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# LETTERS TO THE EDITOR

## APPLICATION OF GREEN FUNCTIONS IN FREQUENCY ANALYSIS OF TIMOSHENKO BEAMS WITH OSCILLATORS

S. Kukla
*Technical University of Częstochowa, Dąbrowskiego 73, 42-200 Częstochowa, Poland*

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### 1. INTRODUCTION

The problem of vibration of a beam carrying oscillators is very interesting from an engineering point of view. Free vibration of such systems was discussed in papers [1–7]. The authors of the papers presented various methods of solution of the problem and investigated the influence of an attached oscillator on vibrations of the combined systems considered.

The effect of concentrated masses elastically mounted to a beam on fundamental frequency of the vibration system was investigated by Ercoli and Laura in reference [1]. The solution of the problem was found by expanding the deflection in terms of characteristic beam functions and by using the Ritz method as well. Gürgöze [2] dealt with a problem of free vibration of a clamped–free beam with an end mass to which a spring mass system is attached. He used the Lagrange multipliers method. Free vibration of a system consisting of a beam and a rigid body elastically mounted to the beam by means of two translational springs was analyzed by Jen and Magrab [3]. The authors used the Laplace transform with respect to the spatial variable. The spectral problem of a free–free beam with oscillators was reported by Pesterev and Tavrizov [4]. The static Green functions were used in the structural analysis method. The dynamic Green function was applied by Bergman and Hyatt [5] and by Kukla and Posiadała [6]. The papers [1–6] are devoted to the free vibration problems of a combined system consisting of oscillators and a Bernoulli–Euler beam. The Timoshenko beam theory was applied by Rossi *et al*. [7]. In this paper, an analytical solution was found by dividing the beam into two segments and using the compatibility conditions at the dividing point.

A system of a beam with rigidly attached mass can be treated as a particular case of a beam with an elastically mounted mass: if the constant of the translational spring connecting a concentrated mass with a beam tends to infinity, a system of the beam with rigidly attached mass is obtained. Likewise, a beam with an elastic support can be treated as a particular case of a beam with an elastically mounted mass [6]. Moreover, the equation for a Bernoulli–Euler beam may be obtained from the Timoshenko equations. Therefore the solution of the problem regarding free vibrations of a Timoshenko beam with oscillators comprises a wide range of issues.

The present note deals with the problem of free vibration of a combined system consisting of a Timoshenko beam and multi-mass oscillators. The formulation and solution of the problem comprises the systems of the beam with many oscillators, which are attached to it at arbitrary points. The solution is found by applying the Green function method. The effect of an oscillator on the frequencies of the combined system is investigated. Exemplary numerical results show the influence of the location of two- and

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three-degree-of-freedom oscillators attached to a cantilever beam on a few first frequencies of the vibration systems.

## 2. THEORY

Consider a Timoshenko beam with $N$ multi-mass oscillators attached to it at points $x_j$, $j = 1, 2, \ldots, N$. The $j$th oscillator consists of $n_j$ spring–mass systems combined in series (see Figure 1). For the beam, the governing coupled differential equations for the total deflection $y$ and the bending slope $\psi$, are

$$K'AG\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) - \rho A \frac{\partial^2 y}{\partial t^2} = \sum_{j=1}^{N} k_{1j}\{y(x_j, t) - z_{1j}(t)\}\, \delta(x - x_j), \tag{1}$$

$$EI\frac{\partial^2 \psi}{\partial x^2} + K'AG\left(\frac{\partial y}{\partial x} - \psi\right) - \rho I \frac{\partial^2 \psi}{\partial t^2} = 0, \tag{2}$$

where $A$ is the area of the cross-section, $E$ is the modulus of elasticity, $G$ is the modulus of rigidity, $I$ is the moment of inertia of the cross-section, $K'$ is a factor depending on the shape of the cross-section, $k_{ij}$ are the stiffness coefficients of the translational springs, $\rho$ is the mass density of the beam material and $\delta()$ is the Dirac delta function. The displacements $z_{ij}(t)$ of the masses $m_{ij}$ for $i = 1, 2, \ldots, n_j$, $j = 1, 2, \ldots, N$, are governed by the differential equations

$$m_{1j}\, \mathrm{d}^2 z_{1j}(t)/\mathrm{d}t^2 + k_{1j}\{z_{1j}(t) - y(x_j, t)\} + k_{2j}\{z_{1j}(t) - z_{2j}(t)\} = 0, \tag{3}$$

$$m_{ij}\, \mathrm{d}^2 z_{ij}(t)/\mathrm{d}t^2 + k_{ij}\{z_{ij}(t) - z_{i-1j}(t)\} + k_{i+1j}\{z_{ij}(t) - z_{i+1j}(t)\} = 0,$$

$$i = 2, 3, \ldots, n_j - 1, \tag{4}$$

$$m_{n_j j}\, \mathrm{d}^2 z_{n_j j}(t)/\mathrm{d}t^2 + k_{n_j}\{z_{n_j j}(t) - z_{n_j - 1j}(t)\} = 0. \tag{5}$$

In order to find the natural frequencies of the system, $\omega$, one assumes that

$$y(x, t) = \bar{Y}(x)\cos\omega t, \qquad \psi(x, t) = \Psi(x)\cos\omega t, \qquad z_{ij}(t) = \bar{Z}_{ij}\cos\omega t. \tag{6}$$

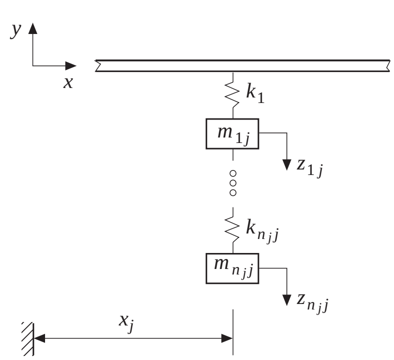


Figure 1. A sketch of the system considered: the $j$th multi-mass oscillator attached at the beam point $x_j$.

Substituting equations (6) into equations (1)–(5) and introducing the dimensionless quantities, one obtains

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}\xi^2} - \frac{\mathrm{d}\psi}{\mathrm{d}\xi} + s^2\beta^4 Y = s^2 \sum_{j=1}^{N} K_{1j}\{Y(\xi_j) - Z_{1j})\,\delta(\xi - \xi_j), \tag{7}$$

$$s^2 \frac{\mathrm{d}^2\Psi}{\mathrm{d}\xi^2} + \frac{\mathrm{d}Y}{\mathrm{d}\xi} + (r^2 s^2 \beta^4 - 1)\Psi = 0, \tag{8}$$

$$-\beta^4 Z_{1j} + \Omega_{1j}^4\{Z_{1j} - Y(\xi_j)\} + \gamma_{1j}\Omega_{1j}^4(Z_{1j} - Z_{2j}) = 0, \tag{9}$$

$$-\beta^4 Z_{ij} + \Omega_{ij}^4(Z_{ij} - Z_{i-1j}) + \gamma_{ij}\Omega_{ij}^4(Z_{ij} - Z_{i+1j}) = 0, \qquad i = 2, 3, \ldots, n_j - 1, \tag{10}$$

$$-\beta^4 Z_{n_j j} + \Omega_{n_j j}^4(Z_{n_j j} - Z_{n_j - 1j}) = 0, \tag{11}$$

where $\xi = x/L$, $\xi_j = x_j/L$, $Y = \overline{Y}/L$, $Z_{ij} = \overline{Z}_{ij}/L$, $\beta^4 = (\rho A L^4/EI)\omega^2$, $r^2 = I/AL^2$, $s^2 = Er^2/K'G$, $\gamma_{ij} = k_{i+1j}/k_{ij}$, $K_{1j} = k_{1j}L^3/EI$, $M_{ij} = m_{ij}/\rho AL$, $\Omega_{ij}^4 = K_{ij}/M_{ij}$ and $L$ is the length of the beam.

The expression $Y(\xi_j) - Z_{1j}$, which occurs in equation (7), may be written by using equations (9)–(11) in the form

$$Y(\xi_j) - Z_{1j} = Q_{n_j j} Y(\xi_j). \tag{12}$$

Here the $Q_{n_j j}$ for $n_j = 1$, 2 and 3, are

$$Q_{1j} = 1 + \frac{1}{\lambda_{1j}}, \qquad Q_{2j} = 1 + \frac{1}{\lambda_{1j} - \gamma_{1j}(1 + 1/\lambda_{2j})},$$

$$Q_{3j} = 1 + \frac{1}{\lambda_{1j} - \gamma_{1j}[1 + 1/\{\lambda_{2j} - \gamma_{2j}(1 + 1/\lambda_{3j})\}]}, \tag{13}$$

where $\lambda_{ij} = (\beta^4/\Omega_{ij}^4) - 1$. The expressions $Q_{2j}$ and $Q_{3j}$ can be written symbolically, by applying the notation used for continued fractions [8], as

$$Q_{2j} = 1 + \frac{1}{\lambda_{1j} - \gamma_{1j} +}\,\frac{-\gamma_{1j}}{\lambda_{2j}}, \qquad Q_{3j} = 1 + \frac{1}{\lambda_{1j} - \gamma_{1j} +}\,\frac{-\gamma_{1j}}{\lambda_{2j} - \gamma_{2j} +}\,\frac{-\gamma_{2j}}{\lambda_{3j}}.$$

Generally, one has

$$Q_{n_j j} = 1 + \frac{1}{\lambda_{1j} - \gamma_{1j} +}\,\frac{-\gamma_{1j}}{\lambda_{2j} - \gamma_{2j} +}\,\frac{-\gamma_{2j}}{\lambda_{3j} - \gamma_{3j} +}\cdots\frac{-\gamma_{n_j - 1j}}{\lambda_{n_j j}}. \tag{14}$$

Equations (7) and (8) can be now written in the matrix form

$$\mathbf{LY} = \mathbf{F}, \tag{15}$$

where

$$\mathbf{L} = \begin{bmatrix} \dfrac{d^2}{d\xi^2} + s^2\beta^4 & -\dfrac{d}{d\xi} \\ \dfrac{d}{d\xi} & s^2\dfrac{d^2}{d\xi^2} + r^2 s^2 \beta^4 - 1 \end{bmatrix},$$

$$\mathbf{Y}(\xi) = \begin{bmatrix} Y(\xi) \\ \Psi(\xi) \end{bmatrix}, \qquad \mathbf{F}(\xi) = \begin{bmatrix} F_Y(\xi) \\ 0 \end{bmatrix}, \tag{16}$$

and $F_Y(\xi) = s^2 \sum_{j=1}^{N} K_{1j} Q_{n_j j} Y(\xi_j)\, \delta(\xi - \xi_j)$.

The solution of equation (15) can be expressed by [10]

$$\mathbf{Y}(\xi) = \int_0^1 \mathbf{G}^{\mathrm{T}}(\xi, \eta)\mathbf{F}(\eta)\, d\eta. \tag{17}$$

Here $\mathbf{G}^{\mathrm{T}}(\xi, \eta)$ denotes the transpose of the Green function matrix

$$\mathbf{G}(\xi, \eta) = \begin{bmatrix} g_Y^f(\xi, \eta) & g_Y^m(\xi, \eta) \\ g_\Psi^f(\xi, \eta) & g_\Psi^m(\xi, \eta) \end{bmatrix},$$

which satisfies the equation

$$\mathbf{LG} = \mathbf{E}\, \delta(\xi - \eta), \tag{18}$$

where $\mathbf{E}$ is the $2 \times 2$ unit matrix. From the equation (17) results, the functions $Y(\xi)$ and $\Psi(\xi)$ can be written in the forms

$$Y(\xi) = s^2 \sum_{j=1}^{N} K_{1j} Q_{n_j j} g_Y^f(\xi, \xi_j) Y(\xi_j), \qquad \Psi(\xi) = s^2 \sum_{j=1}^{N} K_{1j} Q_{n_j j} g_Y^m(\xi, \xi_j) Y(\xi_j). \tag{19, 20}$$

By substituting $\xi = \xi_i$ $(i = 1, 2, \ldots, N)$ successively into equation (19), the set of $N$ homogeneous, linear equations with respect to displacements $Y(\xi_i)$, is obtained. The determinant of the coefficient matrix of this equation system is set equal to zero yielding the frequency equation. This equation appears in the form

$$|a_{ij}(\beta)| = 0, \tag{21}$$

where $a_{ij}(\beta) = s^2 K_{1j} Q_{n_j j} g_Y^f(\xi_i, \xi_j) - \delta_{ij}$, $|a_{ij}|$ denotes the determinant of the matrix $[a_{ij}]$ and $\delta_{ij}$ is the Kronecker delta. Equation (21) is then solved numerically.

## 3. THE GREEN FUNCTION

In the recent paper by Lueschen *et al*. [9] the closed form expressions for Green functions of an uniform Timoshenko beams are given. The functions are obtained for six cases of the end attachments of the beam by using the fourth order differential equation. Taking into consideration the coupled differential equations of motion of the Timoshenko beam, one can derive a functional matrix (Green function matrix), which plays the same part as the Green function with reference to one equation. This approach is presented below.

The Green function matrix as a solution of equation (18) can be written as a sum of two matrices,

$$\mathbf{G} = \mathbf{G}_h + \mathbf{G}_p, \tag{22}$$

where $\mathbf{G}_h$ is the general solution of homogeneous equation, obtained from equation (18), and $\mathbf{G}_p$ is the particular solution of this equation. The frequency equation (21) is expressed by one element of the Green function matrix. That is why further consideration for the first column of the matrices $\mathbf{G}$, $\mathbf{G}_h$ and $\mathbf{G}_p$ is presented. The functions $g_Y^f(\xi, \eta)$ and $g_\Psi^f(\xi, \eta)$ with respect to $\xi$ satisfy the same boundary conditions as the functions $Y$ and $\Psi$, respectively. Depending on the attachments of the beam ends, the conditions for $\xi = 0$ and $\xi = 1$, are: $Y = 0$, $\mathrm{d}\Psi/\mathrm{d}\xi = 0$ for a simply supported end; $Y = 0$, $\Psi = 0$ for a clamped end; $\mathrm{d}Y/\mathrm{d}\xi - \Psi = 0$, $\mathrm{d}\Psi/\mathrm{d}\xi = 0$ for a free end; $\mathrm{d}Y/\mathrm{d}\xi - \Psi = 0$, $\Psi = 0$ for a sliding end.

The column-matrix $\mathbf{G}_h$ appears in the form

$$\mathbf{G}_h = \begin{bmatrix} \cosh \alpha_1\xi \\ a_1 \sinh \alpha_1\xi \end{bmatrix} C_1 + \begin{bmatrix} \sinh \alpha_1\xi \\ a_1 \cosh \alpha_1\xi \end{bmatrix} C_2 + \begin{bmatrix} \cos \alpha_2\xi \\ -a_2 \sin \alpha_2\xi \end{bmatrix} C_3 + \begin{bmatrix} \sin \alpha_2\xi \\ a_2 \cos \alpha_2\xi \end{bmatrix} C_4, \tag{23}$$

where $C_1, C_2, C_3, C_4$ are integral constants, $a_1 = (1/\alpha_1)(\alpha_1^2 + s^2\beta^4)$, $a_2 = (1/\alpha_2)(\alpha_2^2 - s^2\beta^4)$, $\alpha_1 = (1/\sqrt{2})[-\beta^4(r^2 + s^2) + \sqrt{\Delta}]^{1/2}$, $\alpha_2 = (1/\sqrt{2})[\beta^4(r^2 + s^2) + \sqrt{\Delta}]^{1/2}$ and $\Delta = \beta^4[\beta^4(r^2 - s^2)^2 + 4]$. Here the case is considered, when $\alpha_1$ is a real value (i.e., $\beta^4 < 1/r^2s^2$).

The column-matrix $\mathbf{G}_p$ is assumed in the form

$$\mathbf{G}_p = \begin{bmatrix} g_Y^{fp}(\xi - \eta) \\ g_\Psi^{fp}(\xi - \eta) \end{bmatrix} \mathrm{H}(\xi - \eta), \tag{24}$$

where H( ) is the Heaviside function. The functions $g_Y^{fp}(\zeta)$ and $g_\Psi^{fp}(\zeta)$ are determined by substituting equation (24) into equation (18). These functions are as follows:

$$g_Y^{fp}(\zeta) = \frac{1}{\alpha_1 a_2 - \alpha_2 a_1}(a_2 \sinh \alpha_1\zeta - a_1 \sin \alpha_2\zeta), \tag{25}$$

$$g_\Psi^{fp}(\zeta) = \frac{a_1 a_2}{\alpha_1 a_2 - \alpha_2 a_1}(\cosh \alpha_1\zeta - \cos \alpha_2\zeta). \tag{26}$$

The integral constants $C_1$, $C_2$, $C_3$ and $C_4$ are then determined from the boundary conditions. For instance, the function $g_Y^f(\xi, \eta)$ corresponding to a clamped–free beam for $\beta^4 < 1/r^2s^2$, assumes the form

$$g_Y^f(\xi, \eta) = (\cosh \alpha_1\xi - \cosh \alpha_2\xi)C_1 + \left(\sinh \alpha_1\xi - \frac{a_1}{a_2}\sin \alpha_2\xi\right)C_2 + g_Y^{fp}(\xi - \eta)\mathrm{H}(\xi - \eta), \tag{27}$$

where

$$C_1 = \frac{1}{D}\left[(\alpha_1 \sinh \alpha_1 + \alpha_2 \sin \alpha_2)Z_1(1 - \eta) - \left(\frac{1}{a_1\alpha_1}\cosh \alpha_1 + \frac{1}{a_2\alpha_2}\cos \alpha_2\right)Z_2(1 - \eta)\right],$$

$$C_2 = \frac{1}{D}\left[-\left(\alpha_1 \cosh \alpha_1 + \frac{a_2}{a_1}\alpha_2 \cos \alpha_2\right)Z_1(1 - \eta) + \left(\frac{1}{\alpha_1}\sinh \alpha_1 - \frac{1}{\alpha_2}\sin \alpha_2\right)Z_2(1 - \eta)\right],$$

$$D = (\alpha_1 a_2 - \alpha_2 a_1)\left[2 + \left(\frac{\alpha_1}{\alpha_2} - \frac{\alpha_2}{\alpha_1}\right)\sinh\alpha_1 \sin\alpha_2 + \left(\frac{a_1\alpha_1}{a_2\alpha_2} - \frac{a_2\alpha_2}{a_1\alpha_1}\right)\cosh\alpha_1\cos\alpha_2\right],$$

$$Z_1(\zeta) = \frac{a_2}{\alpha_1}\cosh\alpha_1\zeta + \frac{a_1}{\alpha_2}\cos\alpha_2\zeta \qquad \text{and} \qquad Z_2(\zeta) = a_2(\cosh\alpha_1\zeta - \cos\alpha_2\zeta).$$

If $\beta^4 > 1/r^2s^2$, then the function $g_Y^f(\xi,\eta)$ can be obtained from equation (27) by using the following relations: $\alpha_1 = \mathrm{j}\bar{\alpha}_1$, $\cosh\alpha_1\zeta = \cos\bar{\alpha}_1\zeta$, $\mathrm{j}\sinh\alpha_1\zeta = -\sin\bar{\alpha}_1\zeta$, where $\bar{\alpha}_1 = (1/\sqrt{2})[\beta^4(r^2+s^2) - \sqrt{\Delta}]^{1/2}$ and $\mathrm{j} = \sqrt{-1}$.

It can be noticed that $\lim_{r\to 0} s^2 g_Y^f(\xi,\eta) = -G(\xi,\eta)$, where $G(\xi,\eta)$ is the Green function corresponding to a Bernoulli–Euler beam. These Green functions are stated in references [5, 6].

## 4. DISCUSSION

The natural frequencies of the considered system, $\beta_n$, are obtained from frequency equation (21). The equation for a case when a single oscillator is attached to a beam ($N = 1$), appears in the form

$$s^2 K_{11} Q_{n_1 1} g_Y^f(\xi_1, \xi_1) - 1 = 0, \tag{28}$$

where $Q_{n_1 1}$ for $n_1 = 1, 2$ and 3 (number of degrees of freedom of the oscillator), is given by equation (13), and for arbitrary $n_1$ by equation (14). If $K_{11}$ tends to infinity in equation (28) for $n_1 = 1$, then the frequency equation obtained corresponds to a system of a beam with rigidly attached mass. This equation has the form

$$\beta^4 M_{11} s^2 g_Y^f(\xi_1, \xi_1) + 1 = 0. \tag{29}$$

Moreover, when $M_{11}\to\infty$, then $Q_{11}\to 1$. Taking into this account in equation (28), one obtains the frequency equation for a beam with intermediate elastic support:

$$K_{11} s^2 g_Y^f(\xi_1, \xi_1) - 1 = 0. \tag{30}$$

The frequency equation for $N = 2$ has the form

$$(s^2 K_{11} Q_{n_1 1} g_Y^f(\xi_1,\xi_1) - 1)(s^2 K_{12} Q_{n_2 2} g_Y^f(\xi_2,\xi_2) - 1) - s^4 K_{11} K_{12} Q_{n_1 1} Q_{n_2 2}[g_Y^f(\xi_1,\xi_2)]^2 = 0. \tag{31}$$

Consider now a single oscillator attached at point $\xi_1$ to a clamped–free beam. If $\xi_1 = 0$ (clamped end of the beam), then the oscillator is grounded. Using equations (27) and (28) one finds that the frequency equation for this oscillator is: $1/Q_{n_1 1} = 0$. The equation for the two degree-of-freedom oscillator ($n_1 = 2$), has the form (the second subscript is omitted):

$$\beta^8 - [(\gamma_1 + 1)\Omega_1^4 + \Omega_2^4]\beta^4 + \Omega_1^4\Omega_2^4 = 0. \tag{32}$$

An elastic element or a concentrated mass attached to a beam causes a change of its vibration frequencies. It is well known that a rigidly attached mass decreases the frequencies of a beam, and an elastic support leads to increase of the frequencies. In the case of a beam with an elastically mounted mass, the frequencies higher than the spring–mass frequency are increased, and the lower ones are decreased (except at discrete points when the frequencies are unchanged) [11]. This result can be shown by comparing equation (28) with equations (29) and (30): a spring–mass system added to a beam alternates the frequencies of the original system in the same way as a rigidly attached mass does when $Q_{11} < 0$ or an elastic support when $Q_{11} > 0$. On the basis of equation (13a), $Q_{11} < 0$ for $0 < \beta < \Omega_1$ and $Q_{11} > 1$ for $\beta > \Omega_1$ are obtained. Therefore frequencies lower than $\Omega_1$ are decreased, and those higher than $\Omega_1$ are increased, and this is in agreement with reference [11].

Likewise, a two-degree-of-freedom oscillator attached to a beam at any point causes an alteration of frequencies of a combined system depending on the sign of the expression $Q_{21}$. The inequality $Q_{21} > 0$ is satisfied for $\beta_1^* < \beta < \nu$ and $\beta > \beta_2^*$, where $\beta_{1,2}^* = \sqrt[4]{\frac{1}{2}[\nu^4 + \Omega_1^4 \mp \sqrt{(\nu^4 + \Omega_1^4)^2 - 4\Omega_1^4\Omega_2^4}]}$ are the frequencies of the grounded oscillator (roots of equation (32)) and $\nu = \sqrt[4]{\gamma_1\Omega_1^4 + \Omega_2^4}$. Hence the frequencies of the combined system, $\beta_n$, within the two intervals, $\beta_1^* < \beta_n < \nu$ and $\beta_n > \beta_2^*$, are increased, and the remaining frequencies are decreased (as compared with the frequencies of the beam without an oscillator). Because the condition $\beta_1^* < \nu < \beta_2^*$ for each possible value of $\Omega_1$, $\Omega_2$ and $\gamma_1$, is satisfied, the result obtained includes all cases of a beam with a two-degree-of-freedom oscillator.

## 5. NUMERICAL EXAMPLES

Exemplary numerical results are presented for a cantilever beam with one or two oscillators attached. The numerical calculations comprise the first four frequencies derived from free vibration of the beam and all additional frequencies derived from vibrations of the attached spring–mass systems. The calculations are performed for $r^2 = 0{\cdot}0025$ and $s^2 = 4r^2$. The dimensionless spring constants for all translational springs are assumed the same: $K = 1000$. The dependence of the eigenfrequencies of the combined system on an attachment point of an oscillator to the beam, is presented in Figures 2 and 3.

The results shown in Figure 2(a) are obtained for a beam with a spring–mass system. The vibration frequency of the spring–mass system (grounded oscillator) in this case is $\Omega_{11} = 5{\cdot}0813$ (the point of the dashed line on the $\beta$-axis). The frequencies of the combined system for $\xi_1 > 0$, lower than $\Omega_{11}$ are decreased, and higher than $\Omega_{11}$ are increased as compared with the beam frequencies (the points of the solid lines on the $\beta$-axis). In these and the next figures the solid lines apply to frequencies of the combined system derived

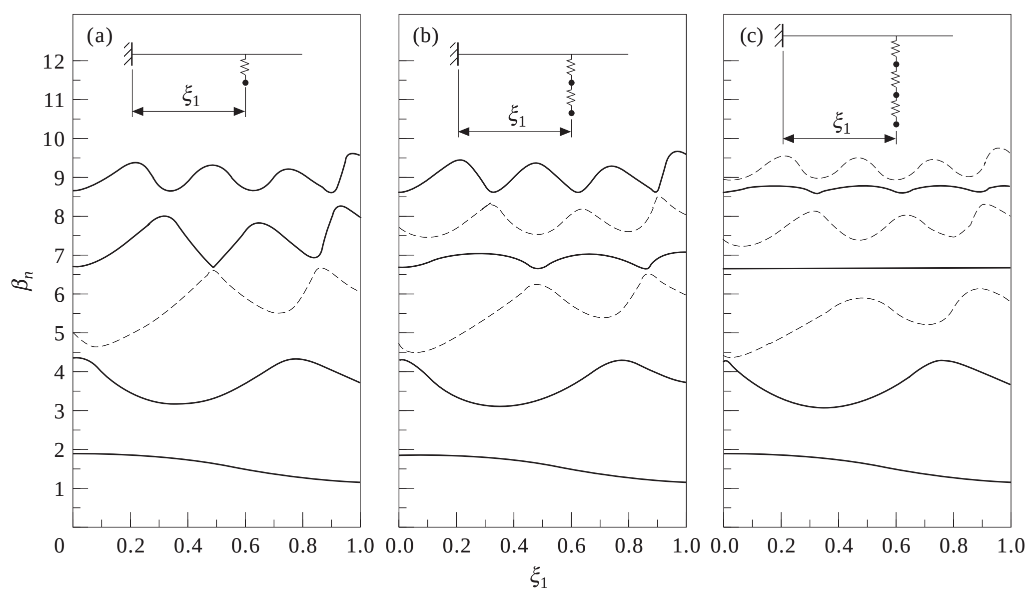


Figure 2. Frequency parameter values, $\beta_n$, for the first modes of vibration of the cantilever versus the location of an oscillator on the beam; (a) one-degree-of-freedom oscillator, $M_{11} = 1{\cdot}5$; (b) two-degree-of-freedom oscillator, $M_{11} = M_{11} = 0{\cdot}75$; (c) three-degree-of-freedom oscillator, $M_{11} = M_{12} = M_{13} = 0{\cdot}5$.

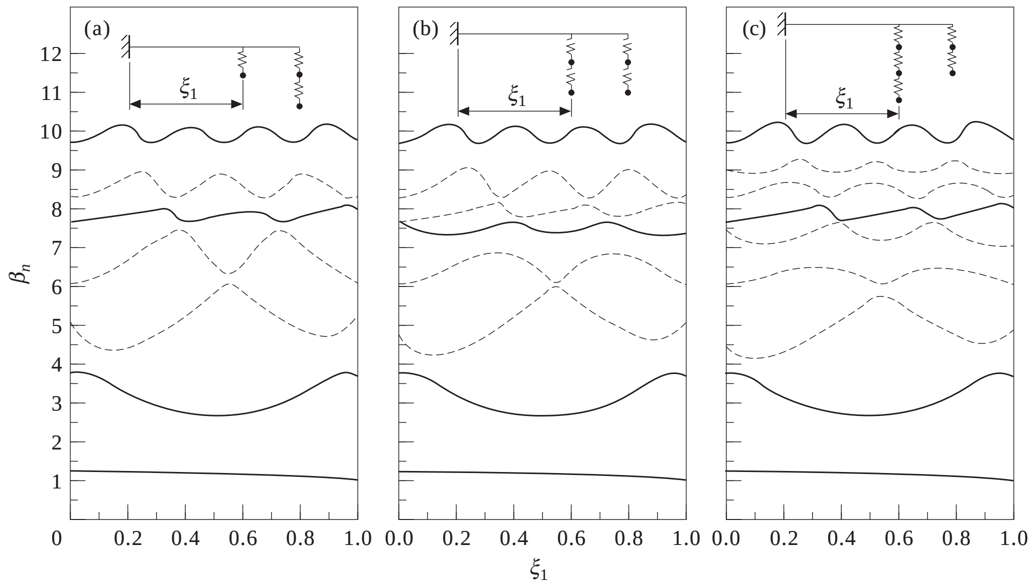


Figure 3. As Figure 2, but for the cantilever beam with two masses oscillator additional mounted at the free end of the beam, $M_{21} = M_{22} = 0{\cdot}5$.

from vibration of the beam and the dashed lines show the changes of the additional frequencies, which are derived from free vibrations of the attached oscillators.

In Figure 2(b), the free vibration frequencies of a combined system consisting of a Timoshenko beam and a two-degree-of-freedom oscillator, are presented. The frequencies of the grounded oscillator, calculated from the equation (32), are $\beta_1^* = 4{\cdot}7501$ and $\beta_2^* = 7{\cdot}6865$. The frequencies of the combined system within either of the two intervals $\beta_1^* < \beta_n < v$ or $\beta_n > \beta_2^*$ ($v = 7{\cdot}1861$), are greater than or equal to the relevant beam frequencies.

The effect of the location of three-degree-of-freedom oscillator attached to the beam on vibration frequencies of the combined system is shown in Figure 2(c). In this case the frequencies of the grounded oscillator are $\beta_1^* = 4{\cdot}4613$, $\beta_2^* = 7{\cdot}4677$ and $\beta_3^* = 8{\cdot}9769$. The inequality $Q_{31} > 0$, is satisfied in the three intervals $\beta_1^* < \beta < v_1$, $\beta_2^* < \beta < v_2$ and $\beta > \beta_3^*$, where $v_1 = 6{\cdot}6874$ and $v_2 = 8{\cdot}8011$. The frequencies of the combined system in these intervals are greater than or equal to the relevant beam frequencies (the remaining frequencies are less than or equal to the relevant beam frequencies).

The vibration frequencies of the system consisting of a beam and two systems of masses attached to it (frequency equation (31)), are shown in Figure 3. The first mass system is established by a one- (Figure 3(a)), two- (Figure 3(b)) or three- (Figure 3(c)) degree-of-freedom oscillator. The other two-mass system is attached to the cantilever at the free end of the beam. The vibration frequencies are shown as functions of the location of the first mass system on the beam.

## 6. CONCLUSIONS

The solution in a closed form of the problem of free vibration of a Timoshenko beam with attached multi-mass oscillators has been presented. Although the number of oscillators considered in the numerical examples was limited to two, the solution can be used for an arbitrary number of oscillators attached to the beam. The solution is obtained by using the Green function method and it includes all possible classical end conditions of the beam.

The oscillators mounted to a beam can cause increases or decreases in the frequencies of the combined system as compared with those of the beam without attached oscillators. In the case of the beam with a spring–mass system attached, the frequencies of the system lower than the spring–mass frequency are decreased, and the higher ones are increased (except at discrete points when the frequencies are unchanged). The frequencies of a combined system of two-mass oscillator attached to a beam, are decreased in the two finite intervals. Similarly, the three-degree-of-freedom oscillator attached to a beam can cause decreases in vibration frequencies that are within any of three finite intervals. The remaining frequencies are increased as compared with the beam frequencies.

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