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SINGULAR VALUES OF CUMULANT MATRICES

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1. INTRODUCTION

The identification of cutting states, associated with the orthogonal cutting of stiff cylinders, was realized in reference [1] through an analysis of the singular values of an unsymmetric Toeplitz matrix, **R**, of third order cumulants, $r(i, j)$, of acceleration measurements. The ratio of the two dominant pairs of singular values of **R**, the R-ratio, was shown to differentiate between light cutting, medium cutting, pre-chatter and chatter states. The **R** matrix is the coefficient matrix in an autoregressive approximation of the bispectrum references [6, 7].

On the basis of an analysis of five sequences of experiments with variable depth of cut and two sequences with variable turning frequency, a total of 42 cutting experiments, it was found that the R-ratio was approximately one for all cases of light cutting and two or more for chatter. For intermediate states the ratio increased as the chatter state was approached. The R-ratio was evaluated for $q = 100$; see equation (6) below.

Relationships between phase coupled trigonometric functions and the singular values of the corresponding **R** matrix were established in reference [1] through a numerical study of three specific phase coupled functions. However, no general assertions regarding relationships between properties of the R-ratio and those of the associated time series were made. Such a relationship is established in what follows for time sequences consisting of sums of phase coupled cosine functions.

The third order cumulant, $r(i, j)$, of a time series consisting of the sum of phase coupled cosine functions may be expressed as a finite sum of cosine functions; see reference [6]. If the sums of cosine functions are periodic for some integral value of the argument of $r(i, i)$, then the Toeplitz matrix $\mathbf{R}(r(i, j))$, of dimension $q + 1$, is circulant for a sequence of values of q. In this case, a simple closed form representation for singular values of R is known (see references [2, 4, 8]), which yields an expression for the R-ratio in terms of the coefficient of the phase coupled cosine functions.

2. HIGHER ORDER SPECTRA

Let $r_3(\tau_1, \tau_2)$ be the third order cumulant of the real third order stationary random process $X(k)$, $k = 0, \pm 1, \pm 2, \ldots$. If the mean of $X(k)$ vanishes, then $r_3(\tau_1, \tau_2) = m_3(\tau_1, \tau_2)$, where $m_3(\tau_1, \tau_2) = \mathrm{E}(X(k)\, X(k + \tau_1)\, X(k + \tau_2))$. E is the expected

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value, which may be estimated as follows:

$$m_3(\tau_1, \tau_2) = (1/2n) \sum_{k=n}^{+n} X(k)X(k+\tau_1)X(k+\tau_2), \tag{1}$$

$n \to +\infty$. The bispectrum of $X(k)$, $R_3(\omega_1, \omega_2)$, is defined by

$$R_3(\omega_1, \omega_2) = \sum_{\tau_1=-\infty}^{+\infty} \sum_{\tau_2=-\infty}^{+\infty} r_3(\tau_1, \tau_2) \exp[-\mathrm{j}(\omega_1\tau_1 + \omega_2\tau_2)]. \tag{2}$$

Consider an autoregressive (AR) estimate of the bispectrum; see reference [6, 7]. A qth order AR process is described by

$$X(k) + \sum_{i=1}^{q} a(i)X(k-i) = W(k), \tag{3}$$

where $W(k)$ is non-Gaussian, $\mathrm{E}(W(k)) = 0$ and $\mathrm{E}(W^3(k)) = \beta$. Multiplying through equation (3), summing and noting equation (1) gives

$$r_3^x(-k, -l) + \sum_{i=1}^{q} a(i) r_3^x(i-k, i-l) = \beta\delta(k, l), \tag{4}$$

where $k, l > 0$. Letting $k = l$ in equation (4), with $k = 0, \ldots, q$, yields $q + 1$ equations for the $q + 1$ unknowns $a(i)$ and β. In matrix notation equation (4) may be expressed as

$$\mathbf{Ra} = \mathbf{b}, \tag{5}$$

where

$$\mathbf{R} = \begin{vmatrix} r(0) & r(1) & r(2) & \cdots & r(q) \\ r(-1) & r(0) & r(1) & \cdots & r(q-1) \\ r(-2) & r(-1) & r(0) & & \\ \vdots & \vdots & & & \vdots \\ r(-q) & r(-q+1) & & \cdots & r(0) \end{vmatrix}, \tag{6}$$

$r(i) \equiv r_3^x(i, i)$, $\mathbf{a} \equiv [1, a(1), \ldots, a(q)]^{\mathrm{T}}$ and $\mathbf{b} \equiv [\beta, 0, \ldots, 0]^{\mathrm{T}}$. $\mathbf{R}$ is in general a non-symmetric Toeplitz matrix.

3. CIRCULANT MATRICES

Circulant matrices are a subset of Toeplitz matrices, in which row i is formed by shifting row $i - 1$ to the right by one element. A circulant matrix $\mathbf{C}$, has the form

$$\mathbf{C} = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_{p-1} \\ c_{p-1} & c_0 & c_1 & c_2 & \vdots \\ \vdots & c_{p-1} & c_0 & & c_2 \\ c_1 & \cdots & \cdots & c_{p-1} & c_0 \end{vmatrix}. \tag{7}$$

It is shown in references [2, 4, 8] that the eigenvalues of $\mathbf{C}$, equation (7), are given by

$$\lambda_m = \sum_{k=0}^{p-1} c_k \exp(-2\pi mki/(p)), \tag{8}$$

where $m = 1, 2, \ldots, p+1$. Each eigenvalue is a discrete Fourier transform of the first row of $\mathbf{C}$.

The Toeplitz matrix, $\mathbf{R}$, equation (6), is a circulant matrix, $\mathbf{C}$, equation (7), if the cumulant $r(k)$ is periodic with integer period α. Then $r(-k) = r(-k + n\alpha)$, $n = 1, \ldots$. $\mathbf{R}$ is circulant, as shown in equation (7), if $r(q-k) = r(-k)$ with $k = 0, 1, \ldots, q$, which is satisfied for $q = n\alpha$. The eigenvalues of R are then given by equation (8) with $p = q + 1$. If $\mathbf{R}$ is symmetric, $r(i) = r(-i)$, then $|\lambda_i| = \sigma_i$, where σ_i are the singular values of $\mathbf{R}$; see references [3, 5]. If $\mathbf{R}$ is circulant, then λ_i occur in pairs. Simple expressions for λ_m (see equation (8)), may be found for sums of commensurate trigonometric functions through the application of the orthogonality conditions

$$\sum_{k=0}^{(r-1)} \sin(2\pi mk/r)\cos(2\pi nk/r) = 0, \tag{9}$$

and

$$\sum_{k=0}^{(r-1)} \cos(2\pi mk/r)\cos(2\pi nk/r) = (0, r/2), \tag{10}$$

for integers m and n, with $m \neq n$ and $m = n$, respectively.

3.1. *Phase coupling*

Consider the function

$$X(t) = \sum_{m=1}^{N}\sum_{i=1}^{3} a_{mi}\cos 2\pi(\theta_{mi}t + \phi_{mi}) + \sum_{m=1}^{M} a'_m \cos 2\pi(\theta'_m t + \phi'_m), \tag{11}$$

where $\theta_{m3} = \theta_{m1} + \theta_{m2}$, $\phi_{m3} = \phi_{m1} + \phi_{m2}$ and ϕ_{mi} and ϕ'_m are independent and uniformly distributed over (0, 1); see references [6, 7]. $X(t)$ consists of the sum of N triplets of phase coupled cosine functions and M non-coupled cosine functions. The cumulants, given in equation (1), associated with equation (11) are [6, 7],

$$\begin{aligned} r(i,j) = \tfrac{1}{4}\sum_{k=1}^{N} b_k[&\cos 2\pi(\theta_{k3}i - \theta_{kl}j) + \cos 2\pi(\theta_{k2}i - \theta_{k3}j) \\ &+\cos 2\pi(\theta_{k3}i - \theta_{k2}j) + \cos 2\pi(\theta_{kl}i - \theta_{k2}j) \\ &+\cos 2\pi(\theta_{kl}i - \theta_{k3}j) + \cos 2\pi(\theta_{k2}i + \theta_{kl}j)], \end{aligned} \tag{12}$$

where $b_k = a_{k1}a_{k2}a_{k3}$ with $a_{k1} = a_{k2} = a_{k3}$.

Letting $i = j$ in equation (12) gives

$$r(i) = \tfrac{1}{2}\sum_{k=1}^{N} b_k(\cos 2\pi\theta_{kl}i + \cos 2\pi\theta_{k2}i + \cos 2\pi\theta_{k3}i). \tag{13}$$

If the frequencies θ_{ki} are commensurate, then $r(i)$ will be periodic with period α. Then equations (8), (9), (10) and (13) imply that the eigenvalues of $\mathbf{R}$ with elements given by equation (13) are

$$\lambda j = (n\alpha/4) \sum_{i=1}^{Nj} b_j(i), \tag{14}$$

where N_j is the number of terms in equation (13) that have the same frequency, $b_j(i)$ is the corresponding coefficient in equation (13), where $n = 1, 2, \ldots$ and $q = n\alpha$. Since $r(i) = r(-i)$ in equation (13), then $|\lambda| = \sigma$. Equation (14) implies that

$$\lambda_k/\lambda_j = \sigma_k/\sigma_j = \sum_{i=1}^{N_k} b_k(i) \Bigg/ \sum_{i=1}^{N_j} b_j(i).$$

4. EXAMPLES

Consider the following test functions that exhibit phase coupling closely approximating that observed in orthogonal cutting data [1]. Let

$$r(i) = 2\cos 2\pi(0{\cdot}1i) + \cos 2\pi(0{\cdot}2i) + 5\cos 2\pi(0{\cdot}3i). \tag{15}$$

Expressing equation (15) as

$$r(i) = \sum_{r=1}^{3} b_r \cos(2\pi i r/\alpha) \tag{16}$$

and noting equations (8), (9) and (10),

$$\lambda_r = n\alpha b_r/2. \tag{17}$$

Then $\lambda_1 = 10\,n$, $\lambda_2 = 5\,n$ and $\lambda_3 = 25\,n$, since $\alpha = 10$ in equation (15). The six largest singular values of equation (6) and equation (15) as functions of q are displayed in Figure 1. For $q = n\alpha$, the values of λ_r are in agreement with equation (17). From equation (17) it follows that $\lambda_i/\lambda_j = \sigma_i/\sigma_j = b_i/b_j$.

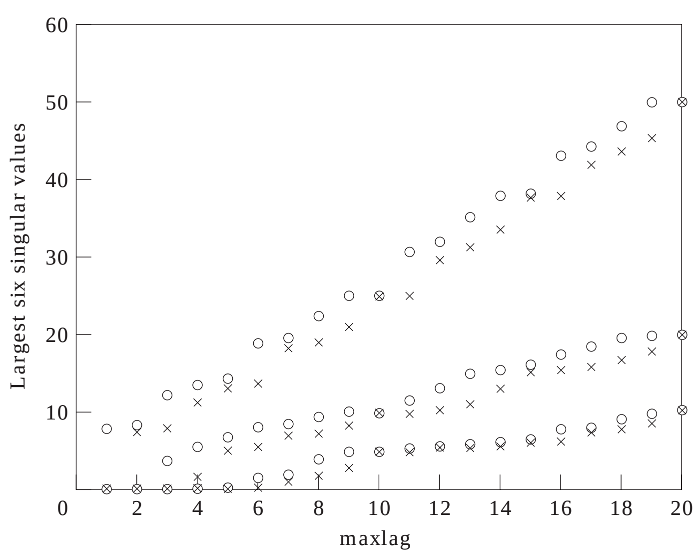

Figure 1. The six largest singular values of equation (6) for equation (15).

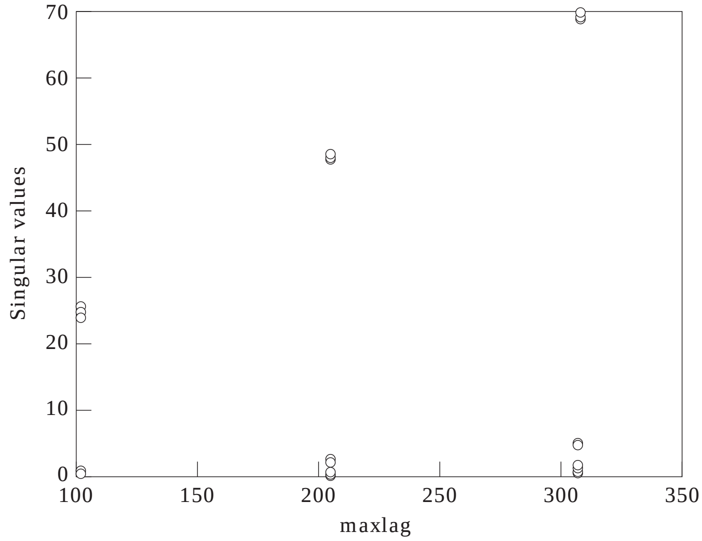

Figure 2. The six largest singular values of the phase coupled function $X(t)$.

The six largest singular values of the phase coupled function X(t), equation (11), with $N = 1$, $M = 0$, $a_{11} = a_{12} = a_{13} = 1$, $\theta_{11} = 90$, $\theta_{12} = 100$ and $\theta_{13} = 190$, are shown in Figure 2. Then, with a sampling rate of 1024 Hz. and a period of 0·1 s, $\alpha = 102{\cdot}4$ lags. With $\alpha \simeq 102$ the repeated eigenvalues $\lambda \simeq 25{\cdot}5\, n$. For $n = 1$, 2 and 3, $\lambda = 25{\cdot}5$, 51·25 and 76·75. λ found from the numerical evaluation of **R** and its singular values gave 25·5, 48·5 and 69·9 for the largest pair of eigenvalues. The errors may be due to a non-integer period and noise associated with the uniformly distributed phases, ϕ, in equation (11). The six largest singular values as functions of q are also shown in Figure 2. For $q = n\alpha$, the values of λ are in approximate agreement with equation (14).

5. CONCLUSIONS

If the elements, $r(i)$, of an unsymmetric Toeplitz matrix, **R**, are periodic, then for a sufficiently large dimension **R** becomes circulant with eigenvalues given by the finite Fourier transform of the first row. The eigenvalues are then expressible in a simple closed form for sums of cosine functions. Then ratios of eigenvalues equal ratios of coefficients of the cosine functions. Application to matrices of third order cumulants of phase coupled cosine functions yields expressions for singular values which are useful in the identification of orthogonal cutting states.

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