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A NOTE ON TRANSVERSE VIBRATIONS OF CIRCULAR, ANNULAR, COMPOSITE MEMBRANES

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1. Introduction

The dynamic analysis of mechanical systems with discontinuously varying material properties has been the subject of many recent investigations. In an excellent paper, Spence and Horgan [1] found upper and lower bounds for the natural frequencies of vibration of a circular membrane with stepped radial density and they showed that eigenvalue estimation techniques based on an integral equation approach are more effective than classical variational techniques. A conformal mapping approach was used in reference [2] in the case of composite membranes of regular polygonal shape whose inner circular core possesses a density ρ_1 while the remaining is characterized by ρ_0.

In general previous investigations deal with composite, simply connected membranes. The present study deals with a doubly connected membrane, fixed at radii $\bar{b}$ (outer radius) and $\bar{a}$ (inner radius).

Two schemes are assumed in the case of continuous variation of the density (Figure 1(a)):

(A-1): $$\rho(r) = \rho_0(1 + \alpha r) = \rho_0 f(r), \qquad \text{where } r = \bar{r}/\bar{b} \text{ and } f(r) = 1 + \alpha r, \tag{1}$$

(A-2): $$\rho(r) = \rho_0[1 + \alpha(r - a)] = \rho_0 f(r), \qquad \text{where } f(r) = 1 + \alpha(r - a);\ a = \bar{a}/\bar{b}. \tag{2}$$

Figure 1(b) depicts the situation where the density varies in a discontinuous fashion.

2. Solution by means of the optimized Rayleigh–Ritz method

The classical Rayleigh–Ritz method requires minimization of the functional

$$J[W] = U_{max} - T_{max}, \tag{3}$$

where

$$U_{max} = \frac{S}{2}\int_a^1\int_0^{2\pi}\left[\frac{\mathrm{d}W}{\mathrm{d}r}\right]^2 r\,\mathrm{d}r\,\mathrm{d}\theta, \qquad T_{max} = \tfrac{1}{2}\omega^2\int_a^1\int_0^{2\pi}\rho W^2 r\,\mathrm{d}r\,\mathrm{d}\theta, \tag{4a, b}$$

W is the displacement amplitude and S is the applied tensile force per unit length, at each boundary.

Expressing W in the form

$$W \simeq W_a = \sum_{n=1}^{N} A_n g(r), \tag{5}$$

where each co-ordinate function $g(r)$ satisfies the boundary conditions

$$W(a) = W(1) = 0, \tag{6}$$

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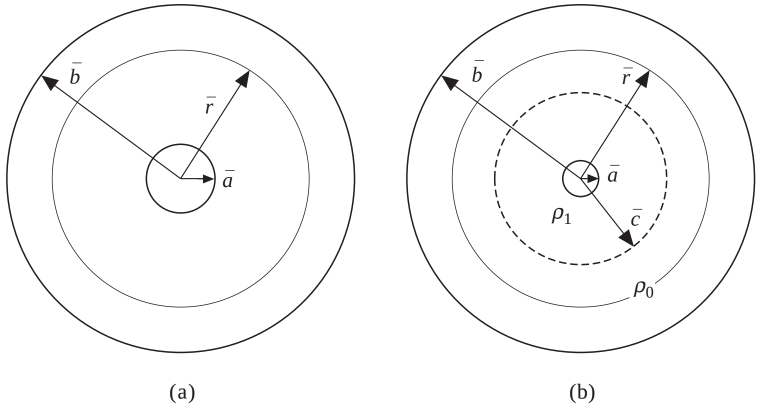

Figure 1. Circular, annular membranes of non-homogeneous density: (a) continuous variation of the membrane density, (b) discontinuous variation of the membrane density; $r = \bar{r}/\bar{b}$, $a = \bar{a}/\bar{b}$; (A-1), $\rho = \rho_0(1 + \alpha r)$; (A-2), $\rho = \rho_0[1 + \alpha(r - a)]$.

substituting in (3) and requiring

$$\partial J/\partial A_n[W_a] = 0 \quad (n = 1, 2, \ldots, N), \tag{7}$$

one finally obtains a determinantal equation in the desired eigenvalues $\sqrt{(\rho_0/S)}\omega\bar{b}$. Including an exponential parameter γ in the co-ordinate functions, one is able to optimize the frequency coefficients by requiring that

$$\mathrm{d}\Omega/\mathrm{d}\gamma = 0. \tag{8}$$

Table 1 depicts the co-ordinate functions used in the numerical experiments performed in the present investigation. A rather interesting feature is the fact that the co-ordinate functions contain a singularity at $r \to 0$.

3. USE OF THE DIFFERENTIAL QUADRATURE METHOD

In the case of normal modes of vibration the governing differential equation is

$$rW'' + W' + (\rho(r)/S)b^2\omega^2 rW = 0, \tag{9}$$

TABLE 1

Co-ordinate functions employed when using the optimized Rayleigh–Ritz method

Approximation	Function
(1)	$W_a = A_1(1 - a/r)[1 - (r)^\gamma]$
(2)	$W_a = A_1(r - a)[1 - (r)^\gamma] \ln r$
(3)	$W_a = A_1(1 - a/r)[1 - (r)^\gamma] + A_2(1 - a/r)[1 - (r)^{\gamma+1}]$
(4)	$W_a = A_1(1 - a/r)[1 - (r)^\gamma] + A_2[1 - (a/r)^2][1 - (r)^{\gamma+1}]$
(5)	$W_a = A_1(r - a)[1 - (r)^\gamma] \ln(r) + A_2(1 - a/r)[1 - (r)^{\gamma+1}]$

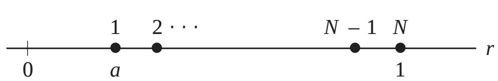

Figure 2. Partition of the interval [0, 1] when applying the differential quadrature method.

TABLE 2

Comparison of values of the first two eigenvalues Ω_{01} and Ω_{02} in the case of axisymmetric modes of vibration (uniform density case)

a/b	Approximation (2)	(I)	Approximation (4)	(5)	(II)	Exact Values
0·10	3·334	3·514	3·325	3·319	3·324	3·3139
	—	—	7·290	6·970	7·254	6·8576
0·20	3·836	3·952	3·822	3·823	3·816	3·8159
	—	—	7·960	7·956	8·146	7·7855
0·30	4·437	4·517	4·417	4·420	4·418	—
	—	—	9·037	9·170	9·286	—
0·40	5·214	5·270	5·186	5·192	5·198	5·1830
	—	—	10·90	10·746	10·822	10·4432
0·50	6·285	6·324	6·248	6·257	6·272	—
	—	—	12·93	12·926	12·976	—
0·60	7·878	7·905	7·830	7·843	7·869	7·8284
	—	—	16·08	16·17	16·212	15·6948
0·70	10·523	10·540	10·456	10·476	10·517	—
	—	—	21·391	21·588	21·609	—
0·80	—	15·811	15·699	15·731	15·797	15·6980
	—	—	32·05	32·39	32·407	31·4110

(I): one term solution (5): $W \simeq A_1(\bar{r} - \bar{a})(\bar{b} - \bar{r})$
(II): two term solution (5): $W \simeq A_1(\bar{r} - \bar{a})(\bar{b} - \bar{r}) + A_2(\bar{r} - \bar{a})(\bar{b}^2 - \bar{r}^2)$
NOTE: Galerkin's method was used in reference [5].

where, as previously stated in the Introduction,

(A-1): $$\rho(r) = \rho_0(1 + \alpha r) = \rho_0 f(r), \qquad f(r) = 1 + \alpha r;$$

(A-2): $$\rho(r) = \rho_0[1 + \alpha(r - a)] = \rho_0 f(r), \qquad f(r) = 1 + \alpha(r - a), \qquad a = \bar{a}/\bar{b}.$$

Expressing equation (9) in the form

$$rW'' + W' + \Omega^2 f(r) r W = 0 \tag{10}$$

which must satisfy $W(a) = W(1) = 0$ and making use of Bert and associates well established notation [3, 4], one obtains the following homogeneous, linear system of equations which allows for the approximate determination of the Ω's:

$$\sum_{k=2}^{N-1} (r_i B_{ik} + A_{ik}) W_k + \Omega^2 f(r_i) r_i W_i = 0, \qquad i = 2, \ldots, N-1. \tag{11}$$

The partition of the interval [0·1] is depicted in Figure 2. The frequency coefficient Ω_{01} was determined making $N = 9$ while Ω_{02} was evaluated taking $N = 12$, in general.

4. NUMERICAL RESULTS

In order to ascertain the accuracy achieved using the different co-ordinate functions shown in Table 1, they were employed first in the case of annular, homogeneous membranes; Table 2. In general the approximation ((4)) did provide the most accurate value of Ω_{01}. The results obtained using approximations ((1)) and ((2)) were not as accurate

TABLE 3

Comparison of values of Ω_{01} $(\rho = \rho_0(1 + \alpha r))$

		α						
a/b		0	0·50	1	1·50	2	2·50	3
0·10	R–R	3·325	2·948	2·674	2·464	2·296	2·158	2·042
	DQ	3·320	2·942	2·668	2·457	2·290	2·152	2·036
0·20	R–R	3·822	3·353	3·021	2·772	2·575	2·414	2·280
	DQ	3·816	3·347	3·016	2·766	2·569	2·409	2·275
0·30	R–R	4·417	3·837	3·437	3·141	2·909	2·722	2·565
	DQ	4·412	3·832	3·433	3·136	2·905	2·718	2·563
0·40	R–R	5·186	4·464	3·976	3·620	3·345	3·124	2·942
	DQ	5·183	4·460	3·973	3·616	3·341	3·121	2·939
0·50	R–R	6·248	5·328	4·722	4·284	3·949	3·682	3·463
	DQ	6·246	5·325	4·719	4·282	3·947	3·680	3·461
0·60	R–R	7·830	6·617	5·835	5·278	4·854	4·519	4·244
	DQ	7·828	6·615	5·833	5·275	4·852	4·517	4·242
0·70	R–R	10·456	8·759	7·687	6·931	6·362	5·914	5·548
	DQ	10·455	8·758	7·685	6·930	6·361	5·912	5·547
0·80	R–R	15·685	13·036	11·387	10·240	9·831	8·707	8·161
	DQ	15·698	13·036	11·387	10·239	9·380	8·706	8·159

R–R: Optimized Rayleigh–Ritz; co-ordinate functions (4). DQ: Differential quadrature

TABLE 4

Comparison of values of Ω_{02} $(\rho = \rho_0(1 + \alpha r))$

		α						
a/b		0	0·50	1	1·50	2	2·50	3
0·10	R–R	7·293	6·549	6·007	5·587	5·246	4·963	4·722
	DQ	6·858	6·078	5·520	5·093	4·754	4·474	4·239
0·20	R–R	7·964	7·027	6·368	5·869	5·473	5·149	4·877
	DQ	7·785	6·832	6·164	5·661	5·265	4·942	4·672
0·30	R–R	9·037	7·873	7·070	6·473	6·008	5·630	5·317
	DQ	8·932	7·763	6·961	6·367	5·903	5·528	5·216
0·40	R–R	10·609	9·107	8·112	7·389	6·831	6·384	6·015
	DQ	10·443	8·990	8·015	7·302	6·751	6·309	5·943
0·50	R–R	12·936	10·956	9·686	8·650	8·088	7·509	7·082
	DQ	12·546	10·702	9·489	8·613	7·943	7·408	6·969
0·60	R–R	16·083	13·820	12·012	10·872	10·006	9·318	8·755
	DQ	15·694	13·266	11·701	10·586	9·739	9·067	8·518
0·70	R–R	21·391	16·672	15·744	14·202	13·039	12·122	11·375
	DQ	20·935	17·539	15·394	13·883	12·745	11·847	11·116
0·80	R–R	32·059	26·632	23·270	20·926	19·173	17·797	16·681
	DQ	31·415	26·090	22·792	20·495	18·776	17·428	16·334

TABLE 5

Comparison of values of Ω_{01} $(\rho = \rho_0[1 + \alpha(r + a)])$

a/b		α = 0	0·50	1	1·50	2	2·50	3
0·10	R–R	3·325	3·007	2·764	2·571	2·413	2·281	2·168
	DQ	3·320	3·002	2·758	2·564	2·406	2·274	2·161
0·20	R–R	3·822	3·490	3·526	3·019	2·844	2·696	2·569
	DQ	3·816	3·484	3·223	3·012	2·837	2·689	2·562
0·30	R–R	4·417	4·075	3·799	3·571	3·380	3·216	3·072
	DQ	4·412	4·069	3·794	3·566	3·374	3·210	3·067
0·40	R–R	5·186	4·836	4·546	4·301	4·091	3·909	3·749
	DQ	5·183	4·832	4·541	4·296	4·086	3·904	3·743
0·50	R–R	6·248	5·890	5·586	5·322	5·092	4·889	4·708
	DQ	6·246	5·887	5·582	5·318	5·088	4·885	4·704
0·60	R–R	7·830	7·466	7·145	6·861	6·608	6·831	6·175
	DQ	7·828	7·463	7·142	6·858	6·604	6·377	6·170
0·70	R–R	10·456	10·084	9·748	9·442	9·162	8·905	8·668
	DQ	10·455	10·083	9·746	9·439	9·159	8·902	8·664
0·80	R–R	15·685	15·319	14·966	14·632	14·317	14·023	13·738
	DQ	15·698	15·319	14·965	14·633	14·321	14·028	13·752

TABLE 6

Comparison of values of Ω_{02} $(\rho = \rho_0[1 + \alpha(r - a)])$

a/b		α = 0	0·50	1	1·50	2	2·50	3
0·10	R–R	7·293	6·685	6·224	5·855	5·549	5·289	5·063
	DQ	6·858	6·201	5·709	5·320	5·003	4·736	4·509
0·20	R–R	7·964	7·318	6·823	6·424	6·091	5·808	5·563
	DQ	7·785	7·111	6·592	6·175	5·830	5·537	5·285
0·30	R–R	9·037	8·364	7·829	7·391	7·024	6·709	6·435
	DQ	8·932	8·245	7·700	7·254	6·878	6·557	6·277
0·40	R–R	10·609	9·865	9·280	8·799	8·391	8·038	7·727
	DQ	10·443	9·742	9·171	8·693	8·284	7·929	7·616
0·50	R–R	12·936	12·099	11·438	10·879	10·417	9·987	9·642
	DQ	12·546	11·833	11·233	10·719	10·272	9·878	9·526
0·60	R–R	16·083	15·359	14·734	14·184	13·636	13·174	12·725
	DQ	15·694	14·967	14·337	13·784	13·291	12·850	12·451
0·70	R–R	21·391	20·650	19·987	19·390	18·846	18·349	17·982
	DQ	20·935	20·194	19·531	18·931	18·386	17·887	17·428
0·80	R–R	32·059	31·031	30·598	29·945	29·335	28·763	28·226
	DQ	31·415	30·660	29·959	29·306	28·696	28·123	27·585

TABLE 7

Values of Ω_{01} in the case of discontinuous variation of the density for $a/b = 0{\cdot}10$, $c/b = 0{\cdot}50$

ρ_1/ρ_0	$\Omega_{01}(2)$	$\Omega_{01}(4)$	Ω_{02}	$\Omega_{01}(5)$	Ω_{02}
0·10	4·115	4·058	13·860	4·078	10·263
0·50	3·707	3·707	9·019	3·697	8·267
0·90	3·401	3·396	7·522	3·389	7·152
1·50	3·059	3·013	6·529	3·013	6·352
2	2·840	2·763	6·075	2·765	5·989
5	2·101	1·947	4·847	1·949	5·157
10	1·586	1·426	4·015	1·429	4·595

NOTE: these values have been determined using the co-ordinate functions (2), (4) and (5), see Table 1.

as those obtained using approximation ((4)) and they are not shown in Table 2. Values of Ω_{01} and Ω_{02} for annular membranes of non-uniform density (case A-1) obtained by means of the optimized Rayleigh–Ritz approach and the differential quadrature technique are shown in Tables 3 and 4, respectively. Excellent agreement is observed.† Tables 5 and 6 depict comparisons of values of Ω_{01} and Ω_{02}, respectively, for the non-uniform density case defined as (A-2). Again very good agreement is observed.

Table 7 shows values of Ω_{01} obtained in the case of discontinuous variation of the density by means of the optimized Rayleigh–Ritz method for a membrane defined by $a/b = 0{\cdot}10$ and $c/b = 0{\cdot}50$. Different co-ordinate functions have been employed and the results have been obtained as a function of ρ_1/ρ_0. For $\rho_1/\rho_0 < 0{\cdot}90$ the value of Ω_{01} determined using approximation ((5)) is slightly lower, hence more accurate, than the value of Ω_{01} obtained employing approximation ((4)). The situation reverses for $\rho_1/\rho_0 > 1{\cdot}50$. A similar situation is observed in the case of Ω_{02} for $\rho_1/\rho_0 \leqslant 2$. The present approach can be extended to more complicated boundary shapes in the case of discontinuous variation of the thickness following the approach developed in reference [2].

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† The approximation ((4)) was used in Tables 3–6.