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# A NON-LINEAR VARIABLE STIFFNESS FEEDBACK CONTROL WITH TUNING RANGE AND RATE SATURATION

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## 1. Introduction

Passive–active and fail-safe systems represent a recent trend of smart materials research. The systems with variable stiffness and damping are examples of such systems. Structural vibration control via variable stiffness and damping has been studied by a number of researchers. Reference [1] contains an extensive literature review on this subject. In reference [2], a fuzzy control rule is proposed for an adaptable stiffness dynamic absorbers; on the other hand, in references [3, 4], classic feedback controls are developed for controlling an adaptive absorber. In references [5, 6], several intuitive switching rules for variable stiffness systems have been proposed to maximize energy dissipation or the damping ratio. These switching rules have been shown to be very effective in reducing the transient structural vibration due to initial conditions. In reference [7], optimal controls are studied for a variable stiffness system with bounded and unbounded stiffness variation. It is interesting to note that the optimal control formulation leads to non-linear stiffness tuning laws that are not quite intuitive [7], while the intuitive tuning rules proposed in references [5, 6] are difficult to show to be optimal in a conventional optimal control formulation. One of the optimal controls from reference [7] is a discontinuous bang–bang control. The bang–bang control implies that the variable stiffness element can be tuned infinitely fast. This is clearly an impractical requirement. Because of the discontinuity, it may also introduce higher order harmonics to the system response. It is desirable to develop a continuous variable stiffness control that takes into account tuning range and rate saturation. The present paper attempts to develop such a continuous non-linear variable stiffness control by following the ideas in references [8, 9]. That is, a piecewise smooth performance index will be constructed in such a way that the resulting optimal control is continuous. A simple one-degree-of-freedom system is used to demonstrate the theoretical development, although the approach is applicable to more complex systems.

The remainder of the paper is organized as follows. In section 2, the continuous non-linear optimal controls for a one-degree-of-freedom variable stiffness system are derived. In section 3, numerical simulations of the proposed controls are presented, and the effect of maximum tuning rate on the damping performance of the closed loop system is studied. Numerical results of power spectral density of the system response under several controls are also presented.

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## 2. CONTINUOUS CONTROL LAWS

By considering a single-degree-of-freedom system the equation of motion in the state space format is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(1+u\alpha) & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \qquad \text{or} \qquad \dot{\mathbf{x}} = \mathbf{A}(u)\mathbf{x}, \qquad t > t_0, \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \quad (1)$$

where $0 < \alpha < 1$ and the control $|u| < 1$ and $|\dot{u}| < v_m$. Note that some scaling has been applied to the equation of motion resulting in a unit resonance frequency of the system.

### 2.1. *Unsaturated controls*

Assume that no saturation occurs. Let one consider a performance index as

$$J(t_0) = \phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} [\beta g(\mathbf{x}) + \tfrac{1}{2} u^2]\, \mathrm{d}t, \quad (2)$$

where $\beta > 0$ is a weighting factor, $g(\mathbf{x})$ is a positive function of the state variables representing, for example, the total energy of the system, $\phi$ is a positive function of its arguments representing the terminal cost of the control. The Hamiltonian function of the system can then be constructed as [10]

$$H = \beta g(\mathbf{x}) + \tfrac{1}{2}u^2 + \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{A}(u), \quad (3)$$

where $\boldsymbol{\lambda}$ is a $2 \times 1$ vector of the Lagrange multipliers, also known as the co-state vector. Hence, one has the co-state equation and the optimality condition as

$$-\dot{\boldsymbol{\lambda}} = \mathbf{A}^{\mathrm{T}}(u)\boldsymbol{\lambda} + \partial g/\partial \mathbf{x}, \qquad \partial H/\partial u = 0. \quad (4, 5)$$

Equation (5) leads to the optimal tuning law without saturation,

$$u^* = \alpha x_1^* \lambda_2^*, \qquad \text{when} \qquad |\alpha x_1^* \lambda_2^*| \leqslant 1 \qquad \text{and} \qquad |\mathrm{d}(\alpha x_1^* \lambda_2^*)/\mathrm{d}t| \leqslant v_m, \quad (6)$$

where the superscript * indicates that the solution is optimal. To determine the optimal control, one needs to specify the terminal condition, and to obtain the solutions for the state and co-state equations. This is a difficult problem on its own right. However, this issue will not be discussed in this paper. Next, control laws are presented when saturation does occur.

### 2.2 *Range-saturated controls*

The tuning range saturation is first considered. When $|\alpha x_1^* \lambda_2^*| > 1$, equation (6) violates the control constraint. In this case, one has to modify the Hamiltonian function such that

$$H = \beta g(\mathbf{x}) + \tfrac{1}{2} + \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{A}(u). \quad (7)$$

Note that when the tuning range saturation occurs, $|u| = 1$. By comparing the Hamiltonian functions in equations (3) and (7), one can see that the Hamiltonian function as well as the performance index is continuous with respect to $u$. By applying Pontryagin's minimum principle [10],

$$H(\mathbf{x}^*, u^*, \boldsymbol{\lambda}^*, t) \leqslant H(\mathbf{x}^*, u, \boldsymbol{\lambda}^*, t) \quad \text{for all} \quad |u| \leqslant 1, \quad (8)$$

and one has

$$-u^* \alpha x_1^* \lambda_2^* \leqslant -u \alpha x_1^* \lambda_2^*. \quad (9)$$

This leads to the well known bang–bang control,

$$u^* = \text{sgn}\,(\alpha x_1^* \lambda_2^*) \qquad \text{when} \qquad |\alpha x_1^* \lambda_2^*| > 1. \tag{10}$$

2.3 *Rate-saturated controls*

The unsaturated control law in equation (6) requires that the tuning rate be smaller than the maximum rate $v_m$. When this condition is violated, a new control law has to be derived. Note that when the control is in the range saturation, the control is either $+1$ or $-1$ and its rate is set to be zero in the time interval when $\alpha x_1^* \lambda_2^*$ does not change its sign, as can be seen from equation (10). Therefore, the rate saturation can only occur when the control is not in the range saturation.

Assume that at a time $t_1$ when the control is at a value $u_1$ such that $-1 < u_1 < 1$, the rate saturation occurs. For $t > t_1$, the control can be written as

$$u(t) = u_1 + \int_{t_1}^{t} \dot{u}(\tau)\,\mathrm{d}\tau, \qquad t > t_1. \tag{11}$$

Here, the rate term in the integrand is to be determined by the formulation of optimal control. By using the same Hamiltonian function as in equation (7) and applying Pontryagin's minimum principle, one has

$$-\int_{t_1}^{t} \dot{u}(\tau)\,\mathrm{d}\tau \cdot \alpha x_1^* \lambda_2^* \leqslant -\int_{t_1}^{t} \dot{u}(\tau)\,\mathrm{d}\tau \cdot \alpha x_1^* \lambda_2^*, \qquad \text{for all} \qquad |\dot{u}| \leqslant v_m. \tag{12}$$

Consider a time interval from $t_1$ to $t_2$ over which $\alpha x_1^* \lambda_2^*$ does not change sign. In this interval the tuning rate given by

$$u^*(t) = v_m\,\text{sgn}\,(\alpha x_1^* \lambda_2^*), \qquad \text{for} \qquad t_1 < t < t_2 \tag{13}$$

will clearly make the left side of equation (12) minimum. Extending this discussion to other time intervals $(t_n, t_{n+1})$ over which $\alpha x_1^* \lambda_2^*$ does not change its sign, one concludes that the optimal control in this case is given by

$$u^*(t) = u^*(t_n) + (t - t_n) v_m\,\text{sgn}\,(\alpha x_1^* \lambda_2^*), \qquad \text{for} \qquad t_n < t < t_{n+1} \qquad \text{and} \qquad |u^*(t)| < 1. \tag{14}$$

In summary, the continuous optimal control law for the system can be presented as

$$u^*(t) = \begin{cases} \text{sgn}\,(\alpha x_1^* \lambda_2^*), & \text{if} \quad |\alpha x_1^* \lambda_2^*| > 1, \\ \alpha x_1^* \lambda_2^*, & \text{if} \quad |\alpha x_1^* \lambda_2^*| \leqslant 1 \text{ and } \mathrm{d}(\alpha x_1^* \lambda_2^*)/\mathrm{d}t \leqslant v_m, \\ u^*(t_n) + (t - t_n) v_m\,\text{sgn}\,(\alpha x_1^* \lambda_2^*), & \text{if} \quad |\alpha x_1^* \lambda_2^*| \leqslant 1 \text{ and } \mathrm{d}(\alpha x_1^* \lambda_2^*)/\mathrm{d}t > v_m, \end{cases} \tag{15}$$

subject to $|u^*(t)| \leqslant 1$, and the integrand of the associated performance index is given by

$$L(\mathbf{x}, u, t) = \begin{cases} \beta \mathrm{g}(\mathbf{x}) + \frac{1}{2}, & \text{if} \quad |\alpha x_1^* \lambda_2^*| > 1 \\ \beta \mathrm{g}(\mathbf{x}) + \frac{1}{2}u^2, & \text{if} \quad |\alpha x_1^* \lambda_2^*| \leqslant 1 \quad \text{and } |\mathrm{d}(\alpha x_1^* \lambda_2^*)/\mathrm{d}t| \leqslant v_m, \\ \beta \mathrm{g}(\mathbf{x}) + \frac{1}{2}, & \text{if} \quad |\alpha x_1^* \lambda_2^*| \quad \text{and } |\mathrm{d}(\alpha x_1^* \lambda_2^*)/\mathrm{d}t| > v_m, \end{cases} \tag{16}$$

Some remarks on the above control law development are in order. Because the system is parametrically controlled, the closed loop system is inherently non-linear. It is not easy to show the stability of the proposed control law explicitly. A rigorous proof of stability is elusive at present. However, one can argue that over each time interval when a particular branch of control is active, the control is truly optimal and attempts to reduce a positive measure of the system dynamics. Hence, the stability should follow. The optimality of the control is guaranteed by Pontryagin's minimum principle in the saturated regions, and can be easily shown to hold in the unsaturated case.

When $\beta = 0$, the performance index defined by $L(\mathbf{x}, u, t)$ in equation (16) describes two distinct optimal control problems: minimum time ($L = 1/2 = \text{constant}$) and minimum control energy ($L = 1/2u^2$). Thus, by switching between these two problems at proper times, the resulting optimal control becomes continuous.

## 3. A SIMPLE NUMERICAL EXAMPLE

As one said earlier, it is very difficult to obtain the solutions for the state and co-state variables in order to compute the control forward in time. It should be pointed out, however, that the research of algorithms for non-linear optimal control problems is a quite active area and there are many effective algorithms for obtaining non-linear optimal controls. See reference [11] and references therein. In this paper, only a simple case is presented that allows backward numerical simulation to determine the state, co-state and control varibles. The objective is to examine the effect of maximum tuning rate on vibration suppression in the rigorous optimal control setting.

Consider a special terminal cost function $\phi(\mathbf{x})$ and the function $g(\mathbf{x})$ in the performance index given by

$$\phi(\mathbf{x}) = (\gamma/2)(x_1^2 + x_2^2), \qquad g(\mathbf{x}) = \tfrac{1}{2}(x_1^2 + x_2^2), \tag{17}$$

where $\gamma > 0$ is a weighting factor for $\phi(\mathbf{x})$ in the performance index. One also assumes that the final time $t_f$ is fixed such that $\mathrm{d}t_f = 0$, and the final state is minimized via the terminal cost term, but otherwise is free such that $\mathrm{d}\mathbf{x}(t_f) \neq 0$. With these assumptions, one has the terminal conditions as [10]

$$\gamma\mathbf{x}(t_f) = \boldsymbol{\lambda}(t_f). \tag{18}$$

In the following, numerical simulations of the continuous optimal control are presented by specifying a terminal value $\mathbf{x}(t_f)$, and integrating the equations backward in time. Three different controls will be compared: a discontinuous bang–bang control with infinitely fast tuning, the complete continuous control in equation (15) and a continuous optimal control using the maximum tuning rate whenever the control is unsaturated. This last control, referred to as the maximum tuning rate control for convenience, is obtained by neglecting the middle branch of the control in equation (15) and is given by

$$u^*(t) = \begin{cases} \operatorname{sgn}(\alpha x_1^* \lambda_2^*), & \text{if} \quad |\alpha x_1^* \lambda_2^*| > 1, \\ u^*(t_n) + (t - t_n)v_m \operatorname{sgn}(\alpha x_1^* \lambda_2^*), & \text{if} \quad |\alpha x_1^* \lambda_2^*| \leqslant 1, \end{cases} \tag{19}$$

subject to $|u^*(t)| \leqslant 1$. The integrand of the associated performance index associated with this control is simply given by

$$L(\mathbf{x}, u, t) = \beta g(\mathbf{x}) + 1/2. \tag{20}$$

In the numerical simulations, $\beta = \gamma = 1$, terminal conditions $x(t_f) = 0{\cdot}0$, $\dot{x}(t_f) = 0{\cdot}2$, and $t_f = 10\pi$ s have been set. The integration time step is 0·0416 s, resulting in 151 points per cycle.

It should be pointed out that the vibration reduction as a function of tuning range $\alpha$ is relatively well understood. In general, the larger the value of $\alpha$, the faster vibration reduction is achievable, meaning the more damping can be introduced by the stiffness

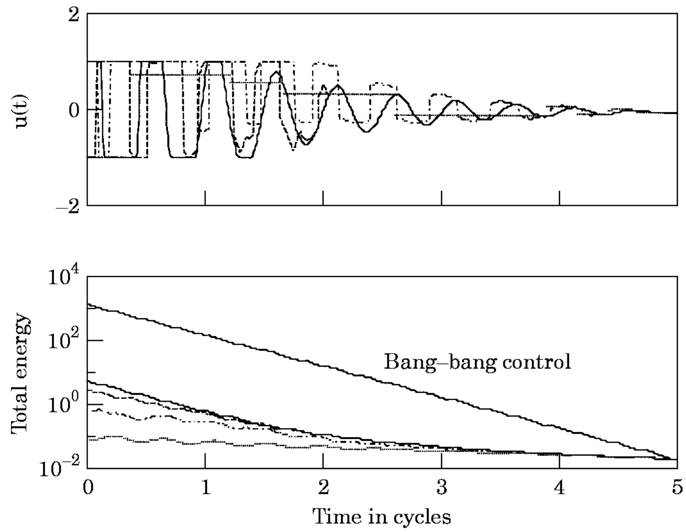


Figure 1. The stiffness tuning time history $u(t)$ and the total energy of the system with different tuning rates for the control law in equation (15): ——, $v_m = 10$; – – – –, $v_m = 1$; – · – · –, $v_m = 0{\cdot}1$; . . . . , $v_m = 0{\cdot}01$.

controller. Less is known about the effect of tuning rate on the vibration control performance. To study the effect of tuning rates, $\alpha$ has been fixed at 0·5 in the simulations. Figure 1 shows the results of this study for the control law in equation (15). As expected, a smaller maximum tuning rate $v_m$ leads to slower energy decay of the system, indicating that a small damping is introduced by the stiffness controller. The damping is proportional to the slope of the energy decay. Even when $v_m$ is large, say $v_m = 10$, the damping effect is still far less than that due to the bang–bang control (see the lower frame in Figure 1). One of the reasons for this light damping effect is that the controller penalizes its energy.

Next, one eliminates the control energy penalty from the performance index, and allows the stiffness element to be changed at the maximum tuning rate whenever it is unsaturated.

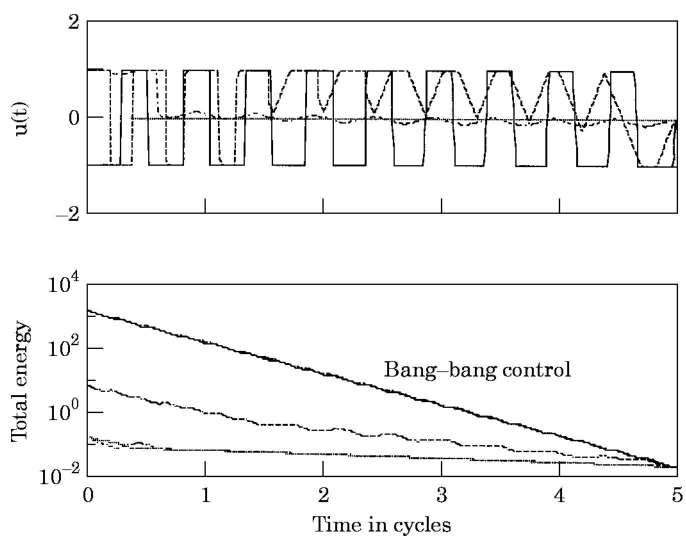


Figure 2. The stiffness tuning time history $u(t)$ and the total energy of the system with different tuning rates for the maximum tuning rate control law in equation (19). Curve identifications as in Figure 1.

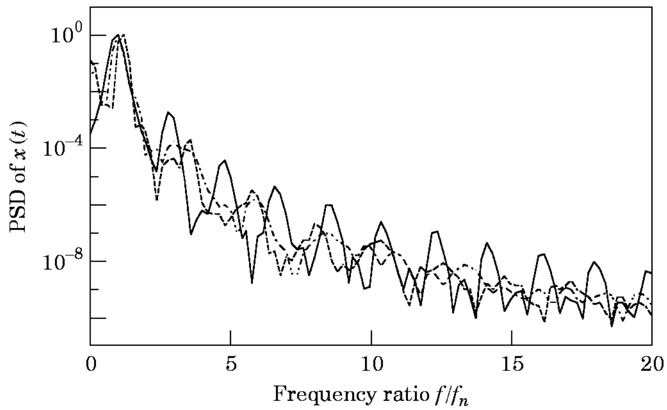


Figure 3. The power spectrum density of $x(t)$ for the purely bang–bang control (——), the maximum tuning rate of control in equation (19) (– – – –), and the control law in equation (15) (–·–·– ); $\alpha = 0{\cdot}5$, $v_m = 0{\cdot}1$. The results are scaled so that the highest peak of the spectrum is unity.

The results are shown in Figure 2. As compared to Figure 1 for the case with control penalty, this control performs much better. For $v_m = 10$, the maximum tuning rate control, while still continuous, approaches the damping level of the discontinuous bang–bang control.

Finally, one shows the power spectral density of the system displacement under all these variable stiffness controls. Figure 3 shows the power spectral density of $x(t)$ for the bang–bang control, and the control in equation (15) with and without the control energy penalty. The power spectral density has been scaled such that the highest peak of the spectrum is unit. This scaling helps to better visualize and compare the relative contribution of higher order harmonic components. It should be pointed out that the bang–bang control always introduces the highest harmonic contents. For the other two controls, the higher order harmonic components grow with $v_m$ for a given $\alpha$.

## 4. CONCLUDING REMARKS

We have presented a continuous nonlinear variable stiffness feedback control, and quantitatively studied the effect of maximum tuning rate on the damping performance of the closed loop system. We have found, as expected, that the damping of the closed loop system increases with the maximum tuning rate of the variable stiffness element for a given tuning range. The continuous control law with control energy penalty performs far less superior than the continuous control law without control energy penalty. Among all the controls studied herein, the discontinous bang–bang control delivers the most damping, and in the meantime, introduces the highest harmonic distortion to the system response. Although the continuous controls don't perform as well as the bang–bang control, they may well provide performance upper bounds for practical implementation of variable stiffness controls.

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