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MODAL COUPLING CONTROLLER DESIGN USING A NORMAL FORM METHOD, PART I: DYNAMICS

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There are several techniques available for vibration suppression of oscillatory systems. However, application of these techniques may not cause a fast rate of decay. This paper is part I of a comprehensive study which uses modal coupling to control vibrations in oscillatory systems (see reference [1] for part II). A second order auxiliary oscillatory system is used as the controller, coupled to the plant via non-linear coupling terms. In part I the dynamics associated with the system is fully investigated using normal form theory.

Earlier studies on **M**odal **C**oupling **C**ontrol (MCC) have resorted to perturbation methods for design purposes. These studies were restricted to linear and undamped plants using specific coupling terms. Also, the selection of controller parameters in these studies was based on trial and error. In this work, by extending MCC to a general class of non-linear systems with a damped or undamped oscillatory linear part, the trial and error in parameter selection is eliminated. The general form of the coupling terms is derived and a phenomenon called neck which is developed in the plant response upon applying the proposed controller is introduced. In part II of this work the authors focus on the control aspects of the method and use the neck phenomenon to define an algorithm for the controller implementation. The controller is then applied to a piezo-actuated flexible beam.

1. INTRODUCTION

In this article, an active control scheme is proposed that uses internal resonance to damp out the vibrations of an oscillatory system. The proposed technique capitalizes on dynamic modal coupling effects. A non-linear system may exhibit modal coupling effects if there exits a state of internal resonance. Internal resonance may occur in a system of coupled non-linear differential equations when natural frequencies of the system are commensurable. That is, there exist constants $\{m_1, m_2, \ldots, m_n \in \mathscr{Z}\}$ such that

$$m_1 w_1 + m_2 w_2 + \cdots + m_n w_n = 0, \tag{1}$$

where w_j are the natural frequencies of the system.

In a state of internal resonance the commensurable modes of vibration are coupled via an energy bridge which facilitates a continuous and periodic transfer of energy between the modes. Theoretically, it seems possible to transfer energy from lightly damped to highly

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damped modes. In other words, the damping characteristics of lightly damped modes can be changed indirectly through other damped modes.

The essence of a MCC design is to provide an energy link between an oscillatory system that is to be controlled (plant) with an auxiliary system (controller). Under strong coupling, energy is transferred back and forth between the plant and the controller. This gives rise to a periodic amplitude modulation or beat phenomenon; and as a result, the envelope of the plant and the controller responses become periodic. In MCC design the oscillatory energy is transferred from the plant to the controller where it is subsequently dissipated.

Modal coupling was first used in the area of control by Golnaraghi [2] and Golnaraghi *et al*. [3]. The authors used modal coupling to control disturbance-induced oscillations in a system. Tuer *et al*. [4] and Oueini and Golnaraghi [5] conducted theoretical and experimental studies in the application of modal coupling control.

In all the above studies any damping in the plant and controller were ignored and the amount of damping for an acceptable response of the plant was found using trial and error. The initial conditions of the controller and other controller gains can change the effectiveness of the controller drastically. However, previous studies did not address these issues and they solely depended on trial and error to find the effect of the controller parameters on MCC.

In references [6] and [7], using center manifold theory along with a normal form method, the issues of MCC design in vibration suppression of a cantilever beam were addressed. The controller was a mass–spring–dashpot mechanism which was free to slide along the beam. The authors used the non-linear coupling terms of the equations to design a modal coupling controller.

Khajepour *et al*. [8] used normal forms to derive a new relation that took into account all the parameters (damping, controller gains and the plant initial conditions). When this relation was used to find the controller initial values, simulation showed that an interesting phenomenon called *neck* is established in the plant response. A neck phenomenon corresponds to almost complete exchange of energy from the plant to the controller. This phenomenon was used to suppress the vibrations of an oscillatory system.

In this paper a systematic approach to design a generalized modal coupling controller is developed. Application of the normal form method enables one to address the main drawbacks of previous studies which were explained earlier. In the following section normal form methods are used to extend MCC to a broader class of non-linear systems and to clarify the unaddressed issues in MCC design.

2. MCC DESIGN USING NORMAL FORM

One considers control of a general second order system. The governing plant equation is defined in state space form as

$$\dot{x}_p = A_p x_p + F_p(x_p) + U_p, \tag{2}$$

where $x_p = (x_{p1}, x_{p2})^{\mathrm{T}}$ is the span of the plant state variables,

$$A_p = \begin{pmatrix} 0 & 1 \\ -w_p^2 & -2\zeta_p w_p \end{pmatrix}, \qquad F_p(x_p) = \begin{pmatrix} f_1(x_p) \\ f_2(x_p) \end{pmatrix}, \qquad U_p = \begin{pmatrix} 0 \\ u_p \end{pmatrix}. \tag{3a, b}$$

The plant natural frequency and damping ratio are w_p and ζ_p respectively. The non-linear terms are included in $F_p(x_p)$ and U_p is the plant input.

The essence of a modal coupling controller design is to provide an energy link between the plant and an auxiliary system (controller). Hence, the following system is introduced as the controller:

$$\dot{x}_c = A_c x_c + U_c, \tag{4}$$

where $x_c = (x_{c1}, x_{c2})^{\mathrm{T}}$ is the span of controller state variables, and

$$A_c = \begin{pmatrix} 0 & 1 \\ -w_c^2 & -2\zeta_c w_c \end{pmatrix}, \qquad U_c = \begin{pmatrix} 0 \\ u_c \end{pmatrix}. \tag{5a, b}$$

The feedback input is U_c and w_c and ζ_c $(0 \leqslant \zeta_c < 1)$† are the controller natural frequency and damping ratio, respectively.

Defining $x = (x_p, x_c)^{\mathrm{T}}$, the closed loop system of (2) and (4) becomes

$$\dot{x} = Ax + F(x_p) + U(x), \tag{6}$$

where

$$A = \begin{pmatrix} A_p & 0 \\ 0 & A_c \end{pmatrix} \tag{7}$$

and

$$F(x_p) = \begin{pmatrix} F_p(x_p) \\ 0 \end{pmatrix}, \qquad U(x) = \begin{pmatrix} U_p(x) \\ U_c(x) \end{pmatrix}. \tag{8}$$

In order to generalize MCC using quadratic coupling terms, one considers $U_p(x)$ and $U_c(x)$ to be general second order functions in x. Furthermore, one assumes that $0 \leqslant \zeta_p < 1$, $F(x_p) \in C^3$ and $F(0) = DF(0) = 0$. This last assumption ensures that the origin is an equilibrium point of equation (6).

As mentioned before, MCC relies on providing a strong coupling link between the plant and the controller. This link does not exist for an arbitrary frequency ratio w_p/w_c or any feedback inputs $U(x)$. In the following normal form theory is used to derive the frequency ratio and the feedback input that result in a strong energy link.

2.1. NORMAL FORM EQUATIONS

A system of differential equations can be written in a simpler form using the normal form method. In general, the normal form method is a series of non-linear co-ordinate transformations used to eliminate or simplify equation non-linearities. Although the transformations are non-linear functions of the state variables, they are found by solving a sequence of linear equations. The transformations are close to identity transformations and therefore the linear part of the system does not change. This indicates that the structure of the normal form equations depends only on the linear part of the system (references [9, 10]).

† The values $\zeta_c \geqslant 1$ are excluded since the controller must be oscillatory in nature.

To transfer equation (6) into normal forms, one first applies a linear transformation to transform A into real Jordan canonical form. Defining $\bar{G}(x) = F(x_p) + U(x)$ and

$$\hat{T} = \operatorname{diag}\left(\begin{pmatrix} 1 & 0 \\ -w_p\zeta_p & -w_p\beta_p \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -w_c\zeta_c & -w_c\beta_c \end{pmatrix}\right), \tag{9}$$

where $\beta_p = \sqrt{1-\zeta_p^2}$ and $\beta_c = \sqrt{1-\zeta_c^2}$, equation (6), using $x \to \hat{T}x$, is transformed to†

$$\dot{x} = \hat{A}x + \hat{G}(x), \tag{10}$$

where $\hat{G}(x) = \hat{T}^{-1}\bar{G}\hat{T}x$ and $\hat{A}$ is

$$\hat{A} = \hat{T}^{-1}A\hat{T} = \operatorname{diag}\left(\begin{pmatrix} -w_p\zeta_p & -w_p\beta_p \\ w_p\beta_p & -w_p\zeta_p \end{pmatrix}, \qquad \begin{pmatrix} -w_c\zeta_c & -w_c\beta_c \\ w_c\beta_c & -w_c\zeta_c \end{pmatrix}\right). \tag{11}$$

Since the linear part of equation (10) has complex eigenvalues, it is easier to calculate the normal form equation using complex co-ordinates. Applying the complex transformation, $x = Tq$ where $q = (q_1, \bar{q}_1, q_2, \bar{q}_2)^{\mathrm{T}}$ and

$$T = \tfrac{1}{2}\operatorname{diag}\left(\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}\right), \tag{12}$$

to equation (10), one obtains

$$\dot{q} = \tilde{J}q + \tilde{G}(q), \tag{13}$$

where $\tilde{J}$ is $\tilde{J} = T^{-1}\hat{A}T = \operatorname{diag}(\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2)$, with

$$\lambda_1 = w_p(-\zeta_p + \mathrm{i}\beta_p), \qquad \lambda_2 = w_c(-\zeta_c + \mathrm{i}\beta_c), \tag{14}$$

and $\tilde{G}(q)$ is

$$\tilde{G}(q) = \begin{Bmatrix} \hat{G}_1(Tq) + \mathrm{i}\hat{G}_2(Tq) \\ \hat{G}_1(Tq) - \mathrm{i}\hat{G}_2(Tq) \\ \hat{G}_3(Tq) + \mathrm{i}\hat{G}_4(Tq) \\ \hat{G}_3(Tq) - \mathrm{i}\hat{G}_4(Tq) \end{Bmatrix}. \tag{15}$$

Using the above transformation, it is clear that the second and fourth equations of (13) are the complex conjugates of the first and third equations. Therefore, all one needs to study is the two dimensional system

$$\dot{z} = Jz + G(z, \bar{z}), \tag{16}$$

where $z = (q_1, q_2)^{\mathrm{T}}$, $J = \operatorname{diag}(\lambda_1, \lambda_2)$ and $G(z, \bar{z}) = (\tilde{G}_1(z, \bar{z}), \tilde{G}_3(z, \bar{z}))^{\mathrm{T}}$. So far, the linear part of equation (2) has been simplified as much as possible.

A non-linear transformation is now used to simplify or even eliminate the non-linear terms of (16). First, using the Taylor series, one expands $G(z, \bar{z})$ about the origin so that (16) becomes

$$\dot{z} = Jz + G_2(z, \bar{z}) + G_3(z, \bar{z}) + \cdots, \tag{17}$$

† In order to have less confusion, the same variables are retained in the calculations.

where $G_j(z, \bar{z})$ represent the ith order term in Taylor expansion of $G(z, \bar{z})$. The non-linear co-ordinate transformation is introduced next†

$$z \to z + h_2(z, \bar{z}), \tag{18}$$

where $h_2(z, \bar{z})$ is second order in z and $\bar{z}$. Substituting equation (18) into equation (17) gives

$$(I + D_z h_2(z, \bar{z}))\dot{z} + D_{\bar{z}} h_2(z, \bar{z})\dot{\bar{z}} = J(z + h_2(z, \bar{z})) + G_2(z, \bar{z}) + \mathcal{O}(3) \tag{19}$$

or

$$\dot{z} = (I + D_z h_2(z, \bar{z}))^{-1}(J(z + h_2(z, \bar{z})) - D_{\bar{z}} h_2(z, \bar{z})\dot{\bar{z}} + G_2(z, \bar{z}) + \mathcal{O}(3)). \tag{20}$$

Note that $\dot{\bar{z}}$ is the complex conjugate of (17) i.e.,

$$\dot{\bar{z}} = \bar{J}\bar{z} + \bar{G}_2(z, \bar{z}) + \mathcal{O}(3). \tag{21}$$

The inverse of $(I + D_z h_2(z, \bar{z}))$ exists for $z, \bar{z}$ sufficiently small and can be represented in a series expansion

$$(I + D_z h_2(z, \bar{z}))^{-1} = I - D_z h_2(z, \bar{z}) + \mathcal{O}(2). \tag{22}$$

Using equations (21) and (22), equation (20) becomes

$$\dot{z} = Jz + (Jh_2(z, \bar{z}) - D_z h_2(z, \bar{z})Jz - D_{\bar{z}} h_2(z, \bar{z})\bar{J}\bar{z} + G_2(z, \bar{z})) + \mathcal{O}(3). \tag{23}$$

Up to this point $h_2(z, \bar{z})$ has been arbitrary. However, one can choose $h_2(z, \bar{z})$ so as to simplify the second order terms in $G_2(z, \bar{z})$ as much as possible. If $h_2(z, \bar{z})$ satisfies

$$D_z h_2(z, \bar{z})Jz + D_{\bar{z}} h_2(z, \bar{z})\bar{J}\bar{z} - Jh_2(z, \bar{z}) = G_2(z, \bar{z}), \tag{24}$$

then all second order terms are eliminated from equation (23). In the following the solution of equation (24) for the unknown function $h_2(z, \bar{z})$ is studied.

Equation (24) can be considered as a special case of a more general *homological* equation (see Arnold [11]) of the form

$$L_J(h_s(z, \bar{z})) \equiv D_z h_s(z, \bar{z})Jz + D_{\bar{z}} h_s(z, \bar{z})\bar{J}\bar{z} - Jh_s(z, \bar{z}) = G_s(z, \bar{z}), \tag{25}$$

where $L_J(\cdot)$ is a linear operator acting on the linear vector space of *vector-valued monomials* of degree s.

Definition: Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathscr{C}^n$ (space of complex numbers), and let $\{z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n\}$ be co-ordinates with respect to this basis. Now let $z^l\bar{z}^m = z_1^{l_1} \ldots z_n^{l_n}\bar{z}_1^{m_1} \ldots \bar{z}_n^{m_n}$ be coefficients in this basis i.e.,

$$z^l\bar{z}^m e_k = (z_1^{l_1} \ldots z_n^{l_n}\bar{z}_1^{m_1} \ldots \bar{z}_n^{m_n})e_k, \qquad \sum_{j=1}^{n} (l_j + m_j) = s, \tag{26}$$

where $l_j, m_j \in \mathscr{N}$. These elements are called vector-valued monomials of degree s. The set of all vector-valued monomials of degree s forms the linear vector space H_s. For instance in $\mathscr{C}^1$ with coefficients z and $\bar{z}$ the space H_2 is the span of

$$\{z^2, z\bar{z}, \bar{z}^2\}. \tag{27}$$

† We retain the same variable for simplicity.

Definition: The set of eigenvalues $\lambda = \{\lambda_1, \ldots, \lambda_n, \bar{\lambda}_1, \ldots, \bar{\lambda}_n\}$ is said to be resonant if among the eigenvalues there exists a relation of the form

$$\lambda_s = \sum_{j=1}^{n} (l_j \lambda_j + m_j \bar{\lambda}_j), \tag{28}$$

where $l_j, m_j \in \mathcal{N}$ and

$$\textstyle\sum_{j=1}^{n} (l_j + m_j) \geqslant 2.$$

Such a relation is called a resonance. The integer

$$s = \textstyle\sum_{j=1}^{n} (l_j + m_j)$$

is called the order of resonance.

To solve equation (25) uniquely, the operator $L_J(\cdot)$ defined by

$$L_J(h_s(z, \bar{z})) = D_z h_s(z, \bar{z}) J z + D_{\bar{z}} h_s(z, \bar{z}) \bar{J} \bar{z} - J h_s(z, \bar{z}) \tag{29}$$

should be invertible in H_s. One now shows that $L_J(\cdot)$ is invertible if the set of eigenvalues of J and $\bar{J}$ i.e., $\lambda = \{\lambda_1, \ldots, \lambda_n, \bar{\lambda}_1, \ldots, \bar{\lambda}_n\}$ are not resonant.

The authors base the argument on the case that J is diagonal. However, if J is not diagonalizable (repeated eigenvalues), the following argument still holds with slight differences (Arnold [9]). Let e_k, $1 \leqslant k \leqslant n$, be an n-vector with 1 in the k^{th} component and zeros in the remaining components. Choosing $\{e_k\}$ as the basis of $\mathscr{C}^n$, H_s will be the space of all possible $z^l \bar{z}^m e_k$. Using the fact that J is diagonal,

$$J e_k = \lambda_k e_k, \qquad \bar{J} e_k = \bar{\lambda}_k e_k, \tag{30a, b}$$

the action of $L_J(\cdot)$ on each $z^l \bar{z}^m e_k$ will be

$$\begin{aligned} L_J(z^l \bar{z}^m e_k) &= D_z(z^l \bar{z}^m e_k) J z + D_{\bar{z}}(z^l \bar{z}^m e_k) \bar{J} \bar{z} - J(z^l \bar{z}^m e_k) \\ &= \left(\sum_{j=1}^{n} \lambda_j l_j + \sum_{j=1}^{n} \bar{\lambda}_j m_j - \lambda_k \right) z^l \bar{z}^m e_k. \end{aligned} \tag{31}$$

Equation (31) shows that the representation of the linear operator $L_J(\cdot)$ with respect to the chosen basis is diagonal with eigenvalues

$$\sum_{j=1}^{n} (l_j \lambda_j + m_j \bar{\lambda}_j) - \lambda_k. \tag{32}$$

Therefore, operator $L_J(\cdot)$ is not invertible if it has a zero eigenvalue for some k, i.e.,

$$\lambda_k = \sum_{j=1}^{n} (l_j \lambda_j + m_j \bar{\lambda}_j). \tag{33}$$

That is, equation (25) can be solved uniquely and all non-linear terms of order s are eliminated by a non-linear transformation ($z \rightarrow z + h_s(z, \bar{z})$) if and only if no resonance of order s exists. Thus, the normal form equation of a non-linear system in the form of equation (17) is

$$\dot{z} = Jz + G_2^r(z, \bar{z}) + \cdots + G_s^r(z, \bar{z}) + \cdots , \tag{34}$$

where $G_s^r(z, \bar{z})$ are non-linear terms which are not eliminated due to resonance of order s.

When a system is in resonance of order s, the equations are strongly coupled and the effect of non-linear terms of order s cannot be ignored. Since in MCC design a strong coupling between the plant and the controller is necessary, one must consider a controller in which the plant and the controller exhibit a resonance.

For the two dimensional system (16), the resonance terms of order 2 are associated with the zero eigenvalues of $L_J(\cdot)$,

$$\lambda_k = \sum_{j=1}^{2} (l_j \lambda_j + m_j \overline{\lambda}_j), \tag{35}$$

where $k \in \{1, 2\}$ and

$$\textstyle\sum_{j=1}^{2} (m_j + l_j) = 2.$$

Substituting λ_j from equation (14) into equation (35) gives

$$\lambda_k = -(l_1 + m_1) w_p \zeta_p - (l_2 + m_2) w_c \zeta_c + \mathrm{i}((l_1 - m_1) w_p \beta_p + (l_2 - m_2) w_c \beta_c). \tag{36}$$

For $k = 1$ where $\lambda_1 = w_p(-\zeta_p + \mathrm{i}\beta_p)$, equation (36) holds if

$$(l_1 + m_1 - 1) w_p \zeta_p + (l_2 + m_2) w_c \zeta_c = 0, \qquad (l_1 - m_1 - 1) w_p \beta_p + (l_2 - m_2) w_c \beta_c = 0, \tag{37}$$

and for $k = 2$ where $\lambda_2 = w_c(-\zeta_c + \mathrm{i}\beta_c)$ (36) holds if

$$(l_1 + m_1) w_p \zeta_p + (l_2 + m_2 - 1) w_c \zeta_c = 0, \qquad (l_1 - m_1) w_p \beta_p + (l_2 - m_2 - 1) w_c \beta_c = 0. \tag{38}$$

In equations (37) and (38) only w_p and ζ_p are known and the rest are unknown. However, since w_p, $w_c \in \mathscr{R}^+$, ζ_c, $\zeta_p \in [0, 1)$ and l_1, l_2, m_1, m_2 are in $\{0, 1, 2\}$ the equations may not have an admissible solution. Investigation for possible solutions indicates that equations (37) and (38) have two solutions each which are listed in Table 1.

As seen in Table 1, equation (37) has a solution if $\zeta_c = 0$ or $\zeta_c = \zeta_p$. However, the goal is to suppress the vibration of a lightly damped system via transferring energy to the controller and therefore ζ_c should not be zero nor as small as ζ_p. Thus, these two resonance cases are not appropriate for the controller design.

The other case that causes resonance in equation (16) is when equation (38) is satisfied. With the same reason the first solution of equation (38) is excluded from the resonance

TABLE 1

Resonance cases of the solutions of equations (37) *and* (38)

#	l_1	l_2	m_1	m_2	ζ_c, ζ_p	w_c, w_p
1st solution of (37)	0	1	1	0	$\zeta_c = 0$	$w_c = 2w_p\beta_p$
2nd solution of (37)	0	2	0	0	$\zeta_c = \zeta_p$	$w_c = \frac{1}{2}w_p$
1st solution of (38)	2	0	0	0	$\zeta_c = \zeta_p$	$w_c = 2w_p$
2nd solution of (38)	1	0	0	1	$\zeta_p = 0$	$w_c = w_p/(2\beta_c)$

cases since $\zeta_c = \zeta_p$. The only resonance case in which ζ_c is arbitrary is the second solution of equation (38).

For this case $L_J(\cdot)$ has a zero eigenvalue and the associated resonance term is $z_1^{l_1} z_2^{l_2} \bar{z}_1^{m_1} \bar{z}_2^{m_2} = z_1 \bar{z}_2$. The normal form equation of equation (16) with this resonance case is

$$\dot{z} = Jz + \begin{pmatrix} 0 \\ z_1 \bar{z}_2 \end{pmatrix} + \mathcal{O}(3). \tag{39}$$

When $\zeta_p \neq 0$ all second order non-linearities in equation (16) can be eliminated. However, the eigenvalue of $L_J(\cdot)$ corresponding to $z_1 \bar{z}_2$ using equation (31) is

$$w_p \zeta_p + w_p(\beta_p - 1), \tag{40}$$

which tends to zero as $\zeta_p \to 0$. Equations (24) and (31) indicate that the unknown coefficient of $h_2(z, \bar{z})$ associated with $z_1 \bar{z}_2$ is found by multiplying the coefficient of $z_1 \bar{z}_2$ in $G_2(z, \bar{z})$ and the inverse of equation (40). Hence, for $\zeta_p \ll 1$ elimination of $z_1 \bar{z}_2$ can lead to a large error in the normal form equation. Thus, one considers equation (39) as the normal form equation of equation (16) for $\zeta_p = 0$ or $\zeta_p \ll 1$.

So far, the normal form equation and the resonance terms of the transformed equation (16) have been studied. The normal form and resonance terms of the original equation (6) can now be determined using equation (39). Since the resonance term of the transformed controller equation (39) is $z_1 \bar{z}_2$, any term in the controller (4) which leads to $z_1 \bar{z}_2$ after using the transformations $\hat{T}$ and T is also a resonance term. Simple calculations show that there are four terms which produce $z_1 \bar{z}_2$. Since in MCC design, resonance terms must be used to make a strong energy link, the most general form of $u_c(x)$ is a linear combination of these terms:

$$u_c(x) = q_1 x_{p1} x_{c1} + q_2 x_{p1} x_{c2} + q_3 x_{p2} x_{c1} + q_4 x_{p2} x_{c2} = x_p^{\mathrm{T}} Q x_c, \tag{41}$$

where

$$Q = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}, \tag{42}$$

and $q_1 \ldots q_4$ are the controller feedback gains. It should be mentioned that since there is only one complex resonance term $(z_1 \bar{z}_2)$, $q_1 \ldots q_4$ are not independent. However, for simplicity one considers equation (41) as the controller feedback input.

At this point it is important to note that equation (39) shows that all second order non-linearities of the plant (2) are eliminated. Using equation (41) in equation (4), the controller equation remains the same (up to second order terms) if one considers the transformation

$$x_p = y_p + h_2(y_p, y_c), \qquad x_c = y_c, \tag{43a, b}$$

where y_p and y_c are the normal form subspaces spanned by the plant and the controller state variables, and $h_2(y_p, y_c)$ is second order in y. Therefore, the normal form equation of equation (6) using equation (43) is

$$\begin{pmatrix} \dot{y}_p \\ \dot{y}_c \end{pmatrix} = \begin{pmatrix} A_p & 0 \\ 0 & A_c \end{pmatrix} \begin{pmatrix} y_p \\ y_c \end{pmatrix} + \begin{pmatrix} 0 \\ U_c(y) \end{pmatrix} + \mathcal{O}(3). \tag{44}$$

Dropping higher order terms ($\mathcal{O}(3)$) from equation (44) the normal form equation (44) is written as

$$\dot{y}_p = A_p y_p, \qquad \dot{y}_c = A_c y_c + U_c(y). \tag{45a, b}$$

Equation (45) with initial conditions obtained from equation (43) approximates the original equation (6) in a neighborhood of the origin through second order terms. Note that the approximated plant normal form equation (45a) is purely oscillatory and independent from the second equation (45b). This implies that when $\zeta_p = 0$, the plant response has permanent oscillations regardless of what controller parameters are chosen. This was observed experimentally by Oueini and Golnaraghi [5].

2.2. CONTROLLER STRATEGY

One now uses the approximated normal form equation (45) along with transformation (43) to develop a new non-linear control strategy. In the closed loop system (6) the initial conditions of the plant are known, and those of the controller are to be selected. One may ask, what should the controller initial values be to yield the fastest decay in the plant response? Equation (45) shows that when the initial conditions of the plant in the normal form space are zero ($y_p(0) = (0, 0)$) the response of the plant in the normal form space, up to third order terms, is zero. This suggests a criterion for finding the controller initial values in the physical space so that the plant initial values in the normal form space become zero. One uses transformation (43) to achieve this task. Setting $y_p(0) = (0, 0)$, the equations

$$x_p(0) = h_2(0, y_c(0)), \qquad x_c(0) = y_c(0), \tag{46a, b}$$

are solved to obtain $y_c(0)$. Setting $y_p(0) = 0$ in equation (46) implies that one needs only that part of the transformation $h_2(\cdot)$ that is solely a function of y_c.

Recalling that the non-linear term $F_p(x_p)$ in the plant equation (3b) is only a function of x_p, the transformation required to eliminate the second order terms of $F_p(x_p)$ also becomes a function of x_p. Setting $y_p(0) = 0$ in equation (46) is independent to that part of the transformation which is responsible for eliminating the second order terms of $F_p(x_p)$. In other words, this method is robust to the non-linearities of the plant. This is an important observation revealing an advantage of the method for real applications.

Using this scheme to choose the controller initial conditions results in the maximum transfer of energy from the plant to the controller. After derivation of $h_2(\cdot)$ so that the controller initial conditions can be calculated, an example is solved in section 4 to obtain a better understanding of this method.

3. DERIVATION OF THE NORMAL FORM TRANSFORMATION

The method that was introduced in the previous section is based on the selection of the initial values of the controller so that the response of the plant in the normal form space becomes zero (up to second order terms). One needs to set $y_p(0) = 0$ and solve equation (46) for the controller initial conditions $y_c(0)$.

The controller input $u_c(x)$ (equation (41)) is the resonance terms of the system. The authors now claim that the most general plant input $u_p(x)$ is

$$u_p(x_c) = p_1 x_{c1}^2 + p_2 x_{c1} x_{c2} + p_3 x_{c2}^2. \tag{47}$$

This claim is justified by the fact that $y_c(0)$ is found by setting $y_p(0) = 0$ in equation (46). Any other second order terms in equation (47) are functions of x_p and thus the non-linear transformation required to eliminate these terms is also a function of x_p (or y_p). Therefore, setting $y_p(0) = 0$ makes this part of the transformation zero. Hence, equation (47) is the most general form of the plant input.

Knowing that $y_c(0)$ is independent of $F_p(x_p)$, one considers

$$\dot{x}_p = A_p x_p + U_p(x_c), \qquad \dot{x}_c = A_c x_c + U_c(x). \tag{48a, b}$$

Transformation (43) eliminates the non-linearities of (48a). One now derives the exact form of function $h_2(y_c)$ in equation (43). The non-linear part of equation (43), $h_2(y_c)$ as well as $y_p(x_c)$ are both members of vector-valued monomials H_2 in $\mathscr{R}^2$ with basis x_{c1}^2, $x_{c1}x_{c2}$, x_{c2}^2 (or y_{c1}^2, $y_{c1}y_{c2}$, y_{c2}^2). Therefore, they can be written as

$$u_p(x_c) = P^{\mathrm{T}} X_c, \qquad h_2(y_c) = \begin{pmatrix} \Delta^{\mathrm{T}} Y_c \\ \Delta'^{\mathrm{T}} Y_c \end{pmatrix}, \tag{49, 50}$$

where X_c, Y_c and P, Δ, Δ' are

$$X_c = (x_{c1}^2 \quad x_{c1}x_{c2} \quad x_{c2}^2)^{\mathrm{T}}, \qquad Y_c = (y_{c1}^2 \quad y_{c1}y_{c2} \quad y_{c2}^2)^{\mathrm{T}}, \tag{51a, b}$$

$$P = (p_1 \quad p_2 \quad p_3)^{\mathrm{T}}, \qquad \Delta = (\delta_1 \quad \delta_2 \quad \delta_3)^{\mathrm{T}}, \qquad \Delta' = (\delta_1' \quad \delta_2' \quad \delta_3')^{\mathrm{T}}. \tag{51c, d, e}$$

Taking the derivative of equation (43) and using equation (50), the transformed equation (48a) becomes

$$\dot{y}_p = A_p y_p + A_p \begin{pmatrix} \Delta^{\mathrm{T}} Y_c \\ \Delta'^{\mathrm{T}} Y_c \end{pmatrix} + \begin{pmatrix} 0 \\ P^{\mathrm{T}} Y_c \end{pmatrix} - \begin{pmatrix} \Delta^{\mathrm{T}} D(Y_c) A_c y_c \\ \Delta'^{\mathrm{T}} D(Y_c) A_c y_c \end{pmatrix} + \mathscr{O}(3), \tag{52}$$

where $D(Y_c)$ is the Jacobian of Y_c,

$$D(Y_c) = \begin{pmatrix} 2y_{c1} & 0 \\ y_{c2} & y_{c1} \\ 0 & 2y_{c2} \end{pmatrix}. \tag{53}$$

To eliminate the second order terms of equation (52), Δ and Δ' should satisfy

$$A_p \begin{pmatrix} \Delta^{\mathrm{T}} Y_c \\ \Delta'^{\mathrm{T}} Y_c \end{pmatrix} + \begin{pmatrix} 0 \\ P^{\mathrm{T}} Y_c \end{pmatrix} - \begin{pmatrix} \Delta^{\mathrm{T}} D(Y_c) A_c y_c \\ \Delta'^{\mathrm{T}} D(Y_c) A_c y_c \end{pmatrix} = 0. \tag{54}$$

Using equations (5a) and (53),

$$D(Y_c) A_c y_c = B Y_c, \tag{55}$$

where

$$B = \begin{pmatrix} 0 & 2 & 0 \\ -w_c^2 & -2w_c\zeta_c & 1 \\ 0 & -2w_c^2 & -4w_c\zeta_c \end{pmatrix}. \tag{56}$$

Now, substituting A_p from equation (3a) and using equation (55), equation (54) is solved to yield

$$\Delta' = B^{\mathrm{T}} \Delta, \qquad \Delta = (w_p^2 I + 2w_p\zeta_p B^{\mathrm{T}} + B^{\mathrm{T}} B^{\mathrm{T}})^{-1} P. \tag{57a, b}$$

When $\zeta_p = 0$, $\delta_1 \ldots \delta_3'$ are

$$\delta_1 = w_c^2 C(9\zeta_c p_1 - w_c(3\zeta_c^2 + 2)p_2 + 3w_c^2\zeta_c p_3), \tag{58a}$$

$$\delta_2 = 2w_c C((2 + 3\zeta_c^2)p_1 - w_c\zeta_c(3\zeta_c^2 - 1)p_2 + w_c^2(3\zeta_c^2 - 2)p_3), \tag{58b}$$

$$\delta_3 = C(3\zeta_c p_1 + w_c(2 - 3\zeta_c)p_2 + w_c^2\zeta_c p_3), \tag{58c}$$

$$\delta_1' = 2w_c^3 C(-(2 + 3\zeta_c^2)p_1 + w_c\zeta_c(3\zeta_c^2 - 1)p_2 + w_c^2(2 - 3\zeta_c^2)p_3), \tag{58d}$$

$$\delta_2' = 4w_c^2 C(\zeta_c(1 - 3\zeta_c^2)p_1 - w_c(2 + \zeta_c^2 - 3\zeta_c^4)p_2 + 3w_c^2\zeta_c\beta_c^2 p_3), \tag{58e}$$

$$\delta_3' = 2w_c C((2 - 3\zeta_c^2)p_1 - 3w_c\zeta_c\beta_c^2 p_2 - w_c^2(2 - \zeta_c^2)p_3), \tag{58f}$$

where

$$C = 1/[8w_c^4\zeta_c(4 - 3\zeta_c^2)]. \tag{59}$$

Where $\zeta_p \neq 0$ the elements of Δ and Δ' are given in Appendix A.

Transformation (43) with $h_2(y_c)$ as defined here can be used to obtain the controller initial values in equation (46).

4. SIMULATION RESULTS

In this section simulation is used to illustrate the controller design method described in section 3. The present results indicate that whenever the plant and controller are coupled through the non-linear terms (41) and (47) and the controller initial conditions are found from equation (46), a strong interaction develops between the systems.

As an example, consider equations (2) and (4) with $f_1(x_p) = 0$ and

$$u_p = p_2 x_{c1} x_{c2} + p_3 x_{c2}^2, \qquad u_c = q_3 x_{p2} x_{c2} + q_4 x_{p2} x_{c2}. \tag{60a, b}$$

Noticing that when $f_1(x_p) = 0$, $x_{p2} = \dot{x}_{p1}$ and $x_{c2} = \dot{x}_{c1}$, the equations in second order form can be written

$$\ddot{x}_{p1} + 2w_p\zeta_p\dot{x}_{p1} + w_p^2 x_{p1} = -C\dot{x}_{p1}^3 - Kx_{p1}^3 + u_p, \tag{61a}$$

$$\ddot{x}_{c1} + 2w_c\zeta_c\dot{x}_{c1} + w_c^2 x_{c1} = u_c, \tag{61b}$$

where $-C\dot{x}_{p1}^3 - Kx_{p1}^3$ represents the non-linearities of the system. For the sake of simulation one chooses $p_2 = 8$, $p_3 = -3$, $q_3 = -1$, and $q_4 = 0{\cdot}4$. Moreover, one sets $w_c = w_p/(2\sqrt{1 - \zeta_c^2})$ so that equation (61) is in resonance.

System (61) is now simulated for the four cases defined in Table 2. The initial conditions of the plant (61a) are assumed to be $x_p = (0{\cdot}1,\ 0{\cdot}2)^{\mathrm{T}}$ for all cases. Using (46a) with $h_2(y_c)$ defined in (50), the corresponding controller initial conditions are calculated for each case and shown in Table 2. It should be noted that equation (46a) is second order in y_c and the controller initial conditions can be obtained from a closed form solution. There are

TABLE 2

System parameters

Case	w_p	ζ_p	ζ_c	C	K	$x_{c1}(0)$	$\dot{x}_{c1}(0)$	$w_c = w_p/(2\sqrt{1 - \zeta_c^2})$
1	20	0	0·15	0	0	0·145	2·699	10·11
2	20	0·01	0·1	0·03	200	0·113	1·955	10·05
3	10	0	0·15	0	0	0·098	1·383	5·06
4	10	0·01	0·1	0·02	40	0·077	1·012	5·03

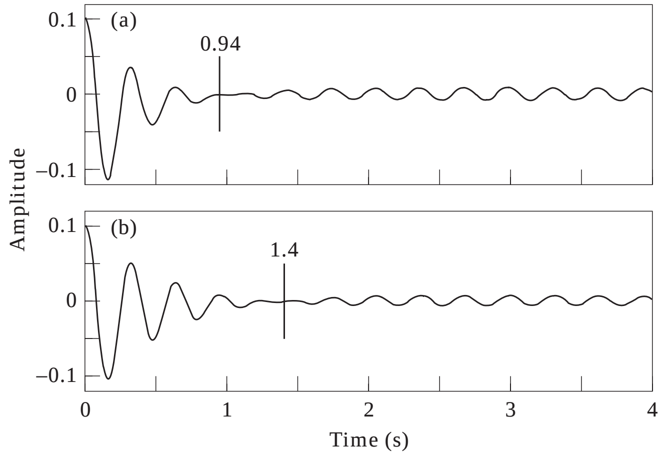

Figure 1. Plant response for (a) case 1 and (b) case 2 of Table 2.

two solutions for each case. These solutions are only different in sign and therefore the controller response, $x_{c1}(t)$ and $\dot{x}_{c1}(t)$, using either solutions are the same except for a 180° phase difference. However, the plant input is second order in x_{c1} and $\dot{x}_{c1}$, and therefore, the plant response is not affected by the choice of either solution.

Figure 1(a) is the plant response $x_{p1}(t)$ for case 1. As seen in the figure, the plant response decays to zero and then rises to steady state oscillations. The point of minimum responses is called the *neck* and the time that the neck occurs *neck time*. The neck is an indication of energy transformation from the plant to the controller and can be used as a means to design a non-linear controller. The truncated normal form equations (45) do not predict the existence of the neck. The neck is explained by the higher order terms which the authors studied in reference [1]. The vertical lines in the figures show the neck times that are found analytically in part II of this work.

Figure 1(b) is the simulation of system (61) for case 2 of Table 2 where the non-linear terms are added to equation (61a). As seen in the figure the occurrence of the neck is not affected by adding the non-linearities or by changing the controller damping. Figure 2(a)

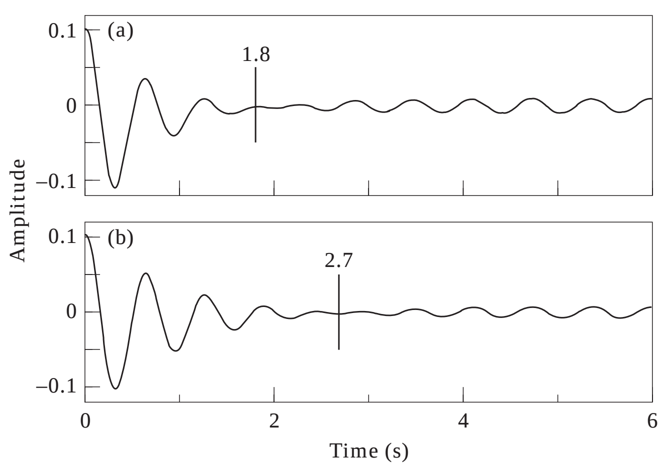

Figure 2. Plant response for (a) case 3 and (b) case 4 of Table 2.

demonstrates the response of the plant for case 3. In this case the plant is a simple linear undamped system with $w_p = 10$ and the same behavior emerges in the plant response.

Finally one examines the case when the plant equation has damping and non-linearities (case 4). Figure 2(b) is the plant response and shows that a neck exists for this general case. In other words, the existence of the neck is independent of the controller or plant damping as well as the system non-linearities.

5. CONCLUSION

This study provides a reliable analytical technique for modal coupling controller design. Upon transferring a closed loop system of non-linear coupled differential equations to normal form, most of the vague concepts associated with previous work on MCC strategy were addressed. Furthermore, the general form of the coupling terms where the transfer of energy is maximized was obtained.

Using system equations in the normal form space a new control scheme was introduced. The proposed control scheme was shown to result in a phenomenon called *neck* which corresponds to almost complete transfer of energy from the plant to the controller. In part II [1] a relation for the time that the plant takes to reach the neck is derived. An algorithm to implement the controller is also introduced. The controller is then examined using an experimental piezo-actuated flexible beam.

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APPENDIX A: ELEMENTS OF Δ AND Δ' WHEN $\zeta_p \neq 0$

When $\zeta_p \neq 0$, equations (58) and (59) become

$$\delta_1 = w_c^2 C((-4(4\zeta_c^4 - 3\zeta_c^2 - 1)\zeta_p^2 - 2\beta_c(6\zeta_c^2 + 7)\zeta_p\zeta_c + 9\zeta_c^2)p_1 + w_c(-8\zeta_c\beta_c^2\zeta_p^2 + 2\beta_c(1 + 5\zeta_c^2)\zeta_p - (3\zeta_c^2 + 2)\zeta_c)p_2 + w_c^2(4\beta_c^2\zeta_p^2 - 6\beta_c\zeta_c\zeta_p + 3\zeta_c^2)p_3), \tag{A.1}$$

$$\delta_2 = 2w_c C((8\beta_c^2\zeta_c\zeta_p^2 - 2\beta_c(5\zeta_c^2+1)\zeta_p + \zeta_c(2+3\zeta_c^2))p_1 + w_c(2\beta_c\zeta_c(2\zeta_c^2-1)\zeta_p - \zeta_c^2(3\zeta_c^2-1))p_2 + w_c^2(2\beta_c^3\zeta_p + \zeta_c(3\zeta_c^2-2))p_3), \quad (A.2)$$

$$\delta_3 = C((4\beta_c^2\zeta_p^2 - 6\zeta_c\beta_c\zeta_p + 3\zeta_c^2)p_1 + w_c(-2\beta_c^3\zeta_p + \zeta_c(2-3\zeta_c))p_2) + w_c^2((4\beta_c^2\zeta_p^2 + 2\beta_c\zeta_c(2\zeta_c^2-3) + \zeta_c^2)p_3), \quad (A.3)$$

$$\delta_1' = 2w_c^3 C((-8\zeta_c\beta_c^2\zeta_p^2 + 2\beta_c(5\zeta_c^2+1)\zeta_p - \zeta_c(2+3\zeta_c^2))p_1 + w_c(2\beta_c\zeta_c(1-2\zeta_c^2)\zeta_p + \zeta_c^2(3\zeta_c^2-1))p_2 + w_c^2(-2\beta_c^3\zeta_p + \zeta_c(2-3\zeta_c^2))p_3), \quad (A.4)$$

$$\delta_2' = 4w_c^2 C((-2\beta_c\zeta_c(1-2\zeta_c^2)\zeta_p + \zeta_c^2(1-3\zeta_c^2))p_1 + w_c(-4\zeta_c\beta_c^2\zeta_p^2 + 2\beta_c(1+3\zeta_c^2-2\zeta_c^4)\zeta_p - \zeta_c(2+\zeta_c^2-3\zeta_c^4))p_2 + w_c^2(-2\beta_c\zeta_c\zeta_p + 3\zeta_c^2\beta_c^2)p_3), \quad (A.5)$$

$$\delta_3' = 2w_c C((-2\beta_c^3\zeta_p + \zeta_c(2-3\zeta_c^2))p_1 + w_c(2\beta_c\zeta_c\zeta_p - 3\zeta_c^2\beta_c^2)p_2 + w_c^2(-8\zeta_c\beta_c^2\zeta_p^2 + 2\beta_c(1+5\zeta_c^2-4\zeta_c^4)\zeta_p - \zeta_c(2-\zeta_c^2))p_3) \quad (A.6)$$

and C is

$$C = (8w_c^4(-8\zeta_c\beta_c^3\zeta_p^3 + 4(1+3\zeta_c^2-6\zeta_c^4+2\zeta_c^6)\zeta_p^2 - 2\beta_c\zeta_c(4+2\zeta_c^2-3\zeta_c^4)\zeta_p + \zeta_c^2(4-3\zeta_c^2))^{-1}. \quad (A.7)$$