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DYNAMICS OF DISORDERED STRUCTURES: EFFECT OF NON-LINEARITY ON THE LOCALIZATION

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1. INTRODUCTION

One-dimensional (1-D) periodic structures are ubiquitous in the area pertinent to the modelling of engineering structures. A periodic structure contains a modular repeat unit with a mass and a stiffness called grounding stiffness. In 1-D periodic structures, repeat units are aligned along a line and connected with springs which may be linear or non-linear. It was first shown in solid state physics by Anderson [1] that presence of disorder in periodic lattices results in localized eigenstates. Later, Hodges [2] utilized this finding in structural vibrations. He showed that distortion of the strict periodicity of a given continuous structure with a small disorder results in normal mode localization. The reader is referred to the review by Ibrahim [3] for developments within the last decade.

An important consequence of this phenomenon for practical applications would follow if one were to determine under what circumstances disorder has a similar effect to damping in that the propagation of vibrations from the external source is confined. In other words, deliberate utilization of disorder may act as a passive control mechanism [4]. Along the same line of thought, the similarities and differences between disorder and structural damping were explored by Langley [5] by comparing the attenuation factors produced for each case. Statistical investigations were presented by Pierre and his collaborators [6, 7] and references cited therein. Effects of disorder in multispan beams were also undertaken by Pierre [8] and Bendiksen and his collaborators [9] who had also initiated localization applications in aerospace structures [10] and studied applications on large space structures [11]. Recently, Pierre and his co-worker [12] have taken another path in that they have calculated Lyapunov exponents of the wave transfer matrix of the disordered periodic linear system.

Literature covering studies which are involved in studying effects of non-linear interactions on the localization is relatively meager. Zaslavsky and his collaborators established an analogy between the disorder in particle chains and dynamical problem of transition to chaos [13]. Vakakis and his co-workers [14] used multiple-scales analysis to show the existence of localized modes due to non-linearity in the grounding stiffness in a periodic structure with cyclic boundary conditions. Contemporary developments on the effects of irregularities in structures are gathered by Beneroya [15].

In this study, the authors' aim is to understand the effect of non-linearity on the localization behavior observed for a linear 1-D periodic structure. Non-linearity is introduced in the nearest neighbor coupling with the aid of a fourth order interaction potential. The similitude between the modal analysis of the linearized system and the principal component analysis of the non-linear system [16] is employed. In order to set the stage for the non-linear analysis, modal analysis for the linearized system is performed first. A global localization measure is defined using the difference in the kurtosis of mode shape distributions between the disordered and perfect configurations. For the non-linear case,

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modes are obtained by the principal component analysis of the covariance matrix of instantaneous position vectors which are obtained when the structure is in thermodynamical equilibrium. In what follows, first, the model and the methodology employed in this study are clarified. The results obtained from the calculations are then discussed. The main finding is that non-linearity in the coupling stiffness delocalizes the modes, which are localized due to disorder in the grounding stiffness, associated with the lower frequencies.

2. METHOD

In this study, linear and non-linear dynamics of 1-D periodic and nearly periodic structures (see Figure 1) are investigated. Non-linear behavior of the system is achieved by introducing a quartic nearest neighbor interaction potential between the substructures [17]. In the linear coupling case, modal analysis is performed; whereas in the non-linear coupling case, principal component analysis is utilized. First, the linear analysis technique is outlined; after that the methodology followed for the non-linear analysis is explained.

2.1. *Linear coupling of the substructures*

In this case, the system consists of N substructures with equal masses m, and equal grounding stiffnesses k. The interactions between the substructures are accomplished by linear nearest neighbor interactions. Let the coupling stiffness of this interaction be k_c. Both ends of the chain are fixed. Connections of the substructures to the fixed ends are also achieved by the coupling stiffness k_c. In this case, the total energy of the system may be written as

$$E_L = \sum_{i=1}^{N} \frac{1}{2} \frac{p_i^2}{m} + \sum_{i=1}^{N} \frac{1}{2} k_i q_i^2 + \sum_{i=1}^{N+1} \frac{1}{2} k_{ci} (q_i - q_{i-1})^2, \tag{1}$$

where $q_0 = q_{N+1} = 0$ due to fixed ends. Here q_i and p_i are the displacement and the momentum of the ith substructure, respectively. In the system, disorder is designed in coupling and in grounding stiffnesses by redistributing these stiffnesses with a uniform random distribution about their mean values which are equal to those of the perfect system. The dynamic characteristics, natural frequencies and the mode shapes, of the perfect and disordered linear system may then be obtained by solving the eigenvalue problem

$$(-\omega^2 m \delta_{ij} + K_{ij}) e_j = 0, \tag{2}$$

where ω^2 is the natural frequency, e is the corresponding mode shape, δ_{ij} is the Kronecker symbol and K_{ij} is the $N \times N$ stiffness matrix of the whole structure. The stiffness matrix is tridiagonal whose $(i-1, i)$, (i, i), and $(i, i+1)$th elements are $(-k_{ci})$, $(k_{ci} + k_{ci+1} + k_i)$, and $(-k_{ci+1})$, respectively. For the perfectly periodic system $k_{ci} = k_{ci+1}$ and $k_i = k_{i+1}$.

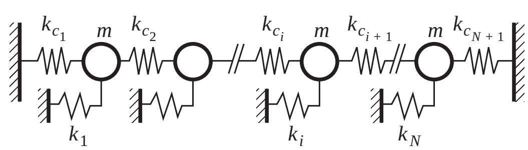

Figure 1. One-dimensional nearly periodic structure with N substructures. Each substructure has a mass m and grounding stiffness k_i. Substructures are connected with linear or non-linear coupling stiffness, k_{ci}.

The complete dynamic behavior of the system is given by

$$q_i(t) = \sum_{j=1}^{N} U_{ij} a_j \cos(\omega_j t + b_j), \tag{3}$$

where U_{ij} is the modal matrix whose columns are e_j; a_j is the amplitude and b_j is the phase of the jth normal mode of the motion.

2.2. *Non-linear coupling of the substructures*

In addition to the harmonic, i.e., second order, nearest neighbor interaction in the potential, an anharmonic, fourth order nearest neighbor interaction between the substructures is considered. The total energy of the system can then be written as

$$E_{NL} = E_L + \sum_{i=1}^{N+1} \frac{1}{4} \alpha_i (q_i - q_{i-1})^4, \tag{4}$$

where $q_0 = q_{N+1} = 0$ and α_i is the coefficient related to the non-linearity. The equations of the motion may be obtained via Hamilton equations, viz.,

$$\dot{q}_i = \partial E_{NL}/\partial p_i, \qquad \dot{p}_i = -\partial E_{NL}/\partial q_i, \tag{5}$$

where the overdot denotes differentiation with respect to time.

Consider the state space description of the system response by constructing the state vector in the form

$$x_I = \begin{pmatrix} q_i \\ p_i \end{pmatrix}, \tag{6}$$

where the capitalized index I takes values from 1–2N; thus, x_I is a $2N \times 1$ column vector characterizing the instantaneous position and the momentum of the ith substructure. The covariance matrix of the state vector is constructed as

$$G_{IJ} = \langle x_I(t) \circ x_J(t) \rangle = \begin{bmatrix} \langle q_i q_j \rangle & \langle q_i p_j \rangle \\ \langle p_i q_j \rangle & \langle p_i p_j \rangle \end{bmatrix}, \tag{7}$$

where G_{IJ} is a $2N \times 2N$ matrix, and the angle bracket and o denote the time average and the outer product, respectively.

To probe the non-linear response of the system when it is in thermodynamical equilibrium, one can excite the system initially with an energy level which is above the equipartition threshold [16]. Non-linearly coupled substructures drive the system to equipartition much more easily than those having non-linearity in the grounding stiffness. In the thermodynamic equilibrium, i.e. when the system satisfies equipartition, equation (7) becomes

$$G_{IJ} = \begin{bmatrix} C_{ij} & 0 \\ 0 & T\delta_{ij} \end{bmatrix}, \tag{8}$$

where T denotes the absolute temperature. The intrinsic structure of C_{ij}, which is the covariance matrix of the instantaneous position vectors, is of interest. One can now apply spectral decomposition [18] to the covariance matrix,

$$C_{ij} = \sum_{k=1}^{N} U_{ik} U_{jk} \Lambda_k^2, \tag{9}$$

where Λ_k^2 are the eigenvalues (principal components) of the covariance matrix C. The analysis associated with the spectral decomposition of the covariance matrix is usually called principal component analysis. In equation (9), U is the eigenvector matrix of the covariance matrix C.

2.3. *A measure for localization*

Let $e_j(i)$ be the component of the eigenvector of the ith substructure for the jth mode. For the non-linear case, $e_j(i)$ is the jth column of the matrix U of equation (9). For the linear system, on the other hand, $e_j(i)$ is the jth eigenvector of the matrix $(-\omega^2 m \delta_{ij} + K_{ij})$ of equation (2). It can be shown that columns of U coincide the eigenvectors of the linear system inasmuch as the trajectories of the linearized system are employed in equation (9). Using equations (3) and (7), one can find the covariance matrix of the displacements, C_{ij}, as

$$\langle q_i(t) \circ q_j(t) \rangle = \frac{1}{2} \sum_{k=1}^{N} U_{ik} U_{jk} a_k^2. \tag{10}$$

The similarity between equations (9) and (10) is noteworthy. They are identical if one sets $\Lambda_k^2 = \frac{1}{2} a_k^2$. Thus, if one takes the displacements or momenta from the linear case and constructs the covariance matrix, the principal component analysis of that matrix results in the eigenvectors which can be calculated by equation (2). Since the decomposition is unique, the eigenvalues (or principal components) turn out to be half of the amplitude squares for each mode. Since the distribution of the initially excited modes is conserved during the time evolution of the structure for the linear case, only excited modes can be obtained by using the principal component analysis. For the non-linear system in thermodynamical equilibrium, however, all of the modes can be probed uniquely by the principal component analysis.

One can define the localization of the jth mode as the difference between the kurtosis of the disordered and the perfect structure,

$$L_j = \kappa_j(\text{disordered}) - \kappa_j(\text{perfect}), \tag{11}$$

where the kurtosis is defined as

$$\kappa_j = \sum_{i=1}^{N} \left(\frac{i - \mu_j}{\sigma_j} \right)^4 |e_j(i)| \Big/ \sum_{i=1}^{N} |e_j(i)|, \tag{12}$$

wherein μ_j and σ_j are the mean and the standard deviation, respectively, and are given in the following forms:

$$\mu_j = \sum_{i=1}^{N} i |e_j(i)| \Big/ \sum_{i=1}^{N} |e_j(i)|, \qquad \sigma_j = \left(\sum_{i=1}^{N} (i - \mu_i)^2 |e_j(i)| \Big/ \sum_{i=1}^{N} |e_j(i)| \right)^{1/2}. \tag{13}$$

3. CALCULATIONS AND RESULTS

A 1-D chain consisting of 16 substructures is considered. Both ends of the chain are fixed. Susceptibility of the structure to localization depends on the degree of coupling between the substructures. Weak coupling (low k_c) results in strong localization manifested in the rearrangement of the displacements for each substructure as may be observed from the eigenvectors of the system. In order to create strong localization, the grounding stiffness, k, and the mass, m, are set to unity, while the coupling stiffness, $k_c = 0{\cdot}01$.

Two different disordered configurations are designed by uniform random distribution of the grounding or coupling stiffnesses about their mean which are the same as the perfect case. In the first configuration, the randomness is given to the coupling stiffness, k_c. The second configuration contains the random distribution for the grounding stiffness k. The following ratio is defined so as to measure the degree of randomness: $\sigma_{kc}/\mu(k_c)$. Here σ is the standard deviation with subscripts designating the variable to which randomness is built in, and $\mu(k_c)$ denotes the mean of a variable that is between the parenthesis following μ. For the first configuration, this ratio is equal to 0·213. In the second case, the numerical value of the ratio for the first case is preserved satisfying $\sigma_{kc}/\mu(k_c) = \sigma_k/\mu(k_c)$. The reason for observing this equality is the following: the results indicated that the dynamic characteristics found are insensitive to changes that may be made in the mean value of the grounding stiffness or that of the mass of the substructure (excluding the differences with three orders of magnitude).

The dynamic characteristics, natural frequencies and the associated mode shapes, both of the perfect and of the disordered chains are obtained for the linear case. To create the non-linear configuration, a small non-linearity with a fourth order potential between the nearest neighbors is added to the energy, which is given by equation (4). The coefficient pertinent to the non-linearity, α, is selected to be one order of magnitude less than the coupling stiffness k_c so that $\alpha_i = 0{\cdot}1k_{ci}$. The non-linear system with 16 substructures is integrated numerically by the Bulirsch–Stoer method [19]. Initial energy is selected in such a way that the thermodynamical equilibrium (equipartition) is accomplished. It is made sure that covariance matrix for the momenta satisfies equation (8), resulting in the identity matrix times the absolute temperature. With this integration scheme, the total energy of the system is preserved up to eight digits during the simulation at each time step. The system is integrated until the equipartition is reached. Thus, the non-linear analysis is performed on the equilibrated structure.

In Figure 2, eigenvectors for the lowest and the highest modes are displayed. The eigenvectors for the linearly and non-linearly coupled structures are obtained from equations (2) and (9), respectively. It is seen that for the first case where the randomness is in the coupling stiffness, the first mode of the disordered configuration is not altered from that of the perfect configuration for the linear coupling. However, in the last mode the eigenvector spans only the first half of the structure. And this picture is almost conserved for the non-linear coupling; the only difference is that the eigenvector associated with the last mode of the disordered configuration extends to the second half of the beam with very small amplitude rather than zero. For the second case, where the randomness is in the grounding stiffness, both the first and the last modes are distorted for the disordered configuration in the linear coupling case. The non-linearity in the coupling removes the distortion in the first mode totally; and extends the distribution to the whole structure similar to the perfect configuration. Figure 2 summarizes the situation for the first and the last modes. In order to understand the changes occurred in all of the modes, one can utilize the measure defined by equation (12).

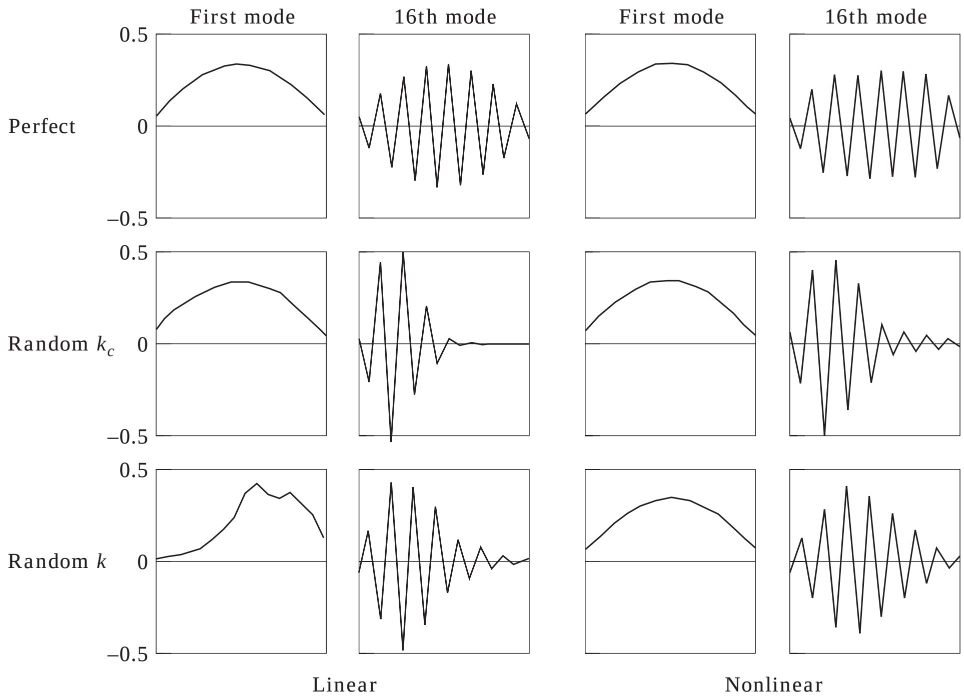

Figure 2. Eigenvector distributions for the lowest and the highest modes of the structure are displayed. The eigenvectors for the linearly and non-linearly coupled substructures are obtained from equations (2) and (9), respectively.

In Figure 3, L_i, the localization defined by equation (12) is shown for the linear and non-linear cases. Dashed and solid lines denote the linear and the non-linear cases, respectively. Here the disorder is in the coupling stiffness with $\sigma_{kc}/\mu(k_c) = 0{\cdot}213$. It may be observed for the linear coupling (dashed line) that the lower modes are not changed up to half of the modal spectrum. After that a pronounced increase with a linear trend in the localization is detected. The additional small non-linearity does not change this

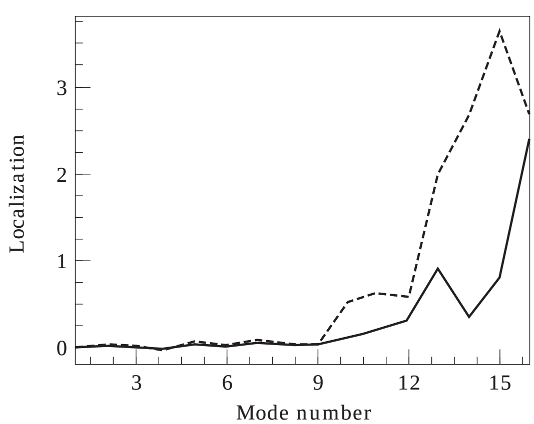

Figure 3. The localization (calculated by equation (12)) due to disorder in the coupling stiffnesses is shown for the linear and non-linear cases. Dashed and solid lines denote the linear and the non-linear configurations, respectively.

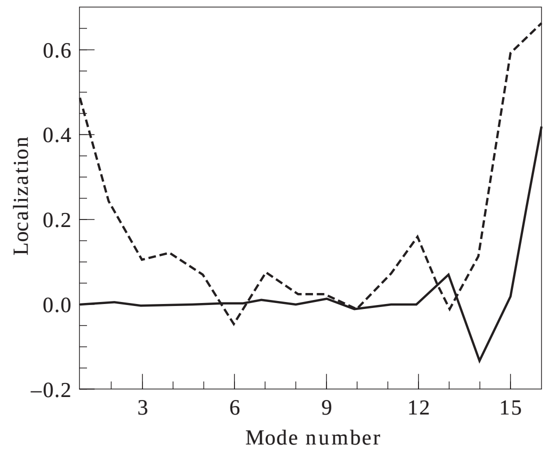

Figure 4. The localization (calculated by equation (12)) due to disorder in the grounding stiffnesses is shown for the linear and non-linear cases. Dashed and solid lines denote the linear and the non-linear configurations, respectively.

observation qualitatively. In the non-linear coupling, the amount of localization is reduced. Figure 4 illustrates the effect of coupling non-linearity on the localization obtained by configuring disorder in the grounding stiffness. Dashed and solid lines denote the linear and the non-linear cases, respectively. It is seen that an almost symmetric localization distribution prevails for the linear configuration, collectively distorting the lower and uppermost parts of the modal spectrum. The non-linearity introduced to the coupling breaks this symmetry and leaves the localization only at the very end of the modal spectrum.

The results displayed in Figures 3 and 4 are obtained by averaging over five different configurations for each curve. Different configurations are created by taking five different seed numbers while distributing uniform random variables for the stiffnesses in both cases. Note that for each configuration the σ/μ ratio is still constant. It may be recognized that perfect non-linear coupling between the substructures delocalizes the modes at the lower end of the spectrum which are created by the random grounding stiffness configuration with linear coupling of substructures. Not shown in this work, the same qualitative picture as in Figure 3 is also obtained when random distribution is considered for the mass of substructures. Also not displayed here, it is also found that longer chains with higher number of substructures behave similarly.

4. CONCLUDING REMARKS

In this study, a localization measure is introduced to quantify the differences in dynamic behavior between perfectly periodic and nearly periodic structures in which a small randomness is designed. For each mode, this measure evaluates the distortion of the mode shape of the disordered structure from that of the perfect one by calculating the kurtosis of the eigenvector distributions. In the linear case, the eigenvalue problem of the characteristic matrix needs to be solved; for the non-linear case, the covariance matrix of the instantaneous state vectors is required to be calculated. While the covariances of instantaneous momenta probe the equipartition, those of the instantaneous position vectors identify the modes associated with the thermodynamical equilibrium.

The results have indicated that non-linear nearest neighbor interactions delocalize the modes corresponding to lower frequencies. The most pronounced delocalization due to non-linearity occurs if the localized modes for the linear structure are obtained by designing for disorder in the grounding stiffness. Since displacements with larger amplitudes are associated with lower modes, obstruction of the collective behavior of these modes with the higher frequencies may be of interest.

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