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FUNDAMENTAL FREQUENCIES OF RECTANGULAR PLATES WITH LINEARLY VARYING THICKNESS

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1. INTRODUCTION

Rectangular plates with two opposite simply supported edges and linearly varying thickness only in one direction are studied, and the natural frequencies of the transverse vibration are investigated.

First, the modal function of the deflection is assumed to be the product of two functions of one variable, and a general solution of the ordinary differential equation for the eigenvalue problem has been obtained by making use of the power series [1].

Fundamental frequencies obtained in the case of a rectangular plate with four simply supported edges have come to be between the two bounds shown by Appl and Byers [2] with only one exception. For the practical use of the readers, fundamental frequencies for various supporting conditions are computed and presented in the tables.

2. EQUATION OF MOTION AND THE SOLUTION

On the central plane of the plate are set x, y co-ordinates and the deflection in z direction is expressed by w. The equation of motion in the case of uni-tapered thickness in the x direction only is [3]

$$D\nabla^4 w + 2\frac{\mathrm{d}D}{\mathrm{d}x}\frac{\partial}{\partial x}\nabla^2 w + \frac{\mathrm{d}^2D}{\mathrm{d}x^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + \rho h\frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where $D = Eh^3/12(1-\nu^2)$, E is Young's modulus, h is the plate thickness, ν is Poisson's ratio, ρ is density, t is time and $\nabla^4 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)^2$, $\nabla^2 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)$.

Edges at $y = 0$ and $y = b$ are simply supported, and the solution $w(x, y, t)$ of equation (1) is assumed to take the form of

$$w(x, y, t) = X(x)\sin([n\pi/b]y)\exp(\mathrm{i}\omega t), \tag{2}$$

where b is the plate length in the y direction, ω is the circular frequency, i the imaginary unit and $n = 1, 2, 3.\ldots.$

Substituting expression (2) into equation (1), the following ordinary differential equation is introduced

$$(1-\xi)^2\,\mathrm{d}^4X/\mathrm{d}\xi^4 - 6(1-\xi)\,\mathrm{d}^3X/\mathrm{d}\xi^3 + [6 - 2(n\pi\delta/c)^2(1-\xi)^2]\,d^2X/\mathrm{d}\xi^2$$
$$+ 6(n\pi\delta/c)^2(1-\xi)\,\mathrm{d}X/\mathrm{d}\xi + \{(n\pi\delta/c)^4(1-\xi)^2 - [\lambda_0/c^4 + 6\nu(n\pi\delta/c)^2]\}X = 0, \tag{3}$$

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where $\xi = (c/a)x$, $c = (h_0 - h_1)/h_0(h_0 > h_1)$, $h(x) = h_0(1 - (c/a)x)$, $h_0 = h(0)$, $h_1 = h(a)$, $\lambda_0 = \rho h_0 a^4 \omega^2 / D_0$, $D_0 = Eh_0^3/12(1 - \nu^2)$, $\delta = a/b$, a is the plate length in the x direction.

The solution $X(\xi)$ of equation (3) can be expressed as the following power series [4]

$$X(\xi) = \sum_{i=0}^{3} C_i g_i(\xi), \tag{4}$$

where

$$g_i(\xi) = \xi^i + \sum_{j=4}^{\infty} G_{ij}\xi^j, \tag{5}$$

TABLE 1

Fundamental frequency parameters $\sqrt{\lambda_0^*} = \sqrt{\rho h_0^* a^4 \omega^2 / D_0^*}$ *for simply supported rectangular plates with linearly varying thickness*; $\nu = 0{\cdot}3$

a/b	α	Akiyama and Kuroda $\sqrt{\lambda_0^*}$	Apply and Byers $\sqrt{R_{max}}$	$\sqrt{R_{min}}$	a/b	Akiyama and Kuroda $\sqrt{\lambda_0^*}$	Appl and Byers $\sqrt{R_{max}}$	$\sqrt{R_{min}}$
0·25	0·1	11·00370	11·00509	11·00305	1·25	26·54719	26·54931	26·54611
	0·2	11·50805	11·50813	11·50800		27·78851	27·80288	27·78100
	0·3	12·00109	12·00600	11·99133		29·01667	29·01782	29·01524
	0·4	12·48409	12·48595	12·48311		30·23312	30·24249	30·22611
	0·5	12·95812	12·95859	12·95728		31·43907	31·44711	31·42629
	0·6	13·42407	13·42546	13·42181		32·63554	32·65289	32·61024
	0·7	13·88268	13·88730	13·87754		33·82336	33·86717	33·76064
	0·8	14·33460	14·35427	14·31604		35·00328	35·07034	34·91215
0·50	0·1	12·94822	12·94824	12·94820	1·50	33·66677	33·67000	33·66509
	0·2	13·54915	13·54935	13·54906		35·23316	35·25022	35·19943
	0·3	14·14105	14·14139	14·14086		36·77837	36·79249	36·75018
	0·4	14·72496	14·76150	14·70585		38·30481	38·31654	38·27858
	0·5	15·30170	15·31218	15·29601		39·81449	39·83326	39·78875
	0·6	15·87198	15·87848	15·86704		41·30907	41·34735	41·25150
	0·7	16·43640	16·44366	16·42812		42·78998	42·86956	42·66874
	0·8	16·99547	17·00932	16·97810		44·25841	44·35729	44·14320
0·75	0·1	16·18751	16·18799	16·18651	1·75	42·07903	42·09013	42·07325
	0·2	16·94498	16·94698	16·94397		44·02451	44·04263	43·98856
	0·3	17·69473	17·70099	17·69150		45·93650	45·94537	45·92292
	0·4	18·43762	18·44961	18·43144		47·81893	47·83689	47·79478
	0·5	19·17434	19·17571	19·17266		49·67502	49·71280	49·61782
	0·6	19·90549	19·91246	19·89032		51·50751	51·58178	51·39129
	0·7	20·63154	20·64127	20·61856		53·31871	53·38730	53·21404
	0·8	21·35293	21·38684	21·31288		55·11058	55·27124	55·05512
1·00	0·1	20·72065	20·72074	20·72050	2·00	51·78344	51·82503	51·76145
	0·2	21·69203	21·69230	21·69150		54·16063	54·17898	54·13811
	0·3	22·65455	22·65787	22·64796		56·48697	56·51058	56·45699
	0·4	23·60919	23·61813	23·59180		58·76856	58·80906	58·70824
	0·5	24·55673	24·55972	24·55282		61·01046	61·11308	60·85006
	0·6	25·49784	25·50614	25·48607		63·21692	63·34341	63·08581
	0·7	26·43308	26·45245	26·40521		65·39157	65·45104	65·31778
	0·8	27·36293	27·41197	27·29489		67·53748	67·49966	67·37930

$\lambda_0^* = (1 + \alpha)^2 \lambda_0$, $h_0^* = (1 - c)h_0$, $D_0^* = (1 - c)^3 D_0$, $c = \alpha/(1 + \alpha)$; α = Taper ratio in reference [2]; R_{max} = Upper bounds of fundamental frequency in reference [2]; R_{min} = Lower bounds of fundamental frequency in reference [2].

TABLE 2

Fundamental frequency parameters $\sqrt{\lambda_0^*} = \sqrt{\rho h_0^* a^4 \omega^2 / D_0^*}$ *for rectangular plates with linearly varying thickness*; $\nu = 0{\cdot}3$

a/b	α	C-S-C-S	C-S-S-S	C-S-F-S	S-S-F-S
0·25	0·1	23·84148	16·86728	4·43817	1·82312
	0·2	24·94212	17·83505	4·83487	1·92997
	0·3	26·02277	18·78887	5·23776	2·04037
	0·4	27·08570	19·73031	5·64627	2·15400
	0·5	28·13279	20·66066	6·05993	2·27057
	0·6	29·16559	21·58100	6·47831	2·38983
	0·7	30·18543	22·49225	6·90107	2·51154
	0·8	31·19344	23·39519	7·32790	2·63549
	0·9	32·19059	24·29050	7·75853	2·76149
	1·0	33·17772	25·17875	8·19273	2·88937
0·50	0·1	24·99344	18·35139	6·10057	4·24237
	0·2	26·14760	19·35584	6·50544	4·45697
	0·3	27·28102	20·34690	6·91783	4·67720
	0·4	28·39604	21·32604	7·33717	4·90273
	0·5	29·49461	22·29447	7·76291	5·13321
	0·6	30·57835	23·25321	8·19460	5·36833
	0·7	31·64864	24·20311	8·63179	5·60778
	0·8	32·70664	25·14492	9·07411	5·85124
	0·9	33·75337	26·07928	9·52121	6·09847
	1·0	34·78971	27·00675	9·97278	6·34918
0·75	0·1	27·10593	20·97607	9·00868	7·62704
	0·2	28·35814	22·05559	9·45389	7·95789
	0·3	29·58814	23·12166	9·90555	8·29449
	0·4	30·79845	24·17574	10·36350	8·63672
	0·5	31·99114	25·21904	10·82750	8·98439
	0·6	33·16796	26·25258	11·29733	9·33730
	0·7	34·33037	27·27722	11·77273	9·69522
	0·8	35·47963	28·29370	12·25345	10·05792
	0·9	36·61683	29·30267	12·73924	10·42517
	1·0	37·74291	30·30468	13·22986	10·79675
1·00	0·1	30·38297	24·87009	13·20802	12·14486
	0·2	31·78695	26·07669	13·73194	12·60847
	0·3	33·16625	27·26823	14·25965	13·07580
	0·4	34·52367	28·44641	14·79149	13·54713
	0·5	35·86152	29·61263	15·32768	14·02263
	0·6	37·18174	30·76806	15·86832	14·50236
	0·7	38·48596	31·91368	16·41344	14·98635
	0·8	39·77557	33·05033	16·96305	15·47457
	0·9	41·05178	34·17874	17·51709	15·96696
	1·0	42·31564	35·29954	18·07548	16·46344
1·25	0·1	34·99035	30·11153	18·70368	17·86460
	0·2	36·60699	31·50651	19·34241	18·48184
	0·3	38·19508	32·88261	19·98086	19·09816
	0·4	39·75786	34·24210	20·62005	19·71463
	0·5	41·29799	35·58680	21·26073	20·33203
	0·6	42·81772	36·91823	21·90347	20·95096
	0·7	44·31893	38·23767	22·54869	21·57188
	0·8	45·80325	39·54618	23·19669	22·19512
	0·9	47·27207	40·84471	23·84770	22·82093
	1·0	48·72660	42·13405	24·50186	23·44950

continued overleaf

TABLE 2—*continued*

a/b	α	C-S-C-S	C-S-S-S	C-S-F-S	S-S-F-S
1·50	0·1	41·02223	36·72863	25·48943	24·81510
	0·2	42·91604	38·37684	26·27543	25·60803
	0·3	44·77548	39·99966	27·05502	26·39216
	0·4	46·60447	41·60022	27·83027	27·16993
	0·5	48·40623	43·18109	28·60276	27·94325
	0·6	50·18344	44·74436	29·37373	28·71359
	0·7	51·93840	46·29180	30·14412	29·48212
	0·8	53·67303	47·82491	30·91469	30·24975
	0·9	55·38904	49·34496	31·68604	31·01722
	1·0	57·08787	50·85304	32·45863	31·78512
1·75	0·1	48·50966	44·72089	33·55788	33·00904
	0·2	50·74553	46·68672	34·52003	33·99904
	0·3	52·93872	48·61735	35·46672	34·96839
	0·4	55·09407	50·51721	36·40169	35·92190
	0·5	57·21560	52·38992	37·32782	36·86325
	0·6	59·30667	54·23848	38·24733	37·79529
	0·7	61·37011	56·06537	39·16197	38·72025
	0·8	63·40835	57·87271	40·07315	39·63991
	0·9	65·42349	59·66230	40·98198	40·55569
	1·0	67·41735	61·43569	41·88936	41·46872
2·00	0·1	57·44627	54·07660	42·90268	42·45211
	0·2	60·08758	56·42207	44·06647	43·65887
	0·3	62·67462	58·71848	45·20194	44·82854
	0·4	65·21359	60·97211	46·31547	45·96968
	0·5	67·70963	63·18807	47·41188	47·08864
	0·6	70·16701	65·37061	48·49488	48·19026
	0·7	72·58938	67·52325	49·56738	49·27826
	0·8	74·97983	69·64900	50·63166	50·35557
	0·9	77·34104	71·75040	51·68957	51·42450
	1·0	79·67535	73·82964	52·74259	52·48689

$$G_{i,j+4} = -\frac{1}{(j+4)_4 P_0}\{T_0 G_{i,j-2} + T_1 G_{i,j-1} + (T_0 + jS_1 + j_2 R_2)G_{ij} + [(j+1)S_0 + (j+1)_2 R_1]G_{i,j+1} + [(j+2)_2 R_0 + (j+2)_3 Q_1 + (j+2)_4 P_2]G_{i,j+2} + [(j+3)_3 Q_0 + (j+3)_4 P_1]G_{i,j+3}\}, \quad (j \geqslant 0)\}; \tag{6}$$

$$G_{i,-2} = G_{i,-1} = 0, \qquad (j+n)_m = (j+n)(j+n-1), \ldots, (j+n-m+1) \tag{7}$$

$$P_0 = 1,\ P_1 = -2,\ P_2 = 1,\ Q_0 = -6,\ Q_1 = 6,\ R_0 = 6 - 2s,\ R_1 = 4s,\ R_2 = -2s,$$

$$S_0 = 6s,\ S_1 = -6s,\ T_0 = s^2 - (\lambda_0/c^4 + 6\nu s),\ T_1 = -2s^2,\ T_2 = s^2,\ s = (n\pi\delta/c)^2$$

G_{0j}, G_{1j}, G_{2j} and G_{3j} are successively determined according to equation (6) by giving the following four first values respectively

$$\begin{aligned} G_{0j}&: \quad G_{00} = 1, \quad G_{01} = G_{02} = G_{03} = 0 \\ G_{1j}&: \quad G_{11} = 1, \quad G_{10} = G_{12} = G_{13} = 0 \\ G_{2j}&: \quad G_{22} = 1, \quad G_{20} = G_{21} = G_{23} = 0 \\ G_{3j}&: \quad G_{33} = 1, \quad G_{30} = G_{31} = G_{32} = 0 \end{aligned} \tag{8}$$

3. NUMERICAL COMPUTATION

Table 1 shows the comparison among fundamental frequencies obtained by the present power series solution for the case of four simply supported edges and those by Appl and Byers. Table 2 gives the computed fundamental frequencies for various supporting conditions.

4. CONCLUSIONS

The present analytical solution gives exact solutions as far as the form of the solution of equation (2) is concerned. There is no room for errors to occur in the course of numerical computation. In other words, numerical values of the fundamental frequencies obtained by the power series solution and given in Table 1 and Table 2 can be said to be the standards for all other computed values.

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