mathrm{d}x - \tfrac{1}{2}\int_0^1 y_t^2\,\mathrm{d}x, \tag{17}$$

for all $t \geqslant 0$. Thus, equation (14c) follows.

Next, one writes

$$\int_0^1 x y_{xx} y_x\,\mathrm{d}x = \tfrac{1}{2}\int_0^1 (x y_x^2)_x\,\mathrm{d}x - \tfrac{1}{2}\int_0^1 y_x^2\,\mathrm{d}x = \tfrac{1}{2}y_x^2(1, t) - \tfrac{1}{2}\int_0^1 y_x^2\,\mathrm{d}x, \tag{18}$$

for all $t \geqslant 0$. Using equation (13b) in equation (18), one obtains equation (14d).

Finally, one writes

$$\int_0^1 x y_{xx} y_x^3\,\mathrm{d}x = \tfrac{1}{4}\int_0^1 (x y_x^4)_x\,\mathrm{d}x - \tfrac{1}{4}\int_0^1 y_x^4\,\mathrm{d}x = \tfrac{1}{4}y_x^4(1, t) - \tfrac{1}{4}\int_0^1 y_x^4\,\mathrm{d}x, \tag{19}$$

for all $t \geqslant 0$. Using equation (13b) in equation (19), one obtains equation (14e). □

Next, the time-derivative of the function E is computed.

Lemma 2.3. The time-derivative of the function E in equation (6), along the solution of the system (1a), (1d), and (13) (equivalently, the system (1)–(4)) satisfies

$$\dot{E}(t) = -k y_t^2(1, t) \leqslant 0, \tag{20}$$

for all $t \geqslant 0$.

Proof. From equation (6), one obtains

$$\dot{E}(t) = \int_0^1 (y_{tt} y_t + y_{xt} y_x)\,\mathrm{d}x + \frac{b}{2}\int_0^1 y_{xt} y_x^3\,\mathrm{d}x, \tag{21}$$

for all $t \geqslant 0$. Substituting y_{tt} from equation (1a) into equation (21), one obtains

$$\dot{E}(t) = \int_0^1 (y_{xx} y_t + y_{xt} y_x)\,\mathrm{d}x + \frac{b}{2}\int_0^1 (3y_{xx}y_x^2 y_t + y_x^3 y_{xt})\,\mathrm{d}x, \tag{22}$$

for all $t \geqslant 0$. Using equation (14a) and (14b) in equation (22), one obtains

$$\dot{E}(t) = -\frac{k y_t^2(1, t)}{1 + b y_x^2(1, t)/2}\left[1 + \frac{b k^2 y_t^2(1, t)}{2[1 + b y_x^2(1, t)/2]^2}\right], \tag{23}$$

for all $t \geqslant 0$. Using equation (13b) in the last term of equation (23), one obtains equation (20). □

Using the preliminary results obtained thus far, it is next proved that the functions V and E tend to zero exponentially.

Theorem 2.4. Let γ in equation (5) satisfy

$$\gamma < 4k/(3k^2 + 2). \tag{24}$$

Then, the functions V and E, along the solution of the system (1a), (1d), and (13) (equivalently, the system (1)–(4)) satisfy

$$0 \leqslant V(t) \leqslant V(0)\,\mathrm{e}^{-(\gamma/K_2)t} \qquad 0 \leqslant E(t) \leqslant (V(0)/K_1)\,\mathrm{e}^{-(\gamma/K_2)t}, \tag{25a, b}$$

for all $t \geqslant 0$, where K_1 and K_2 are given in equations (10).

Proof. From equation (5), one obtains

$$\dot V(t) = \dot E(t) + \gamma \int_0^1 (x y_{tt} y_x + x y_t y_{xt})\,\mathrm{d}x, \tag{26}$$

for all $t \geqslant 0$. Substituting y_{tt} from equation (1a) into equation (26), one obtains

$$\dot V(t) = \dot E(t) + \gamma \int_0^1 (x y_{xt} y_t + x y_{xx} y_x + \tfrac{3}{2} b x y_{xx} y_x^3)\,\mathrm{d}x, \tag{27}$$

for all $t \geqslant 0$. Using equations (20), (14c), (14d), and (14e) in equation (27), one obtains

$$\dot V(t) = -\gamma E(t) - \frac{\gamma b}{4}\int_0^1 y_x^4(x,t)\,\mathrm{d}x - k y_t^2(1,t) + \frac{\gamma}{2} y_t^2(1,t) - \frac{\gamma k^2 y_t^2(1,t)}{4[1 + b y_x^2(1,t)/2]^2}$$
$$+\frac{3\gamma k^2 y_t^2(1,t)}{4[1 + b y_x^2(1,t)/2]^2}\left[1 + \frac{b k^2 y_t^2(1,t)}{2[1 + b y_x^2(1,t)/2]^2}\right], \tag{28}$$

for all $t \geqslant 0$. Neglecting the second and fifth terms of equation (28) and using equation (13b) in the last term of this equation, one obtains

$$\dot V(t) \leqslant -\gamma E(t) - k y_t^2(1,t) + \frac{\gamma}{2} y_t^2(1,t) + \frac{3\gamma k^2 y_t^2(1,t)}{4[1 + b y_x^2(1,t)/2]}, \tag{29}$$

for all $t \geqslant 0$. Therefore,

$$\dot V(t) \leqslant -\gamma E(t) - F(t), \tag{30}$$

for all $t \geqslant 0$, where

$$F(t) := [k - \gamma(3k^2 + 2)/4] y_t^2(1,t). \tag{31}$$

Having inequality (24), one concludes that $F(t) \geqslant 0$ for all $t \geqslant 0$. Using the non-negativeness of F in inequality (30), one obtains

$$\dot V(t) \leqslant -\gamma E(t), \tag{32}$$

for all $t \geqslant 0$. Since $4k/(3k^2 + 2) \leqslant \sqrt{(2/3)} < 1$ for all $k > 0$, from inequality (24), one concludes that $\gamma < 1$. Therefore, inequalities (8) and (9) hold. Using inequality (9) in inequality (32), one obtains the differential inequality

$$\dot V(t) \leqslant -(\gamma/K_2) V(t), \tag{33}$$

for all $t \geqslant 0$, with the initial condition $V(0) > 0$ given in equation (7b). By a comparison theorem given in references [14, p. 2] or [15, p. 3], one concludes that V in inequality (33) satisfies $V(t) \leqslant V(0)\,e^{-(\gamma/K_2)t}$ for all $t \geqslant 0$. Note that by inequality (9), one has $V(t) \geqslant 0$ for all $t \geqslant 0$. Thus, inequality (25a) holds. By inequalities (9) and (25a), one concludes that inequality (25b) holds. □

Finally, it is shown that the boundary control u in equation (4) stabilizes the non-linear string in equations (1)–(3).

Corollary 2.5. The solution of the system (1a), (1d), and (13) (equivalently, the system (1)–(4)) $y(x, t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x \in [0, 1]$.

Proof. For the system (1a), (1d), and (13), one chooses the Lyapunov function V in equation (5), and lets γ in equation (5) satisfy inequality (24). Then, by Theorem 2.4, the function E tends to zero exponentially. From equation (6), one concludes that $y_x(x, t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x \in [0, 1]$. Since $y(0, t) = 0$ for all $t \geqslant 0$, one concludes that $y(x, t) \rightarrow 0$ as $t \rightarrow \infty$ for all $x \in [0, 1]$. □

3. CONCLUSION

In this note, it was proved that the non-linear stretched string represented by equations (1)–(3) can be stabilized by the linear boundary control in equation (4). The boundary control is the negative feedback of the transversal velocity of the string at one end.

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