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NON-LINEAR NORMAL MODES OF A LUMPED PARAMETER SYSTEM

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A method based on the power series technique is developed for the computation of normal modes and frequencies of a non-linear conservative lumped parameter system. The power series analysis is facilitated upon transforming the time variable into an harmonically oscillating time. Recurrence relations are derived from the governing equations of motion and used to determine the normal modes and frequencies iteratively. The oscillating time frequency is obtained by satisfying Rayleigh's energy principle. Excellent accuracy is demonstrated by the method in predicting the modal amplitudes and frequencies.

1. INTRODUCTION

The concept of non-linear normal modes was introduced in 1966 by Rosenberg [1] and co-workers in relation to conservative lumped parameter systems, and formed the basis of the works that followed. It was postulated that, during a modal motion, all the masses move periodically with the same period and that they pass through the maximum displacement position or through the static equilibrium position at the same time. This concept was later generalized to include continuous systems.

For weakly non-linear systems, the perturbation method has commonly been used to construct the mode shapes and natural frequencies [2–6]. The first order homogeneous solution is normally assumed to be a single linear mode and corrections are made at higher order due to other modes. Recently, Shaw and Pierre [7–9] have developed an invariant manifold approach for determining the normal modes of conservative and non-conservative weakly non-linear systems. The approach was first developed in relation to lumped parameter problems [8] and later applied to continuous systems [9]. The normal mode was defined as a motion which takes place on a two-dimensional invariant manifold in the phase space. The manifold has the property that it is tangent to the linear eigenspace as it passes through the static equilibrium position.

In this paper, the non-linear normal modes of a two-degree-of-freedom conservative system, previously analyzed by the invariant manifold approach [8], are obtained using the power series technique. The time variable is first transformed into an oscillating time, which transforms the governing differential equations into a form suitable for power series analysis. An important feature of this approach is that the form of the system's time behaviour in any mode is not prescribed *a priori*, but is generated from the differential equations governing its modal dynamics. Additionally, whereas the invariant manifold and perturbation methods involve asymptotic expansions about the equilibrium position, the present method assumes a power series expansion about the maximum displacement position and seeks to obtain accurate solutions iteratively.

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In a recent monograph [10], a method of non-smooth temporal transformation is discussed which resembles, in principle, the present method. Such a transformation reduces the problem of computing non-linear periodic solutions to the solution of a set of non-linear boundary value problems which are solved using regular perturbation expansions.

2. FORMULATION

The non-linear conservative system to be considered is shown in Figure 1. Without loss of generality, unit values are assigned to the inertia and stiffness elements. The leftmost spring has, in addition to the linear stiffness of unity, a cubic non-linearity of coefficient g. This system has been analyzed by Shaw and Pierre [8] using the invariant manifold concept. The equations of motion are given by

$$\ddot{x}_1 + 2x_1 - x_2 + gx_1^3 = 0, \qquad \ddot{x}_2 - x_1 + 2x_2 = 0, \tag{1, 2}$$

where x_1 and x_2 are the displacement co-ordinates and the overdot denotes differentiation with respect to time t. In order to facilitate the use of the power series method in the analysis of the periodic motion of non-linear conservative systems, the time variable t is transformed into an harmonically oscillating time τ as

$$\tau = \sin \omega t, \tag{3}$$

wherein the infinite time domain $0 \leqslant t \leqslant \infty$ is reduced to a finite time scale $-1 \leqslant \tau \leqslant 1$ within which τ oscillates harmonically at a frequency ω to be determined. Upon using equation (3) to transform equations (1) and (2), the governing equations of motion in the new time variable become

$$\omega^2(1 - \tau^2)x_1'' - \omega^2 \tau x_1' + 2x_1 - x_2 + gx_1^3 = 0, \tag{4}$$

$$\omega^2(1 - \tau^2)x_2'' - \omega^2 \tau x_2' - x_1 + 2x_2 = 0, \tag{5}$$

where the prime denotes differentiation with respect to τ. This reduction of the independent time variable into a finite scale permits power series expansions of the dependent displacements x_1 and x_2 in terms of τ. According to the theory of ordinary differential equations [11], equations (4) and (5) have one ordinary point at $\tau = 0$ and two regular singular points at $\tau = \pm 1$. One may therefore write power series expansions for x_1 and x_2 about the ordinary point as

$$x_1(\tau) = a_1 + a_2\tau^2 + a_3\tau^4 + \cdots = \sum_{n=1}^{\infty} a_n \tau^{2n-2} \tag{6}$$

$$x_2(\tau) = b_1 + b_2\tau^2 + b_3\tau^4 + \cdots = \sum_{n=1}^{\infty} b_n \tau^{2n-2} \tag{7}$$

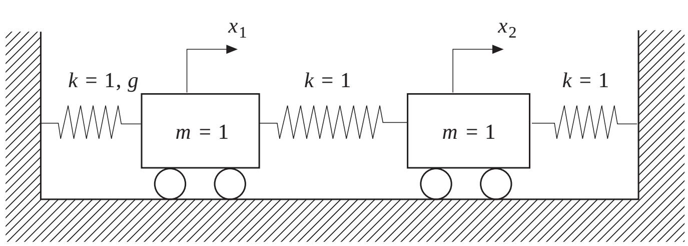

Figure 1. The non-linear vibratory system.

where a_i, b_i are constant coefficients to be determined. Equations (6) and (7) are capable of representing periodic motion since they repeat themselves every time $\tau = 0$. Furthermore, only even powers of τ are admitted in the series so that the same motion is repeated every half-cycle (positive or negative) of the oscillating time. This requires the oscillating time frequency to be equal to one-half the vibration frequency

$$\omega = \Omega/2. \tag{8}$$

By substituting equations (6) and (7) into equation (4), one obtains

$$\omega^2(1-\tau^2)\sum_{n=1}^{\infty} a_n(2n-2)(2n-3)\tau^{2n-4} - \omega^2\tau\sum_{n=1}^{\infty} a_n(2n-2)\tau^{2n-3}$$

$$+ 2\sum_{n=1}^{\infty} a_n\tau^{2n-2} - \sum_{n=1}^{\infty} b_n\tau^{2n-2} + g\sum_{n=1}^{\infty} c_n\tau^{2n-2} = 0, \tag{9}$$

in which the non-linear term x_1^3 is expanded as

$$x_1^3 = c_1 + c_2\tau^2 + c_3\tau^4 + \cdots = \sum_{n=1}^{\infty} c_n\tau^{2n-2}, \tag{10}$$

which results from the triple multiplication of equation (6). It will be noticed that the constant c_n can be computed once the constants $a_1, a_2, \ldots, a_n$ are known. A shift of indices in the first two terms in equation (9) may now be introduced so that all terms have the same power, as follows

$$\sum_{n=1}^{\infty} [\omega^2(2n)(2n-1)a_{n+1} - \omega^2(2n-2)(2n-3)a_n - \omega^2(2n-2)a_n$$

$$+ 2a_n - b_n + gc_n]\tau^{2n-2} = 0. \tag{11}$$

If equation (4) is to be satisfied exactly for all time, the coefficient of each power in equation (11) must be identically zero. This condition introduces the recurrence relation

$$a_{n+1} = \frac{[\omega^2(2n-2)^2 - 2]a_n + b_n - gc_n}{2n(2n-1)\omega^2}, \qquad n = 1, 2, \ldots \tag{12}$$

between the series coefficients. By substituting equations (6) and (7) into equation (5) and following the same steps described above, the following recurrence relation is established

$$b_{n+1} = \frac{[\omega^2(2n-2)^2 - 2]b_n + a_n}{2n(2n-1)\omega^2}, \qquad n = 1, 2, \ldots. \tag{13}$$

The recurrence relations (12) and (13) ensure that the modal dynamics satisfies the equations of motion. Consequently, the system's time behaviour is not prescribed *a priori* but is generated by the differential equation governing its motion. The system of differential equations (4) and (5) represents a non-linear initial value problem and requires for its solution the initial displacements and initial velocities to be prescribed. It is convenient to let the motion start at $t = \tau = 0$ from the maximum displacement position in which the velocities vanish. Consequently, the two constants a_1 and b_1 in equations (6) and (7) are assigned the maximum displacement values $a_1 = x_1(0)$ and $b_1 = x_2(0)$. Also, the zero initial velocity conditions are satisfied by equations (6) and (7) since $\dot{x}_1(0) = \omega x_1'(0) = 0$ and $\dot{x}_2(0) = \omega x_2'(0) = 0$. The remaining constants a_n and b_n depend recursively on the fundamental constants a_1 and b_1 and on the oscillating time frequency in accordance with equations (12) and (13).

The oscillating time frequency is an auxiliary unknown that may be determined by invoking Rayleigh's energy principle. This principle states that, for conservative systems, the maximum potential and kinetic energies are equal. The maximum potential energy U_{max} is associated with the maximum displacement position, assumed to occur at the start of the motion. For the system under consideration, this is given by

$$U_{max} = \tfrac{1}{2}b_1^2 + \tfrac{1}{2}(a_1 - b_1)^2 + \tfrac{1}{2}a_1^2 + \tfrac{1}{4}ga_1^4. \tag{14}$$

The kinetic energies of the two masses are given by

$$T = \tfrac{1}{2}\dot{x}_1^2 + \tfrac{1}{2}\dot{x}_2^2 = \tfrac{1}{2}\omega^2(1 - \tau^2)(x_1'^2 + x_2'^2). \tag{15}$$

The maximum kinetic energy occurs at the equilibrium position for which $\omega t = \pi/4$, $3\pi/4$, $5\pi/4, \ldots$ etc. From equation (3), this position is reached at $\tau = \pm 1/\sqrt{2}$. By using this result in equation (15), the maximum kinetic energy becomes

$$T_{max} = \tfrac{1}{4}\omega^2(x_1'^2 + x_2'^2)|_{\tau = 1/\sqrt{2}} \tag{16}$$

3. RESULTS AND DISCUSSION

The non-linear normal modes and frequencies of the system shown in Figure 1 were computed using an iterative scheme. Because the results are dependent on amplitude, they were obtained for specified values of the amplitude of the displacement x_1. For a given mode, the amplitude of x_1 is specified and an initial guess is made on x_2. The modal motion is assumed to start at $\tau = 0$ with maximum displacements and zero velocities so that $a_1 = x_1$ and $b_1 = x_2$. The recurrence relations (12) and (13) are then used to compute the ensuing motion, which satisfies the equations of motion for a range of values of ω. For each candidate motion, the error $\varepsilon = U_{max} - T_{max}$, has only one stationary minimum at a single value of ω. This value of ω is taken as the first approximation of the oscillating time frequency and used in conjunction with the initial guess on the mode shape a_1 and b_1 to compute the amplitudes x_1 and x_2 at $\tau = 1$ from equations (6) and (7). It turns out that these amplitudes, which represent maximum displacements on the opposite side of the equilibrium position, provide an improved estimate of the mode shape. This improvement results from satisfying the equations of motion exactly and Rayleigh's energy principle approximately. The computed values of x_1 and x_2 at $\tau = 1$ are used, after being normalized

TABLE 1

Convergence of second mode for amplitude $x_1 = 2$, $g = 0{\cdot}5$

Iteration number	Amplitudes at $\tau = 0$		ω	$\varepsilon = U_{max} - T_{max}$	Amplitudes at $\tau = 1$	
	x_1	x_2			x_1	x_2
1	2	0·0000	1·0231	1·4830E-1	−1·6284	0·6133
2	2	−0·7533	1·0060	8·0786E-3	−1·9144	0·8951
3	2	−0·9351	0·9998	4·8065E-4	−1·9795	0·9694
4	2	−0·9794	0·9982	2·5749E-5	−1·9949	0·9881
5	2	−0·9906	0·9975	3·6602E-6	−1·9991	0·9930
6	2	−0·9934	0·9979	1·7490E-7	−1·9994	0·9937
7	2	−0·9940	0·9979	4·2551E-8	−1·9996	0·9941

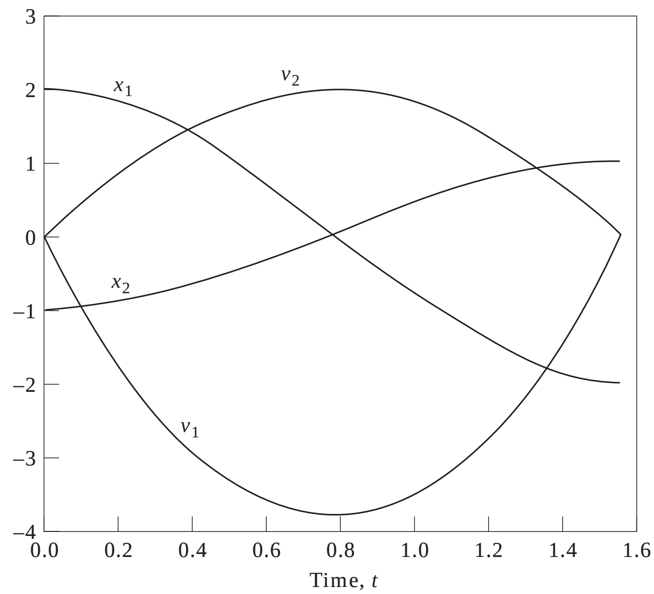

Figure 2. Second modal displacements x_1 and x_2 and velocities v_1 and v_2 versus time, for amplitude $x_1 = 2$, $g = 0{\cdot}5$.

to the specified value of x_1, as initial amplitudes in the next iteration, and the process is repeated until the desired degree of accuracy is achieved.

In Table 1 is demonstrated the convergence of the second non-linear mode and frequencies for amplitude $x_1 = 2$ and $g = 0{\cdot}5$. The number of terms used in the power series was 20. It can be seen that as the results converge to the exact solution, the error ε becomes vanishingly small, which ensures that Rayleigh's energy principle is satisfied. The modal amplitudes at $\tau = 0$ must then be identical to those at $\tau = 1$, in accordance with Rosenberg's definition of non-linear normal modes [1]. It is shown in Table 1, that after seven iterations the error in modal amplitude x_1 is 0·02% and that in x_2 is 0·01%.

In Figure 2 are shown the modal displacements and velocities in the second mode for amplitude $x_1 = 2$ and $g = 0{\cdot}5$, simulated over one-half of the vibration cycle. As expected, both masses attain their maximum displacements or maximum velocities simultaneously. The series coefficients a_i and b_i are shown in Table 2. Two points are noted here. First, the progressive decrease in absolute value of the coefficients characterizes a convergent solution. Second, the number of significant coefficients for x_1 is greater than those for x_2,

TABLE 2

Series coefficients of second mode for amplitude $x_1 = 2$, $g = 0{\cdot}5$

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
a_i	2·0000	−4·5159	1·6856	−1·6019	0·8485
a_{i+5}	−0·6291	0·3785	−0·2587	0·1635	−0·1084
a_{i+10}	0·0698	−0·0458	0·0296	−0·0194	0·0126
a_{i+15}	−0·0082	0·0053	−0·0035	0·0022	−0·0015
b_i	−0·9940	2·0024	−0·0456	0·0352	−0·0074
b_{i+5}	0·0043	−0·0016	0·0009	−0·0004	0·0002
b_{i+10}	−0·0001	0·0001	0·0000	0·0000	0·0000

TABLE 3

Comparison of first modal amplitudes and frequency, $g = 0{\cdot}5$

Amplitude x_1	Amplitude x_2 Invariant manifold	Amplitude x_2 Present method	Vibration frequency Invariant manifold	Vibration frequency Present method
0·25	0·2526	0·2526	1·0059	1·0060
0·50	0·5208	0·5214	1·0234	1·0226
0·75	0·8203	0·8236	1·0527	1·0486
1·00	1·1666	1·1805	1·0937	1·0804
1·25	1·5755	1·6163	1·1465	1·1164
1·50	2·0625	—	1·2109	—
1·75	2·6432	—	1·2871	—
2·00	3·3333	—	1·3750	—

which may be attributed to the direct connection of x_1 to the non-linear spring. This last point prompted an examination of the harmonic content of the series solution, equations (6) and (7), by a Fourier expansion over one cycle. The solutions were integrated using Simpson's rule with 50 grid points. The results showed that the displacements were dominated by the fundamental and third harmonic components, as approximated by

$$x_1(t) = 1{\cdot}96780 \cos \Omega t + 0{\cdot}03245 \cos 3\Omega t,$$

$$x_2(t) = -0{\cdot}99346 \cos \Omega t - 0{\cdot}00173 \cos 3\Omega t, \tag{17}$$

wherein the third harmonic component is significant in x_1 and x_2 is almost harmonic.

In Tables 3 and 4 are compared, for the first and second modes respectively, the computed modal amplitude x_2 and vibration frequency for specified values of x_1 with those obtained by Shaw and Pierre [8] using the invariant manifold approach. The coefficient g was 0·5, which represents a significant cubic non-linearity. For the linearized system ($g = 0{\cdot}0$), the first mode shape $x_2/x_1 = 1$ and the vibration frequency $\Omega = 1$, whereas for the second mode $x_2/x_1 = -1$ and $\Omega = \sqrt{3}$. In all the results, the initial guess used for x_2 was $x_2 = x_1$ in the first mode and $x_2 = 0{\cdot}0$ in the second mode. The errors in the computed modal amplitudes were all within 0·04%. The corresponding errors in the vibration frequency were smaller than 0·04%. In Tables 3 and 4 it is shown that the agreement between the two methods in modal amplitudes and frequency is good up to an amplitude x_1 of unity. To estimate the error in the invariant manifold method, the predicted modal

TABLE 4

Comparison of second modal amplitudes and frequency, $g = 0{\cdot}5$

Amplitude x_1	Amplitude x_2 Invariant manifold	Amplitude x_2 Present	Vibration frequency Invariant manifold	Vibration frequency Present
0·25	−0·2470	−0·2467	1·7354	1·7354
0·50	−0·4759	−0·4761	1·7455	1·7460
0·75	−0·6689	−0·6733	1·7624	1·7638
1·00	−0·8077	−0·8271	1·7861	1·7898
1·25	−0·8744	−0·9315	1·8166	1·8248
1·50	−0·8509	−0·9892	1·8538	1·8725
1·75	−0·7194	−1·0068	1·8978	1·9280
2·00	−0·4616	−0·9940	1·9485	1·9958

amplitudes and frequency were used to simulate the modal motion in accordance with equations (12) and (13), which was used to determine the modal amplitudes at $\tau = 1$ from equations (6) and (7) and then compared with the initial amplitudes. In the first mode and at $x_1 = 1$, the errors in x_1 and x_2 by the invariant manifold method were 1·48% and 0·086% respectively, compared with 0·001% and 0·017% respectively by the present method. At amplitudes larger than $x_1 = 1$, the accuracy of the invariant manifold prediction of the non-linear normal modes deteriorates rapidly. For example, in the second mode with $x_1 = 2$, the invariant manifold errors in x_1 and x_2 were 6·61% and 66·66% respectively, with the present method giving corresponding errors of 0·02% and 0·01% respectively. However, for the first mode, convergent power series solutions could not be obtained for amplitudes between $x_1 = 1{\cdot}5$ and $x_1 = 2$. This indicates the existence of an unstable region in which bifurcation of the normal mode occurs [6]. To investigate the nature of such bifurcations, a stability of the periodic solution, equations (6) and (7), must be carried out, but this is beyond the scope of this paper. Finally, to demonstrate the applicability of the power series solution to strongly non-linear problems, the value of g was increased to $g = 2$. The corresponding amplitudes for the second mode, $x_1 = 1$ and $x_2 = -0{\cdot}4978$, were accurate to within 0·06% and the vibration frequency was 1·9952.

4. CONCLUSIONS

A power series method has been developed for the computation of non-linear normal modes and frequencies of two-degree-of-freedom conservative systems. The time variable is transformed into an oscillating time, which transforms the equations of motion into a form suitable for power series analysis. Recurrence relations derived from the equations of motion enable the normal modes to be computed iteratively. Extension of the method to multi-degree-of-freedom systems is straightforward. The computational labour involved is minimal and convergence of the normal modes can be obtained over a broad range of vibration amplitudes.

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