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VIBRATION ANALYSIS OF CIRCULAR MINDLIN PLATES USING THE DIFFERENTIAL QUADRATURE METHOD

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Axisymmetric free vibrations of moderately thick circular plates described by the linear shear-deformation Mindlin theory are analyzed by the differential quadrature (DQ) method. The first fifteen natural frequencies of vibration are calculated for uniform circular plates with free, simply-supported and clamped edges. Through these computations, the capability and simplicity of the differential quadrature method for moderately thick plate eigenvalue analysis is demonstrated, and convergence and accuracy are thoughtfully examined. The case of a rigid point support at the plate centre is also considered in the present paper, for which special attention is paid to the capability and convergence of the current method.

1. INTRODUCTION

As kinds of basic structural components, thin and moderately thick circular plates are extensively used in mechanical, civil, nuclear and aerospace structures. Considerable studies have been reported in the open literature on the axisymmetric vibration analysis of circular plates, e.g., [1–7]. Excellent reviews have been made by Leissa [8–14]. Recently, the differential quadrature (DQ) method [15–19] has been applied to free vibration analysis of circular and annular thin plates with uniform or non-uniform thickness [20–24]; these applications concentrate mainly on the fundamental frequency.

The differential quadrature method is a rather efficient numerical method for the rapid solution of linear and non-linear partial differential equations. It was originated by Bellman and Casti [15] and Bellman *et al*. [16], and generalized and simplified further by Quan and Chang [17, 18] and Shu and Richards [19] through introducing simple algebraic expressions to calculate directly the weighting coefficients associated with derivatives. Thanks to the efforts of Bert *et al*. [25, 26], Striz *et al*. [27], Sherbourne and Pandey [28], and Kukreti *et al*. [29], the method is becoming increasingly popular in the solution of bending, buckling and free vibration problems of structures. In all the aforementioned studies [15–29], the DQ method appears to be a potential alternative to the conventional numerical approaches, and has been claimed to have the capability of yielding highly accurate solutions to initial and boundary value problems with minimal computational effort.

In view of the fact that no publications are concerned with the free vibration analysis of moderately thick plates using the DQ method, in this paper, the method is employed to analyze the axisymmetric free vibration of moderately thick circular Mindlin plates with free, simply-supported and clamped edges. The first fifteen natural frequencies of the plates

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are calculated. The capability and simplicity of the DQ method in moderately thick plate eigenvalue analysis are demonstrated through these studies. The accuracy and convergence of the method for the vibration analysis of moderately thick plates are investigated through directly comparing DQ results with corresponding exact solutions in the open literature. A particular advantage involved in the present solution procedures, i.e., using the simplified version of the DQ method along with the Mindlin plate theory, is noted. The case of a rigid point support at the plate centre is also included in the present investigation. The capability and convergence characteristics of the method for this particular problem are explored.

2. MATHEMATICAL FORMULATIONS

2.1. GOVERNING EQUATIONS

Consider a circular plate of radius a and thickness h (Figure 1). The equations governing the axisymmetric free vibration of a uniform circular plate of isotopic material can be derived using Mindlin's theory [30] as [31, 32]:

$$D\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\psi}{r^2}\right)-\kappa Gh\left(\frac{\partial w}{\partial r}+\psi\right)-\frac{\rho h^3}{12}\frac{\partial^2\psi}{\partial t^2}=0, \tag{1a}$$

$$\kappa Gh\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial\psi}{\partial r}+\frac{\psi}{r}\right)-\rho h\frac{\partial^2 w}{\partial t^2}=0, \tag{1b}$$

where w is the transverse deflection; ψ is the rotation of the normal about the r-axis; $D = Eh^3/[12(1-v^2)]$, E, G and v are the plate flexural rigidity, Young's modulus, shear modulus and Poisson's ratio, respectively; ρ and κ are the density of the plate material and the shear correction factor respectively.

According to the relationship between force resultants and deformation variables, the following formulae are obtained:

$$M_r = D\left(\frac{\partial\psi}{\partial r}+v\frac{\psi}{r}\right); \qquad M_\theta = D\left(v\frac{\partial\psi}{\partial r}+\frac{\psi}{r}\right); \qquad Q_r=\kappa Gh\left(\frac{\partial w}{\partial r}+\psi\right), \tag{2}$$

in which M_r, M_θ and Q_r are the moment resultants and the shear resultant.

Using the following non-dimensional parameters and relation:

$$R = r/a; \qquad \delta = h/a; \qquad W = w/a; \qquad \psi=\psi; \qquad T = t\sqrt{E/\rho a^2(1-v^2)}; \tag{3}$$

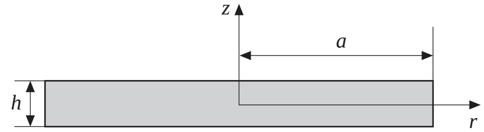

Figure 1. Configuration of a circular thick plate.

the governing equations given by equation (1) can be normalized as:

$$\delta^2\left(R^2\frac{\partial^2\Psi}{\partial R^2}+R\frac{\partial\Psi}{\partial R}-\Psi\right)-6\kappa(1-\nu)R^2\left(\Psi+\frac{\partial W}{\partial R}\right)-\delta^2R^2\frac{\partial^2\Psi}{\partial T^2}=0, \tag{4a}$$

$$R\frac{\partial^2W}{\partial R^2}+\frac{\partial W}{\partial R}+R\frac{\partial\Psi}{\partial R}+\Psi-\frac{2R}{(1-\nu)\kappa}\frac{\partial^2W}{\partial T^2}=0, \tag{4b}$$

and the stress–displacement relationships are given by:

$$M_r=\frac{D}{a}\left[\frac{\partial\Psi}{\partial R}+\frac{\nu}{R}\Psi\right];\qquad M_\theta=\frac{D}{a}\left[\nu\frac{\partial\Psi}{\partial R}+\frac{\Psi}{R}\right];\qquad Q_r=\kappa Gh\left[\Psi+\frac{\partial W}{\partial R}\right]. \tag{5}$$

For free vibration, the solution can be assumed as:

$$W(R,T)=W_j(R)e^{\mathrm{i}\Omega_jT}\qquad\text{and}\qquad\Psi(R,T)=\Psi_j(R)e^{\mathrm{i}\Omega_jT}. \tag{6}$$

Substitution of these solutions into the homogeneous differential equations leads to

$$\delta^2\left(R^2\frac{\mathrm{d}^2\Psi}{\mathrm{d}R^2}+R\frac{\mathrm{d}\Psi}{\mathrm{d}R}-\Psi\right)-6\kappa(1-\nu)R^2\left(\Psi+\frac{\mathrm{d}W}{\mathrm{d}R}\right)+\delta^2R^2\Omega^2\Psi=0, \tag{7a}$$

$$R\frac{\mathrm{d}^2W}{\mathrm{d}R^2}+\frac{\mathrm{d}W}{\mathrm{d}R}+R\frac{\mathrm{d}\Psi}{\mathrm{d}R}+\Psi+\frac{2R\Omega^2}{(1-\nu)\kappa}W=0, \tag{7b}$$

W, Ψ and Ω, should have been taken as $W_j(R)$, $\Psi_j(R)$ and Ω_j of equation (6), respectively for the jth mode of vibration. Here and in the following, the suffix j is dropped for the sake of convenience.

According to the DQ procedure (refer to the Appendix for details) and by setting $R_1=0$ and $R_N=1$, equations (7) take the following discrete forms:

$$\delta^2\sum_{k=1}^{N}(C_{ik}^{(2)}R_i^2+C_{ik}^{(1)}R_i)\Psi_k-[\delta^2+6\kappa(1-\nu)R_i^2]\Psi_i$$

$$-6\kappa(1-\nu)R_i^2\sum_{k=1}^{N}C_{ik}^{(1)}W_k+\delta^2R_i^2\Omega^2\Psi_i=0, \tag{8a}$$

$$\sum_{k=1}^{N}(C_{ik}^{(2)}R_i+C_{ik}^{(1)})W_k+\Psi_i+R_i\sum_{k=1}^{N}C_{ik}^{(1)}\Psi_k+\frac{2R_i}{(1-\nu)\kappa}\Omega^2W_i=0, \tag{8b}$$

where $i=1,2,\ldots,N$. $C_{rs}^{(n)}$, determined by equations (A2)–(A5), are the weighting coefficients for the nth order derivatives of W and Ψ with respect to R.

2.2. BOUNDARY CONDITIONS

For the edge of the circular plate, the boundary conditions can be divided into the following three kinds:

(1) Clamped edge (C): $w=0$; $\psi=0$, (9)

(2) Simply–supported edge (S): $w=0$; $M_r=0$, (10)

(3) Free edge (F): $Q_r=0$; $M_r=0$. (11)

These conditions can be further expressed as:

$$\text{(C)}\ \ W = 0; \qquad \Psi = 0, \tag{12}$$

$$\text{(S)}\ \ W = 0; \qquad \partial\Psi/\partial R + (v/R)\Psi = 0, \tag{13}$$

$$\text{(F)}\ \ \Psi + \partial W/\partial R = 0; \qquad \partial\Psi/\partial R + (v/R)\Psi = 0, \tag{14}$$

Thus, the discretized forms on the edge of the plate are:

$$\text{(C)}\ \ W_N = 0; \qquad \Psi_N = 0, \tag{15}$$

$$\text{(S)}\ \ W_N = 0; \qquad \sum_{k=1}^{N} C_{Nk}^{(1)} \Psi_k + v\Psi_N = 0, \tag{16}$$

$$\text{(F)}\ \ \Psi_N + \sum_{k=1}^{N} C_{Nk}^{(1)} W_k = 0; \qquad \sum_{k=1}^{N} C_{Nk}^{(1)} \Psi_k + v\Psi_N = 0. \tag{17}$$

For the central point ($r = 0$), the restraint conditions and their discretized forms are:

(4) Regularity conditions (R) $\quad Q_r = 0; \quad \psi = 0.$ (18)

(5) Rigid centre support (C) $\quad w = 0; \quad \psi = 0.$ (19)

and

$$\text{(R)}\ \ \Psi_1 + \sum_{k=1}^{N} C_{1k}^{(1)} W_k = 0; \qquad \Psi_1 = 0, \tag{20}$$

$$\text{(C)}\ \ W_1 = 0; \qquad \Psi_1 = 0. \tag{21}$$

2.3. SOLUTION PROCEDURES

Since both the discretized governing equations and the discretized constraint conditions are written out on a point-wise basis, for the boundary and centre points, the discretized governing equations and the discretized constraint conditions should be satisfied simultaneously. In order to get solutions of the problem, however, one has to use the discretized constraint conditions instead of the discretized governing equations on both the boundary and the centre points. Thus, the solutions of the problems are acquired by solving the set of secular equations which consists of $2 \times (N - 2)$ governing equations at all the non-boundary/centre points and 2×2 constraint conditions at both the edge and centre points.

When using the differential quadrature method together with thin plate theory, a difficulty in dealing with boundary constraints arises. The reason is that there is only one governing equation but two boundary conditions which should be satisfied at each boundary point. To overcome this difficulty, the δ-method has been introduced [33] in which the two respective boundary conditions are applied both at the boundary and at a very small distance δ from the boundary. However, due to the DQM being a polynomial approach [15–17], applying boundary conditions on non-boundary points will cause the polynomial to oscillate, and these oscillations will be strengthened with the increasing order of the approximating polynomial (i.e., by increasing the number of grid points employed). This problem is naturally evaded by using the Mindlin plate theory, in which there are two governing equations and two constraint conditions at both central and edge points for axisymmetric problems.

3. RESULTS AND DISCUSSION

Based on the formulations presented in the previous section, a program has been developed. For simplicity, only uniform thickness plates are considered in the present study; the Poisson's ratio is taken as $\nu = 0{\cdot}3$, and the shear correction factor κ is taken as $\pi^2/12$ [30]. The grid points employed in the computations are designated by:

$$R_i = \frac{1}{2}\left[1 - \cos\left(\frac{(i-1)\pi}{N-1}\right)\right], \qquad i = 1, 2, \ldots, N. \tag{22}$$

A non-dimensional frequency parameter λ^2 is adopted for the results presentation, and is defined as:

$$\lambda^2 = \omega a^2 \sqrt{\rho h/D} \qquad \text{and} \qquad \omega = \Omega\sqrt{E/\rho a^2(1-\nu^2)}. \tag{23}$$

3.1. PLATES WITHOUT RIGID CENTRE SUPPORT

In this sub-section, the free vibration analyses of circular Mindlin plates subject to completely free, simply-supported and clamped boundary conditions without a rigid point support at the plate centre (Figure 2a, 2b and 2c) are carried out.

In accordance with previous experience [34], convergence and accuracy studies must be carried out to reveal the convergence characteristics of the differential quadrature method for a particular problem as well as to ensure accuracy of the results. Thus, in Figure 3, the normalized frequency parameters $\lambda^2/\lambda^2_{ext}$ of the first four mode sequences are presented with an increasing number of grid points for the completely free circular plates of $h/a = 0{\cdot}001$ (Figure 3a) and $h/a = 0{\cdot}250$ (Figure 3b). Here, the values λ^2_{ext} are the exact solutions taken from [3]. For the simply-supported and clampled circular plates, similar convergence and accuracy studies are conducted, which are described in Figures 4 and 5. In Tables 1–3, the first fifteen non-dimensional frequency parameters λ^2, are tabulated for the various relative thickness plates subject to the free, simply-supported and clamped boundary conditions, respectively. There, the minimum numbers of grid points required for achieving convergent results with five significant digits are also exhibited for the first fifteen non-dimensional frequency parameters of various relative thickness plates with different boundary conditions. From these figures and tables, the following remarks on the convergence characteristics and the accuracy of the method for the present problem can

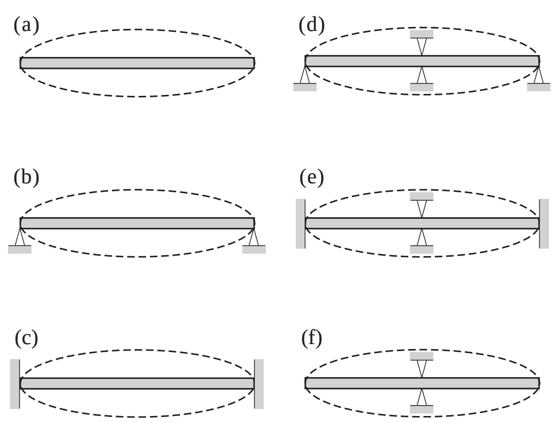

Figure 2. Various circular plates analysed: (a) completely free plate; (b) simply-supported plate; (c) clamped plate; (d) simply-supported plate with a rigid centre support; (e) clamped plate with a rigid centre support; and (f) free plate with a rigid centre support.

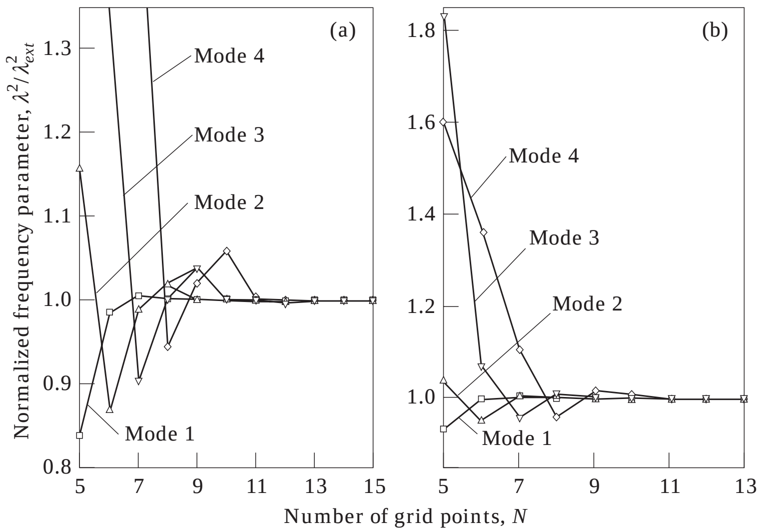

Figure 3. Convergence and accuracy of the normalized frequency parameter, $\lambda^2/\lambda^2_{ext}$, for the first four mode sequences with grid refinement for the free circular plates: (a) $h/a = 0{\cdot}001$ and (b) $h/a = 0{\cdot}250$. λ^2_{ext}—the exact solution [3].

be made: (1) When increasing the number of grid points, the DQ results converge to the corresponding exact solutions, which is true for all three kinds of boundary conditions considered and for various natural frequencies (at least for the first fifteen natural frequencies). (2) Whatever the relative thickness of a plate, the convergence of the DQ results with grid refinement demonstrates a fluctuation characteristic for the free and simply-supported plates, while for the clamped plate, the frequency parameters obtained

TABLE 1

Convergent results† of frequency parameters, $\lambda^2 = \omega a^2(\rho h/D)^{1/2}$, for free circular plates‡

Mode sequences	h/a					
	0·001	0·050	0·100	0·150	0·200	0·250
1	9·0031 (10)	8·9686 (11)	8·8679 (10)	8·7095 (10)	8·5051 (10)	8·2674 (10)
2	38·443 (13)	37·787 (13)	36·401 (12)	33·674 (13)	31·111 (13)	28·605 (13)
3	87·749 (16)	84·443 (14)	76·676 (16)	67·827 (14)	59·645 (14)	52·584 (14)
4	156·81 (16)	146·76 (15)	126·27 (15)	106·40 (15)	90·645 (15)	76·936 (17)
5	245·62 (18)	222·38 (18)	181·46 (16)	146·83 (16)	120·57 (16)	99·545 (17)
6	354·17 (21)	308·98 (19)	239·98 (21)	187·79 (18)	149·63 (18)	114·53 (16)
7	482·45 (22)	404·44 (22)	300·38 (22)	228·39 (20)	171·18 (20)	126·34 (17)
8	630·46 (25)	506·96 (23)	361·73 (23)	267·32 (22)	183·36 (22)	138·59 (19)
9	798·19 (28)	615·01 (26)	423·41 (24)	297·08 (22)	199·04 (22)	154·77 (20)
10	985·65 (29)	727·37 (27)	484·93 (27)	310·03 (24)	217·13 (24)	166·06 (20)
11	1192·8 (30)	843·04 (30)	545·74 (28)	330·92 (24)	231·82 (25)	182·35 (23)
12	1419·7 (31)	961·25 (31)	604·75 (29)	351·70 (25)	251·78 (26)	197·61 (23)
13	1666·3 (32)	1081·4 (31)	653·91 (32)	372·16 (25)	268·68 (25)	208·73 (24)
14	1932·6 (33)	1202·9 (33)	667·41 (30)	397·54 (30)	285·12 (26)	228·95 (25)
15	2218·6 (36)	1325·5 (34)	695·93 (30)	416·62 (30)	308·16 (28)	238·48 (26)

† Convergent results with five significant digits.
‡ A number in parentheses refers to the minimum number of grid points needed to obtain the convergent result with five significant digits.

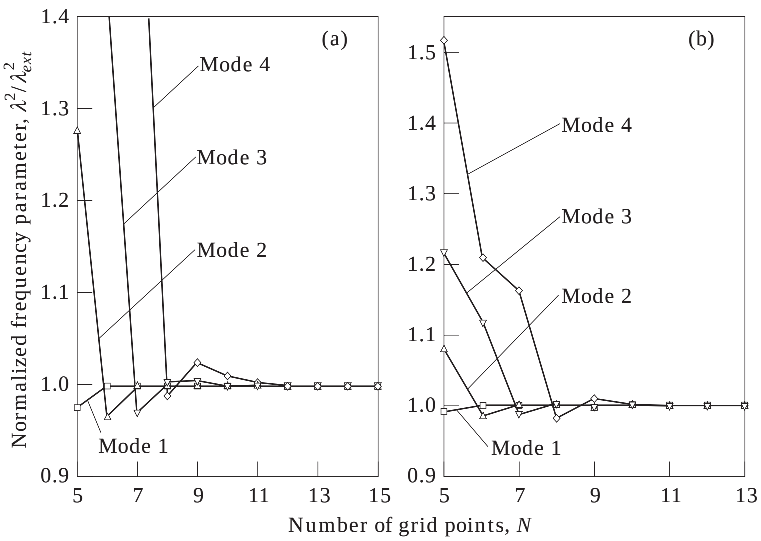

Figure 4. Convergence and accuracy of the normalized frequency parameter, $\lambda^2/\lambda^2_{ext}$, for the first four mode sequences with grid refinement for the simply-supported circular plates: (a) $h/a = 0{\cdot}001$ and (b) $h/a = 0{\cdot}250$. λ^2_{ext}—the exact solution [3].

using the DQ method show essentially monotonic convergence. (3) For various boundary conditions and relative thicknesses, more grid points are required to acquire a convergent result for a higher frequency than for a lower one. (4) Using the same number of grid points, the thicker a plate ($h/a = 0{\cdot}001$–$0{\cdot}250$), the more accurate the results by the DQ method will be. When a higher frequency is required, this thickness effect becomes more

TABLE 2

Convergent results† of frequency parameters, $\lambda^2 = \omega a^2(\rho h/D)^{1/2}$, for simply-supported circular plates‡

Mode sequences	h/a 0·001	0·050	0·100	0·150	0·200	0·250
1	4·9351 (9)	4·9247 (8)	4·8938 (8)	4·8440 (8)	4·7773 (8)	4·6963 (8)
2	29·720 (11)	29·323 (11)	28·240 (11)	26·715 (11)	24·994 (9)	23·254 (9)
3	74·155 (14)	71·756 (14)	65·942 (14)	59·062 (12)	52·514 (12)	46·775 (12)
4	138·31 (16)	130·35 (14)	113·57 (15)	96·775 (15)	82·766 (16)	71·603 (15)
5	222·21 (18)	202·81 (18)	167·53 (16)	136·98 (15)	113·87 (16)	96·609 (16)
6	325·83 (19)	286·79 (19)	225·34 (17)	178·23 (17)	145·13 (16)	108·27 (14)
7	499·18 (22)	380·13 (20)	285·44 (20)	219·86 (20)	166·29 (18)	121·50 (16)
8	592·27 (23)	480·94 (23)	346·83 (21)	261·51 (20)	176·28 (18)	131·65 (14)
9	755·08 (26)	587·65 (24)	408·91 (23)	291·55 (20)	191·38 (18)	146·17 (18)
10	937·61 (27)	698·97 (27)	471·31 (25)	303·05 (22)	207·23 (19)	163·30 (15)
11	1139·9 (28)	813·85 (30)	533·80 (28)	318·34 (21)	227·28 (20)	170·65 (18)
12	1361·8 (29)	931·50 (29)	596·23 (27)	344·39 (24)	237·98 (22)	194·94 (21)
13	1603·5 (30)	1051·2 (31)	649·29 (30)	359·27 (22)	268·49 (24)	198·98 (21)
14	1864·9 (32)	1172·6 (31)	658·55 (30)	385·53 (25)	269·03 (24)	219·11 (23)
15	2145·9 (35)	1295·1 (32)	677·58 (27)	408·98 (23)	298·91 (26)	236·92 (20)

† Convergent results with five significant digits.
‡ A number in parentheses refers to the minimum number of grid points needed to obtain the convergent result with five significant digits.

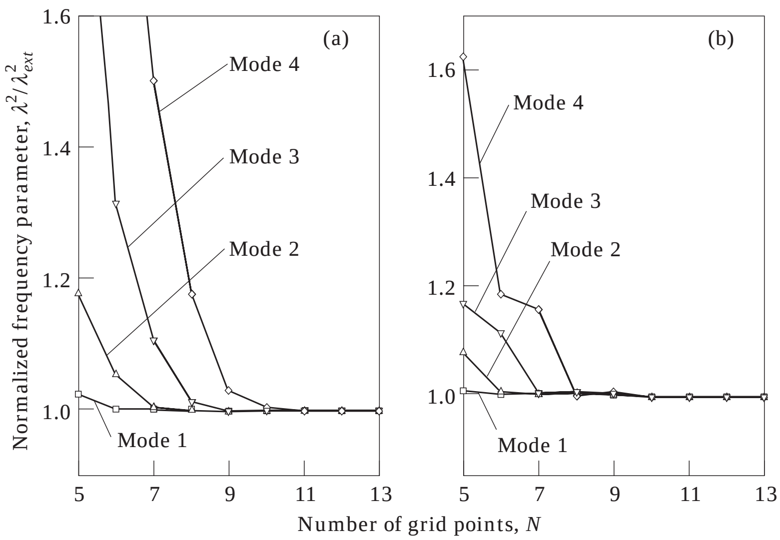

Figure 5. Convergence and accuracy of the normalized frequency parameter, $\lambda^2/\lambda^2_{ext}$, for the first four mode sequences with grid refinement for the clamped circular plates (a) $h/a = 0{\cdot}001$ and (b) $h/a = 0{\cdot}250$. λ^2_{ext}—the exact solution [3].

dominant. (5) For all three kinds of boundary conditions considered herein, the DQ solutions of the first fifteen natural frequencies by 36 grid points are convergent with at least five significant digits; and the DQ solutions of the first four natural frequencies by 11 grid points can be regarded as ones with enough accuracy (3–4 significant digits). In order to further demonstrate the accuracy of the DQ method, Table 4 introduces a comparison between the DQM results and the exact solutions obtained by Irie *et al*. [3]

TABLE 3

Convergent results† of frequency parameters, $\lambda^2 = \omega a^2(\rho h/D)^{1/2}$, for clamped circular plates‡

Mode sequences	h/a					
	0·001	0·050	0·100	0·150	0·200	0·250
1	10·216 (8)	10·145 (8)	9·9408 (8)	9·6286 (8)	9·2400 (8)	8·8068 (8)
2	39·771 (11)	38·855 (11)	36·479 (9)	33·393 (9)	30·211 (9)	27·253 (9)
3	89·102 (12)	84·995 (12)	75·664 (12)	65·551 (11)	56·682 (10)	49·420 (11)
4	158·18 (13)	146·40 (13)	123·32 (13)	102·09 (12)	85·571 (13)	73·054 (13)
5	246·99 (15)	220·73 (15)	176·41 (15)	140·93 (13)	115·55 (16)	97·198 (14)
6	355·54 (19)	305·71 (16)	232·97 (16)	180·99 (15)	145·94 (16)	117·90 (14)
7	483·82 (20)	399·32 (19)	291·71 (17)	221·62 (19)	174·97 (17)	122·43 (15)
8	631·83 (22)	499·82 (20)	351·82 (20)	262·45 (21)	178·76 (17)	144·42 (17)
9	799·57 (23)	605·78 (23)	412·77 (22)	301·11 (21)	205·32 (18)	148·75 (17)
10	987·03 (25)	716·07 (24)	474·18 (23)	305·15 (21)	210·53 (18)	170·37 (20)
11	1194·2 (25)	829·74 (25)	535·81 (25)	336·52 (22)	237·46 (22)	181·05 (18)
12	1421·1 (27)	946·07 (28)	597·43 (26)	345·58 (24)	248·18 (21)	195·12 (21)
13	1667·7 (29)	1064·5 (27)	657·60 (27)	380·88 (25)	268·60 (23)	216·40 (23)
14	1934·0 (31)	1184·5 (29)	662·37 (27)	388·16 (25)	290·67 (24)	220·58 (22)
15	2220·0 (32)	1305·7 (31)	698·63 (28)	425·43 (26)	299·71 (24)	243·02 (24)

† Convergent result with five significant digits.
‡ A number in parentheses refers to the minimum number of grid points needed to obtain the convergent result with five significant digits.

TABLE 4

Comparison study of frequency parameters, $\lambda^2 = \omega a^2(\rho h/D)^{1/2}$, for circular plates with different boundary conditions

Boundary conditions	Mode sequences	$h/a = 0{\cdot}001$ DQM	$h/a = 0{\cdot}001$ Exact [3]	$h/a = 0{\cdot}250$ DQM	$h/a = 0{\cdot}250$ Exact [3]
S	1	4·935	4·935	4·696	4·696
	2	29·720	29·720	23·254	23·254
	3	74·155	74·156	46·775	46·775
	4	138·314	138·318	71·603	71·603
C	1	10·216	10·216	8·807	8·807
	2	39·771	39·771	27·253	27·253
	3	89·102	89·104	49·420	49·420
	4	158·180	158·184	73·054	73·054
F	1	9·003	9·003	8·267	8·267
	2	38·443	38·443	28·605	28·605
	3	87·749	87·750	52·584	52·584
	4	156·808	156·818	76·936	76·936

for the first four natural frequencies. It is found that, except for some rounding errors, the DQ results are identical to the corresponding exact solutions.

3.2. PLATES WITH A RIGID CENTRE SUPPORT

To further explore the capability, convergence and accuracy of the differential quadrature method for plate vibration problems, free vibrations of circular Mindlin plates with a rigid point support at the plate centre (Figure 2d, 2e and 2f) are investigated. It is noted that there is a stress singularity at the centre of such plates due to the rigid centre support.

In Figures 6 and 7, the variations of the first five frequency parameters for simply-supported and clamped plates with a rigid centre support via the number of grid

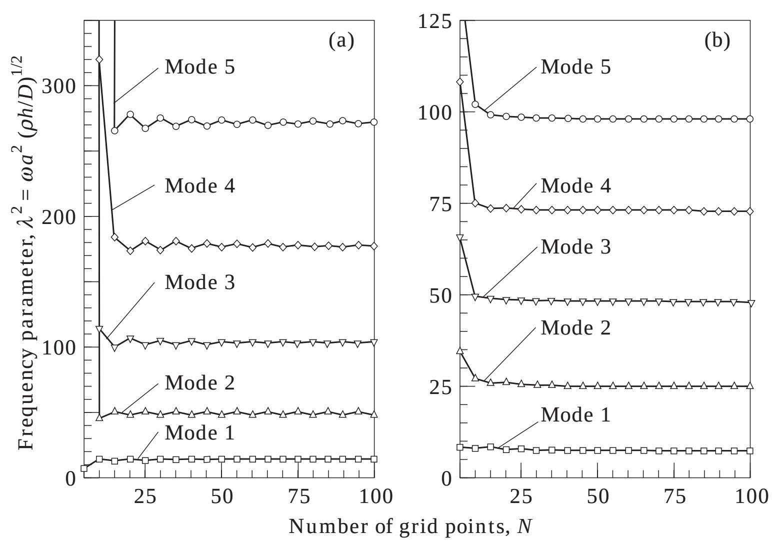

Figure 6. Convergence of the first five frequency parameters with grid refinement for simply-supported plates with a rigid centre support: (a) $h/a = 0{\cdot}001$ and (b) $h/a = 0{\cdot}250$.

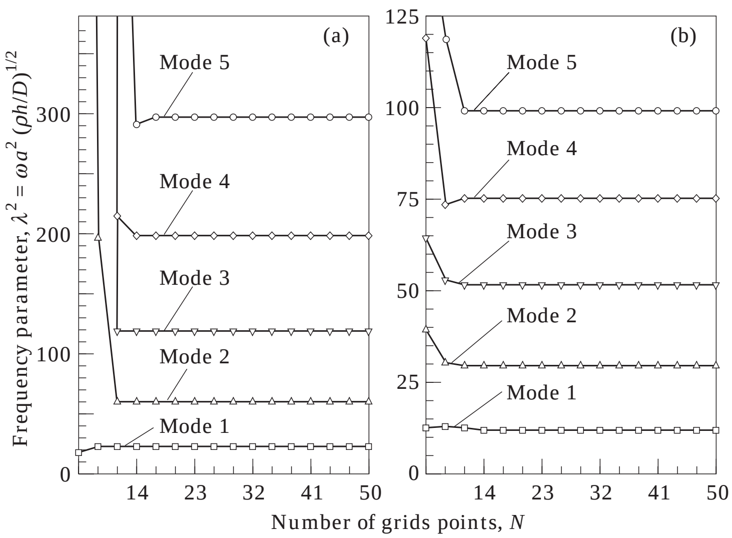

Figure 7. Convergence of the first five frequency parameters with grid refinement for clamped plates with a rigid centre support: (a) $h/a = 0{\cdot}001$ and (b) $h/a = 0{\cdot}250$.

points are demonstrated respectively. These figures expose the following convergence characteristics of the differential quadrature method for such special problems: (1) For a simply-supported or clamped plate with a rigid centre support, the DQ results converge to a steady value when increasing the number of grid points. (2) For all the frequencies considered in the figures, the DQ results for a clamped plate with grid refinement converge much faster than those for a simply-supported plate with the same relative thickness. (3) The thicker a plate, the faster the DQ results advance to corresponding convergent values.

Table 5 presents some results of the first and fifth frequency parameters obtained using different numbers of grid points for simply-supported and clamped plates of $h/a = 0{\cdot}001$ and $h/a = 0{\cdot}250$. It is found from the table that, for clamped plates, the results by 50 grid points can be regarded as ones with enough accuracy (3–4 significant digits), whereas, for simply-supported plates, a grid with 200 points is required to obtain about the same accuracy. In order to obtain results with acceptable accuracy (about 2 significant digits), 14 and 24 grid points are needed for the clampled plates and simply-supported plates respectively. It is noticed that, for the simply-supported plates, even using as many as 200 grid points, one still cannot obtain fully convergent results.

In Table 5, the thin plate exact solutions for the fundamental frequency [8] are also presented for comparison. This comparison shows that the DQ results converge to the corresponding exact solutions when the number of grid points is increased.

For a free plate with a grid centre support, convergence is completely different from those for simply-supported or clamped plates. The variations of the first five frequency parameters via the grid number are illustrated in Figure 8. It is found from the figure, that, even using as many as 200 grid points, the DQ solutions for the first two mode sequences do not display any sign of convergence. In fact, after carefully examining the results, one finds, that for the first frequency, the DQ results obtained using even numbers of grid points converge, with grid refinement, to a given value. When odd numbers of grid points are used, the results converge to another value. The authors have also tried to solve for

TABLE 5

Frequency parameters, $\lambda^2 = \omega a^2(\rho h/D)^{1/2}$, of simply-supported and clamped circular plates with a rigid centre support calculated by different numbers of grid points†

Boundary conditions	$h/a = 0{\cdot}001$		$h/a = 0{\cdot}250$	
	Mode 1	Mode 5	Mode 1	Mode 5
S	13·709 (15)	264·95 (15)	8·4773 (15)	99·104 (15)
	15·904 (16)	280·49 (16)	7·8702 (16)	98·739 (16)
	14·133 (23)	266·30 (23)	8·0972 (23)	98·698 (23)
	15·487 (24)	276·73 (24)	7·7267 (24)	98·500 (24)
	14·514 (49)	269·22 (49)	7·6207 (49)	98·248 (49)
	14·113 (50)	273·82 (50)	7·4679 (50)	98·117 (50)
	14·672 (99)	270·44 (99)	7·3093 (99)	97·982 (99)
	14·953 (100)	272·60 (100)	7·2416 (100)	97·954 (100)
	14·740 (199)	271·04 (199)	7·0718 (199)	97·796 (199)
	14·872 (200)	271·96 (200)	7·0411 (200)	97·784 (200)
Ref. [8]	14·8	—	—	—
C	22·809 (13)	288·56 (13)	12·248 (13)	99·560 (13)
	22·714 (14)	291·26 (14)	12·193 (14)	99·538 (14)
	22·754 (19)	299·31 (19)	11·968 (19)	99·303 (19)
	22·730 (20)	298·63 (20)	11·934 (20)	99·250 (20)
	22·740 (29)	298·89 (29)	11·710 (29)	99·027 (29)
	22·734 (30)	298·74 (30)	11·691 (30)	99·003 (30)
	22·737 (49)	298·79 (49)	11·446 (49)	98·776 (49)
	22·736 (50)	298·77 (50)	11·437 (50)	98·766 (50)
Ref. [8]	22·7	—	—	—

† A number in parentheses refers to the number of grid points with which the DQM result is obtained.

the natural frequencies of a free plate with relative thickness of 0·100 or less using the present method. It is found that there are some negative values among the eigenvalues obtained by solving the corresponding determinant equations. Therefore, it may be

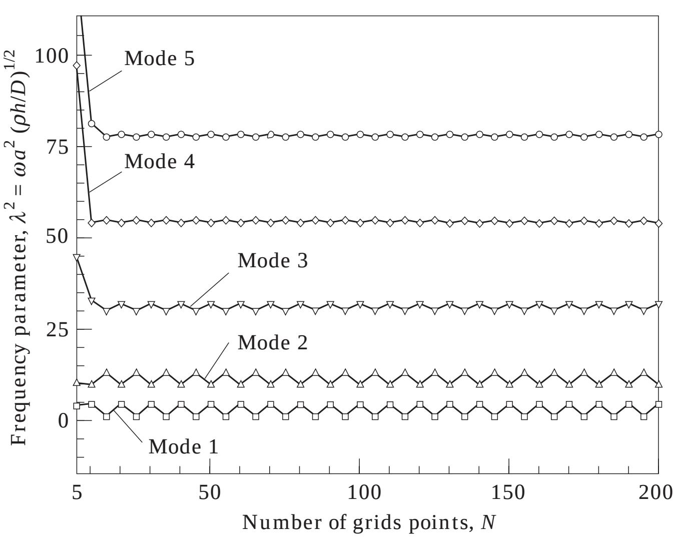

Figure 8. Variation of the first five frequency parameters via the number of grid points for a free plate of $h/a = 0{\cdot}250$ with a rigid centre support.

concluded that, for a free plate with a rigid centre support, the differential quadrature method fails to produce convergent and correct solutions.

4. CONCLUSIONS

In this paper, the differential quadrature method has been applied successfully to solve the axisymmetric free vibration problem of moderately thick circular Mindlin plates. Free, simply-supported and clamped plates without or with a rigid centre support have been considered in the present study. The applicability, convergence properties and accuracy of the present method for moderately thick plate eigenvalue problems have been carefully examined.

For plates without a rigid centre support, the first fifteen frequency parameters have been calculated for various relative thicknesses and for different boundary conditions. For such plates, the DQ method yields convergent and accurate solutions even for a small number of grid points. The results show that different boundary conditions, relative plate thicknesses and mode sequences have significant influences on the convergence properties of the method.

For simply-supported and clamped plates with a rigid centre support, the DQ results approach the corresponding correct solutions slowly with increasing grid refinement. Using a grid of about 25 points, the method can provide solutions with acceptable accuracy (about 2 significant digits). However, for free plates with a rigid centre support, the method seems to fail to produce convergent and correct solutions.

The two advantages involved in the present solution procedure are: (1) By using the Mindlin plate theory, the problem that two grid points are required at each boundary point is naturally avoided. (2) By using algebraic expressions to calculate the weighting coefficients, the number of grid points can be greatly increased when necessary.

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APPENDIX: THE DIFFERENTIAL QUADRATURE METHOD

Supposing that there are N grid points along the r-axis with $r_1, r_2, \ldots, r_N$ as the co-ordinates, the nth order derivative of $f(r)$ can be expressed discretely at the point r_i as:

$$f_r^{(n)}(r_i) = \sum_{k=1}^{N} C_{ik}^{(n)} f(r_k); \qquad n = 1, 2, \ldots, N-1, \tag{A1}$$

where $C_{ij}^{(n)}$ are the weighting coefficients associated with the nth order derivative of $f(r)$ at the discrete point r_i.

According to references [17, 19], the weighting coefficients in equation (A1) can be determined as follows:

$$C_{ij}^{(1)} = M^{(1)}(r_i)/[(r_i - r_j)M^{(1)}(r_j)]; \qquad i, j = 1, 2, \ldots, N, \text{ but } j \neq i, \tag{A2}$$

where

$$M^{(1)}(r_i) = \prod_{j=1, j \neq i}^{N} (r_i - r_j), \qquad i = 1, 2, \ldots, N \tag{A3}$$

and

$$C_{ij}^{(n)} = n\left(C_{ii}^{(n-1)} C_{ij}^{(1)} - \frac{C_{ij}^{(n-1)}}{r_i - r_j}\right); \qquad i, j = 1, 2, \ldots, N, \text{ but } j \neq i; \text{ and } n = 2, 3, \ldots, N-1 \tag{A4}$$

$$C_{ii}^{(n)} = -\sum_{j=1, j \neq i}^{N} C_{ij}^{(n)}; \qquad i = 1, 2, \ldots, N, \quad \text{and} \quad n = 1, 2, \ldots, N-1. \tag{A5}$$