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TRANSVERSE VIBRATION OF SKEW PLATES WITH VARIABLE THICKNESS

B. SINGH

Department of Mathematics, University of Roorkee, Roorkee (U.P.) 247667, India

AND

V. SAXENA

Priyadarshini College of Computer Sciences, 273 Shaheed Captain Gaur Marg, Sriniwaspuri, New Delhi 110065, India

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The Rayleigh–Ritz method has been used to study the transverse vibrations of skew plates of variable thickness with different combinations of boundary conditions at the four edges. The two-dimensional thickness variation is taken as the Cartesian product of linear variations along the two concurrent edges of the plate. The first three frequencies and mode shapes have been computed by using successive approximations. Convergence of results is ensured by working out several approximations until the results converge to four significant digits. In special cases, comparisons have been made with results that are available in the literature. Mode shapes have also been plotted for some selected cases.

1. INTRODUCTION

The study of transverse vibration of plates of various shapes under different boundary conditions is important owing to a wide variety of applications in engineering design. Leissa [1] and subsequent review articles [2–7] provide an extensive source of information on the subject. In the present discussion we shall confine our discussion to skew plates only. The special case of rectangular plates is of great importance but, again, we shall not touch upon this case, as there are a large number of papers already available on the subject, and bringing it into discussion will unnecessarily increase the bulk of this paper. We suggest that the reader see Gorman [8] for more information on rectangular plates. More recently, Singh and Chakraverty [9] have also given 87 references about rectangular and skew plates of uniform thickness. Singh and Saxena [10] have studied rectangular plates with variable thickness under different boundary conditions.

A large number of papers dealing mainly with the vibration of skew plates with uniform thickness have appeared in the literature from time to time. Dalley and Ripperger [11, 12] are perhaps two of the earlier references on skew plates. They gave the experimental results for skew and rectangular cantilever plates of uniform thickness. Barton [13] studied the vibration of rectangular and skew cantilever plates. Kaul and Cadambe [14] used the Rayleigh–Ritz method to compute the natural frequencies of thin skew plates for different combinations of boundary conditions. Hasegawa [15] has given the lowest natural frequency for isotropic clamped parallelogrammic plates using the Ritz method. Hamada [16] has obtained the fundamental frequency of rhomboidal plates with all edges clamped by employing the Lagrangian multiplier method. Plass *et al*. [17] studied Reissner's

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variational principle to find the frequencies of cantilever plate. Conway and Farnham [18] have found the values of the first natural frequency of skew plates of uniform thickness using the point-matching technique. Durvasula *et al*. [19–22] have studied the natural frequencies and modes of skew membranes and skew plates. They used the Galerkin and partition method to compute the first six to eight natural frequencies of plates of uniform thickness. Mizusawa *et al*. [23] studied the transverse vibrations of skew plates of uniform thickness with different combinations of boundary conditions. They used B-spline functions to obtain the natural frequencies. Liew and Lam [24, 25] have investigated the flexural vibration of skew plates by using the two-dimensional orthogonal plate function and vibration of multi-span plates having orthogonal straight edges with various combinations of boundary conditions. Fan and Luah [26] studied the free vibration analysis of general plates by a newly developed nine-node spline element method. The formulation of the problem is based on Kirchoff's thin plate theory. They employed B-spline shape functions to form the two-dimensional displacement functions and biquadratic Lagrangian functions for geometric interpolation.

The basic aim of the present investigation is to study the vibrations of skew plates of variable thickness under different boundary conditions at the edges. Some references that deal with skew plates of variable thickness and have come to the authors' notice are Dokainish and Kumar [27], Banerjee [28] and Liu and Chang [29]. Dokainish and Kumar [27] have computed the natural frequencies for isotropic fully clamped skew plates with linearly varying thickness in the y direction. They give the results for various combinations of aspect ratio, skew angle and taper parameters. Banerjee [28] has determined the natural frequencies of vibrating skew plates with thickness varying linearly in the x direction. In reference [28], the results for fully clamped skew plates with different aspect ratios, skew angles and taper constants are given. Liu and Chang [29] have studied non-uniform skewed cantilever plates by the finite element transfer matrix method and the conventional finite element method. The above papers deal with thickness variation parallel to one edge of the plate. The present paper considers two-dimensional thickness variation, which is the Cartesian product of two different linear variations parallel to the two adjacent edges. This brings about two taper parameters. The basis functions are chosen to satisfy the essential boundary conditions. Computations are continued until the two consecutive approximations converge to four significant digits. Since a large number of combinations of boundary conditions are possible, results are reported for CCCC, SSSS, CFFF and CSCS plates with different values of α, β, aspect ratio and skew angle. The first three frequencies and mode shapes are computed. Comparisons have been made with known results in special cases.

2. METHOD OF SOLUTION

Let the skew plate R be defined by three numbers a, b and θ, as shown in Figure 1(a). The transformation that maps the plate into the unit square R' is given by

$$x = a\xi + b\cos(\theta)\eta, \qquad y = b\sin(\theta)\eta, \tag{1}$$

where ξ and η are new co-ordinates as shown in Figure 1(b).

Let the thickness h at the point (ξ, η) of R' be given by

$$h = ah_0(1 + \alpha\xi)(1 + \beta\eta), \tag{2}$$

where α and β are taper parameters controlling the thickness variation and h_0 is the non-dimensional thickness at (0, 0).

For free vibration of the plate, the displacement is assumed to be of the form

$$w(x, y, t) = W(x, y) \sin \omega t, \quad (3)$$

where $W(x, y)$ is the maximum displacement at time t and ω is the angular frequency. The maximum strain and the maximum kinetic energies are given by

$$V_{max} = \tfrac{1}{2} \iint_R D[(W^{xx})^2 + 2\nu W^{xx} W^{yy} + (W^{yy})^2 + 2(1-\nu)(W^{xy})^2]\, \mathrm{d}x\, \mathrm{d}y, \quad (4)$$

$$T_{max} = \frac{\rho\omega^2}{2} \iint_R h W^2\, \mathrm{d}x\, \mathrm{d}y, \quad (5)$$

where

$$D = Eh^3/(12(1-\nu^2)), \quad (6)$$

is the flexural rigidity, E, ν and ρ are Young's modulus, the Poisson ratio and the density of the plate material, respectively, and superscripts indicate differentiation.

Equating the two energies yields the Rayleigh quotient:

$$\omega^2 = \frac{\iint_R D[(\nabla^2 W)^2 + 2(1-\nu)\{(W^{xy})^2 - W^{xx} W^{yy}\}]\, \mathrm{d}x\, \mathrm{d}y}{\iint_R \rho h W^2\, \mathrm{d}x\, \mathrm{d}y}. \quad (7)$$

Let us consider an N-term approximation,

$$W(x, y) = \sum_{j=1}^{N} c_j \phi_j(x, y), \quad (8)$$

where $\phi_j(x, y)$ are the basis functions satisfying the essential boundary conditions and the c_j are constants. Now minimizing ω^2 as a function of the constants $c_1, c_2, \ldots, c_N$ after substituting equation (8) in equation (7) and changing the variables from x, y to ξ, η, we

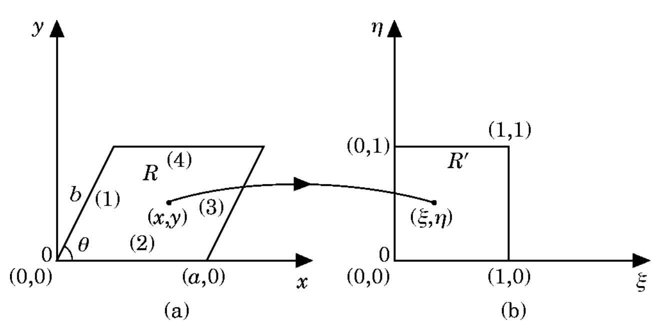

Figure 1. Mapping of the plate into a unit square plate: (a) Plate R; (b) Unit square plate R'.

TABLE 1

The first three frequencies ($\mu = 1{\cdot}0$)

		$\theta = 60^\circ$			$\theta = 45^\circ$			$\theta = 30^\circ$		
α	β	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
CCCC plate										
−0·5	−0·5	24·371	42·525	56·220	34·795	55·872	79·955	65·763	99·189	135·95
−0·5	0·0	33·742	59·399	76·714	48·501	78·256	110·94	92·698	137·46	196·27
−0·5	0·5	42·102	74·216	94·470	60·841	97·813	139·53	117·15	171·09	255·62
0·0	−0·5	33·742	59·399	76·714	48·501	78·256	110·94	92·698	137·46	196·27
0·0	0·0	46·166	81·602	105·52	66·330	106·72	156·34	127·06	184·58	282·95
0·0	0·5	57·187	100·95	130·45	82·171	132·44	190·66	157·24	230·44	341·95
0·5	−0·5	42·102	74·216	94·470	60·841	97·813	139·53	117·15	171·09	255·62
0·5	0·0	57·187	100·95	130·45	82·171	132·44	190·66	157·24	230·44	341·95
0·5	0·5	70·549	124·14	161·87	101·09	163·39	230·95	192·39	287·33	402·81
SSSS plate										
−0·5	−0·5	13·756	27·768	39·481	19·834	35·851	56·102	38·308	62·676	94·284
−0·5	0·0	18·688	38·704	53·191	27·210	49·574	78·920	53·475	84·115	139·82
−0·5	0·5	23·289	48·535	65·296	34·039	61·854	99·107	67·460	103·11	184·75
0·0	−0·5	18·688	38·704	53·191	27·210	49·574	78·920	53·475	84·115	139·82
0·0	0·0	25·314	52·660	72·714	36·971	66·707	112·49	73·135	111·38	209·84
0·0	0·5	31·467	65·403	90·112	45·902	83·224	137·29	90·565	139·79	249·45
0·5	−0·5	23·289	48·535	65·296	34·039	61·854	99·107	67·460	103·11	184·75
0·5	0·0	31·467	65·403	90·112	45·902	83·224	137·29	90·565	139·79	249·45
0·5	0·5	39·073	80·686	112·40	56·768	103·57	164·23	111·08	177·20	286·73
CFFF plate										
−0·5	−0·5	2·8700	6·1354	13·013	3·0764	7·1458	14·301	3·6193	9·3995	18·270
−0·5	0·0	4·1296	8·4818	19·342	4·5676	10·570	20·948	5·4610	15·162	27·379
−0·5	0·5	5·5507	11·029	24·272	6·2031	14·336	27·393	7·4323	21·448	36·913
0·0	−0·5	2·8427	7·2062	17·026	3·2702	8·4346	19·335	4·2830	11·215	24·194
0·0	0·0	3·9454	9·6209	26·011	4·6371	11·847	27·920	6·1125	17·215	35·275
0·0	0·5	5·2274	12·347	31·922	6·1681	15·747	35·567	8·0923	23·943	46·939
0·5	−0·5	2·8405	8·2203	20·528	3·4148	9·6864	24·054	4·8037	12·957	29·831
0·5	0·0	3·8756	10·755	30·606	4·7471	13·183	34·550	6·7071	19·169	42·807
0·5	0·5	5·0960	13·666	38·768	6·2525	17·266	43·505	8·7655	26·280	55·881
CSCS plate										
−0·5	−0·5	19·735	34·177	48·876	28·266	45·238	66·924	53·901	80·856	115·07
−0·5	0·0	27·125	47·015	68·386	39·091	62·246	95·660	75·083	110·08	168·20
−0·5	0·5	33·985	58·750	83·844	49·226	77·666	121·74	95·111	136·28	221·09
0·0	−0·5	27·419	47·516	66·581	39·698	62·820	93·854	76·969	110·00	169·52
0·0	0·0	37·193	64·389	93·577	53·840	84·446	136·02	104·53	146·39	248·26
0·0	0·5	46·210	80·083	114·64	66·892	105·36	163·59	129·80	183·28	298·21
0·5	−0·5	34·223	59·186	82·603	49·728	78·117	119·80	96·759	135·89	222·82
0·5	0·0	46·038	79·745	115·93	66·527	105·02	165·39	128·58	183·56	296·96
0·5	0·5	56·930	98·805	142·11	82·044	130·73	196·00	157·91	230·77	345·94

Table 2

The first three frequencies ($\mu = 2{\cdot}0$)

		$\theta = 60^\circ$			$\theta = 45^\circ$			$\theta = 30^\circ$		
α	β	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
CCCC plate										
−0·5	−0·5	66·457	85·787	114·79	97·283	120·30	153·27	189·80	225·47	275·15
−0·5	0·0	91·787	118·27	157·84	135·09	166·95	212·35	264·33	314·96	387·09
−0·5	0·5	114·35	147·29	197·13	168·84	208·36	269·04	331·47	392·30	499·53
0·0	−0·5	94·169	116·84	157·19	138·78	164·37	213·06	271·85	307·71	392·60
0·0	0·0	128·90	159·73	215·29	190·00	223·98	294·67	372·53	416·33	552·09
0·0	0·5	159·65	197·93	266·43	235·30	277·93	363·00	461·15	518·04	675·24
0·5	−0·5	116·18	146·21	195·76	171·64	206·33	270·64	337·32	386·07	508·28
0·5	0·0	158·14	198·86	266·46	232·88	279·84	361·35	455·71	524·06	667·99
0·5	0·5	195·27	245·69	328·50	287·06	345·70	440·20	560·67	650·06	799·92
SSSS plate										
−0·5	−0·5	34·126	52·294	80·131	49·859	71·792	103·59	96·575	132·69	181·42
−0·5	0·0	46·430	71·478	109·95	67·646	98·708	145·78	130·21	182·64	264·23
−0·5	0·5	57·469	89·396	139·18	83·499	123·49	187·92	159·97	227·38	347·14
0·0	−0·5	46·905	71·557	112·04	68·529	98·242	149·21	132·93	179·08	271·47
0·0	0·0	64·070	96·553	153·76	93·773	132·07	209·83	182·45	239·90	394·64
0·0	0·5	79·410	120·23	189·95	116·15	164·70	256·63	225·75	299·53	476·35
0·5	−0·5	57·836	89·490	141·13	84·167	123·12	191·96	162·06	224·28	357·55
0·5	0·0	79·119	120·24	188·02	115·58	165·22	253·21	223·77	302·64	468·60
0·5	0·5	98·194	149·23	230·01	143·57	205·38	303·36	278·28	377·99	547·78
CFFF plate										
−0·5	−0·5	2·9370	10·796	14·953	3·1112	11·359	19·098	3·5577	12·651	28·340
−0·5	0·0	3·9588	14·082	21·274	4·2575	15·466	27·480	4·8945	17·904	42·200
−0·5	0·5	5·1911	18·197	28·396	5·6071	20·442	36·963	6·4472	24·043	57·755
0·0	−0·5	2·7994	12·821	18·374	3·0899	14·038	24·069	3·8070	16·503	38·526
0·0	0·0	3·7332	16·150	25·812	4·1649	18·292	34·208	5·1278	22·289	55·873
0·0	0·5	4·8651	20·548	34·242	5·4275	23·732	45·586	6·6440	29·250	75·292
0·5	−0·5	2·7502	14·368	21·481	3·1317	16·043	28·648	4·0739	19·540	47·386
0·5	0·0	3·6474	17·855	29·955	4·1972	20·492	40·548	5·4476	25·937	67·945
0·5	0·5	4·7373	22·533	39·614	5·4424	26·335	53·828	7·0025	33·755	90·908
CSCS plate										
−0·5	−0·5	37·690	61·423	94·235	54·239	83·071	120·54	103·09	150·40	208·44
−0·5	0·0	51·005	83·868	129·48	73·241	113·43	168·71	138·90	204·07	299·49
−0·5	0·5	63·135	104·88	163·36	90·376	141·62	216·43	170·65	252·94	389·86
0·0	−0·5	51·555	84·824	131·35	73·963	114·22	173·62	140·66	201·67	315·99
0·0	0·0	70·166	114·46	179·91	100·95	153·58	240·85	192·94	271·18	444·56
0·0	0·5	87·075	142·50	222·48	125·15	191·46	296·36	238·78	338·05	544·22
0·5	−0·5	63·577	105·67	164·99	90·961	142·15	221·69	172·11	250·54	408·75
0·5	0·0	86·751	141·94	221·09	124·72	191·16	292·09	237·55	340·27	529·66
0·5	0·5	107·87	176·15	271·25	155·16	237·90	352·93	295·50	425·53	630·02

TABLE 3

A comparison of the results: *U* ≡ *upper bound*; *L* ≡ *lower bound*; *E* ≡ *experimental results*; *E*! ≡ *experimental results from Leissa* [1], *who has taken data from* [11, 12, 15]

Case	α	β	Reference	λ_1	λ_2	λ_3
$\theta = 60^\circ$, $\mu = 1{\cdot}0$						
CCCC	0	0	Ours	46·166	81·602	105·52
			[9]	46·166	81·613	105·56
		U→	[14]	50·867	—	—
		L→	[14]	43·945	—	—
			[16]	46·384	—	—
			[23]	46·081	81·602	105·11
			[22]	46·150	81·602	105·90
			[20]	46·140	81·691	105·51
			[15]	46·170	—	—
			[27]	46·119	—	—
			[28]	46·154	—	—
			[26]	46·090	—	—
	0·0	0·2	Ours	50·689	89·574	115·81
			[27]	48·495	—	—
	0·0	0·4	Ours	55·053	97·225	125·66
			[27]	54·420	—	—
	0·0	0·02	Ours	46·627	82·416	106·57
			[28]	46·999	—	—
$\theta = 45^\circ$, $\mu = 1{\cdot}0$						
CCCC	0	0	Ours	66·330	106·72	156·34
			[9]	66·330	106·77	156·34
		U→	[14]	80·16	—	—
		L→	[14]	61·276	—	—
			[16]	65·59	—	—
			[23]	65·495	106·49	156·53
			[22]	66·383	106·59	162·35
			[20]	65·929	106·59	158·90
			[26]	65·652	106·50	148·34
			[28]	70·390	—	—
	0·0	0·02	Ours	66·992	107·79	157·89
			[28]	71·531	—	—
$\theta = 30^\circ$, $\mu = 1{\cdot}0$						
CCCC	0	0	Ours	127·06	184·58	282·95
			[9]	127·06	185·00	282·94
			[16]	121·29	—	—
			[23]	120·90	—	—
$\theta = 60^\circ$, $\mu = 1{\cdot}0$						
SSSS	0	0	Ours	25·314	52·660	72·714
			[9]	25·314	52·765	73·006
			[18]	23·7	—	—
			[19]	24·871	52·704	71·752
			[25]	25·069	52·901	72·344
$\theta = 45^\circ$, $\mu = 1{\cdot}0$						
SSSS	0	0	Ours	36·971	66·707	112·49
			[9]	36·970	67·023	113·26
			[18]	31·9	—	—
			[19]	34·840	66·323	100·47
			[24]	34·938	66·422	100·87
$\theta = 30^\circ$, $\mu = 1{\cdot}0$						
SSSS	0	0	Ours	73·135	111·38	209·84
			[9]	73·135	112·64	209·84

continued opposite

Table 3—*continued*

Case	α	β	Reference	λ_1	λ_2	λ_3
$\theta = 60^\circ,\ \mu = 1{\cdot}0$						
CFFF	0	0	Ours	3·9454	9·6209	26·011
			[9]	3·9454	9·6209	26·011
			[13]	3·961	10·190	—
			E!	3·82	9·23	24·51
			[24]	3·9478	9·4748	25·069
		E→	[13]	3·8491	9·2774	25·464
	$\nu = 0{\cdot}33$		Ours	3·9303	9·4645	25·637
			[29]	3·925	9·306	25·101
	0·0	−0·2	Ours	3·4571	8·4673	22·441
			[29]	3·623	8·464	22·539
	0·0	−0·4	Ours	3·0224	7·5312	18·709
			[29]	3·395	7·714	19·793
$\theta = 45^\circ,\ \mu = 1{\cdot}0$						
CFFF	0	0	Ours	4·6371	11·847	27·920
			[9]	4·6373	11·847	27·938
			[13]	4·824	13·750	—
			E!	4·26	11·07	26·52
			[24]	4·639	11·251	27·240
		E→	[13]	4·2439	11·054	26·549
			[19]	4·12	11·26	27·12
$\theta = 30^\circ,\ \mu = 1{\cdot}0$						
CFFF	0	0	Ours	6·1125	17·215	35·275
			[9]	6·1126	17·223	35·275
$\theta = 60^\circ,\ \mu = 1{\cdot}0$						
CSCS	0	0	Ours	37·193	64·389	93·577
			[9]	37·193	64·390	93·626
			[21]	37·475	65·120	94·146
$\theta = 45^\circ,\ \mu = 1{\cdot}0$						
CSCS	0	0	Ours	53·840	84·446	136·02
			[9]	53·840	85·087	136·01
			[21]	54·382	86·260	127·32
$\theta = 30^\circ,\ \mu = 1{\cdot}0$						
CSCS	0	0	Ours	104·53	146·39	248·26
			[9]	104·53	148·64	248·26
$\theta = 60^\circ,\ \mu = 2{\cdot}0$						
CCCC	0	0	Ours	128·90	159·73	215·29
			[9]	128·90	159·72	215·29
			[23]	128·74	159·41	213·38
			[22]	128·78	159·69	214·29
			[20]	128·90	159·93	214·64
			[28]	126·79	—	—
$\theta = 45^\circ,\ \mu = 2{\cdot}0$						
CCCC	0	0	Ours	190·00	223·98	294·67
			[9]	190·00	223·98	294·67
			[23]	189·18	222·07	279·78
			[22]	189·50	224·24	284·24
			[28]	190·64	—	—
$\theta = 30^\circ,\ \mu = 2{\cdot}0$						
CCCC	0	0	Ours	372·53	416·33	552·09
			[9]	372·52	416·35	552·09
			[23]	369·28	405·44	470·19

continued overleaf

TABLE 3—*continued*

Case	α	β	Reference	λ_1	λ_2	λ_3
$\theta = 60°,\ \mu = 2{\cdot}0$						
SSSS	0	0	Ours	64·070	96·553	153·76
			[9]	64·069	96·558	153·76
$\theta = 45°,\ \mu = 2{\cdot}0$						
SSSS	0	0	Ours	93·773	132·07	209·83
			[9]	93·772	132·09	209·83
$\theta = 30°,\ \mu = 2{\cdot}0$						
SSSS	0	0	Ours	182·45	239·90	394·64
			[9]	182·44	240·11	394·64
$\theta = 60°,\ \mu = 1{\cdot}0$						
CFFF	0	0	Ours	3·7332	16·150	25·812
			[9]	3·7331	16·154	25·813
$\theta = 45°,\ \mu = 2{\cdot}0$						
CFFF	0	0	Ours	4·1649	18·292	34·208
			[9]	4·1649	18·291	34·219
$\theta = 30°,\ \mu = 2{\cdot}0$						
CFFF	0	0	Ours	5·1278	22·289	55·873
			[9]	5·1282	22·308	55·989
$\theta = 60°,\ \mu = 2{\cdot}0$						
CSCS	0	0	Ours	70·166	114·46	179·91
			[9]	70·165	114·46	179·91
$\theta = 45°,\ \mu = 2{\cdot}0$						
CSCS	0	0	Ours	100·95	153·58	240·85
			[9]	100·95	153·64	240·85
$\theta = 30°,\ \mu = 2{\cdot}0$						
CSCS	0	0	Ours	192·94	271·18	444·56
			[9]	192·94	271·56	444·55

obtain an eigenvalue problem

$$\sum_{j=1}^{N} (a_{ij} - \lambda^2 b_{ij}) c_j = 0, \qquad i = 1, 2, \ldots, N, \tag{9}$$

where

$$a_{ij} = \frac{1}{\sin^4(\theta)} \iint_{R'} f^3 [\phi_i^{\xi\xi}\phi_j^{\xi\xi} - 2\mu\cos(\theta)(\phi_i^{\xi\eta}\phi_j^{\xi\xi} + \phi_i^{\xi\xi}\phi_j^{\xi\eta})$$
$$+ \mu^2(\nu\sin^2(\theta) + \cos^2(\theta))(\phi_i^{\eta\eta}\phi_j^{\xi\xi} + \phi_i^{\xi\xi}\phi_j^{\eta\eta}) + 2\mu^2(1 + \cos^2(\theta)$$
$$- \nu\sin^2(\theta))\phi_i^{\xi\eta}\phi_j^{\xi\eta} - 2\mu^3\cos(\theta)(\phi_i^{\eta\eta}\phi_j^{\xi\eta} + \phi_i^{\xi\eta}\phi_j^{\eta\eta}) + \mu^4\phi_i^{\eta\eta}\phi_j^{\eta\eta}]\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{10}$$

$$b_{ij} = \iint_{R'} f\phi_i\phi_j\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{11}$$

$$\lambda^2 = 12(1-\nu^2)\rho a^2\omega^2/Eh_0^2, \qquad \mu = a/b \qquad \text{and} \qquad f(\xi,\eta) = (1+\alpha\xi)(1+\beta\eta). \tag{12}$$

To satisfy the essential boundary conditions, the following basis functions $\phi_i(\xi, \eta)$ have been chosen:

$$\phi_i(\xi, \eta) = \xi^p \eta^q (1-\xi)^r (1-\eta)^s (1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \ldots), \tag{13}$$

where $p = 0$, 1 or 2 depending upon whether the side $\xi = 0$ is free (F), simply supported (S) or clamped (C). Similarly, q, r and s are controlling the boundary conditions at $\eta = 0$, $\xi = 1$ and $\eta = 1$, respectively. Thus the basis functions in equation (13) satisfy all of the essential boundary conditions. After substituting f and ϕ_i in equations (10) and (11), we obtain fairly complicated expressions for the integrands. Fortunately, all of the integrals contain the polynomials in ξ, η, $1-\xi$ and $1-\eta$. The following formula helps to evaluate them in closed form:

$$\iint_{R'} \xi^p \eta^q (1-\xi)^r (1-\eta)^s \, d\xi \, d\eta = \frac{p!q!r!s!}{(p+r+1)!(q+s+1)!}. \tag{14}$$

3. NUMERICAL WORK AND DISCUSSION

For all the computations that have been carried out on Tata-Elxsi at NCF, Roorkee, the following parameters and constants have been considered.

(1) The Poisson ratio v has been taken to be 0·3 for all computations.

(2) The parameters α and β that control the thickness variation have been given the values −0·5, 0 and 0·5. Other values of taper parameters are also considered for the sake of comparison.

(3) The aspect ratio μ is taken to be either 1 or 2.

(4) The order of approximation N is varied from 1 to 21, which accommodates polynomials up to the fifth degree in ξ and η. The first three frequencies and associated mode shapes are seen to converge to four significant digits for all values of the parameters.

(5) Each one of the parameters p, q, r and s is taken to be either 0, 1 or 2. The values $p = 0$, 1 or 2 correspond to side 1 ($\xi = 0$) being free, simply supported or clamped, respectively. The same interpretation can be given for the other three sides. When the plate is CSCS, this means that sides 1, 2, 3 and 4 (see Figure 1) are clamped, simply supported, clamped and simply supported, respectively.

TABLE 4

Convergence of the results ($\alpha = \beta = 0·5$, $\theta = 60°$)

	$\mu = 1·0$					$\mu = 2·0$				
	$N = 5$	$N = 10$	$N = 15$	$N = 20$	$N = 21$	$N = 5$	$N = 10$	$N = 15$	$N = 20$	$N = 21$
CCCC	71·764	71·019	70·564	70·538	70·538	197·93	196·29	195·38	195·25	195·25
	134·97	126·72	124·18	124·14	124·14	259·75	250·54	246·81	245·66	245·66
	180·08	165·69	163·52	161·86	161·85	408·98	350·36	338·10	328·47	328·47
SSSS	42·010	39·613	39·122	39·074	39·073	107·45	99·194	98·328	98·195	98·194
	107·87	85·884	80·770	80·702	80·686	179·68	157·42	150·38	149·23	149·23
	143·40	117·97	114·49	112·54	112·40	395·60	306·99	249·97	230·02	230·01
CFFF	5·4056	5·1786	5·1197	5·0961	5·0960	5·1932	4·8648	4·7651	4·7373	4·7373
	16·919	14·915	14·085	13·668	13·666	26·977	24·385	23·144	22·535	22·533
	46·718	41·550	39·856	38·894	38·768	52·444	44·843	41·154	39·615	39·614
CSCS	59·458	57·443	56·973	56·931	56·930	117·30	109·25	108·02	107·88	107·87
	125·89	103·73	99·220	98·849	98·805	199·10	182·97	177·43	176·15	176·15
	161·44	146·87	144·43	142·53	142·11	363·16	304·28	285·68	271·28	271·25

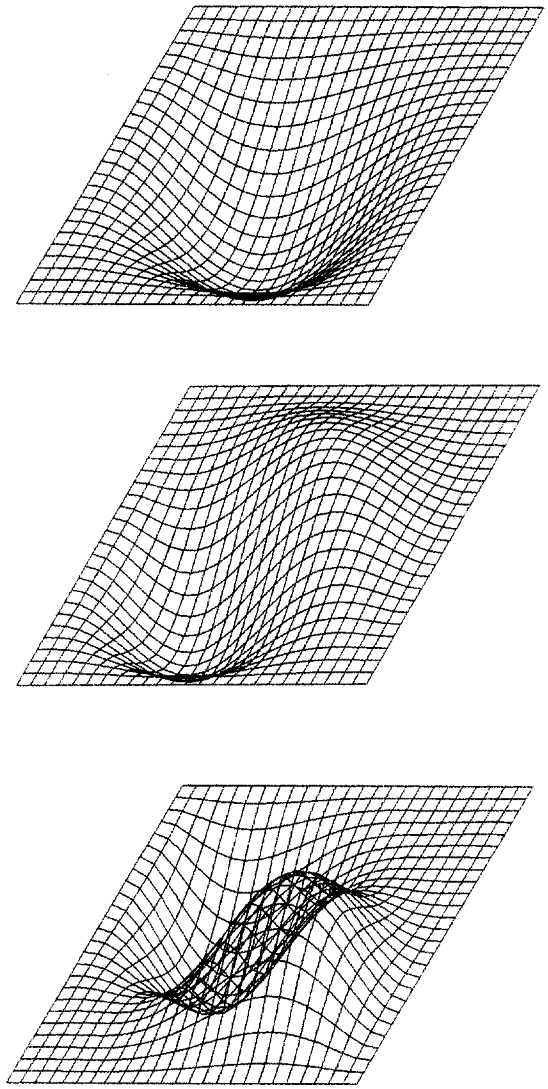

Figure 2. Mode shapes for the first three frequencies for fully clamped plate with $\alpha = \beta = 0{\cdot}5$, $\theta = 60^\circ$ and $\mu = 1{\cdot}0$.

The results for CCCC, SSSS, CFFF and CSCS plates with an aspect ratio of $\mu = 1{\cdot}0$ are given in Table 1. It is easily seen from the table that the frequencies increase with a decrease in the skew angle θ. It is also observed from the table that as α and/or β is increased, the frequencies increase. These are all expected, as the stiffness increases with an increasing α and/or β. Similar results are demonstrated in Table 2 for CCCC, SSSS, CFFF and CSCS plates with an aspect ratio of $\mu = 2{\cdot}0$. It is easily seen that the frequency increases with an increase in the aspect ratio in most of the cases. However, there are also some exceptional cases. For example, for the CFFF case frequencies do not always increase with an increasing aspect ratio. This has also been observed earlier by Leissa [1] and Gorman [8].

Table 3 has been prepared for comparison with the existing results in the literature. Comparison has been made with references [9, 11–29] for uniform and variable thickness.

It is clear from the table that the results agree well with known results in most of the cases. However, there are some discrepancies in some cases. This is because the methods employed in the cited references vary from crude estimates to the very accurate ones. Our results are based upon successive approximations which converge quite rapidly in certain cases but very slowly in some others. Therefore, although care has been taken to carry out sufficiently large number of approximations to make consecutive values agree to at least four significant figures, this does not mean that the last values will be correct to the same accuracy.

In Table 4 is given the convergence of the results for selected values of α and β ($\alpha = \beta = 0{\cdot}5$) and $\theta = 60^\circ$. It can be seen from the table that the frequencies converge almost to four significant digits in 21 approximations. The same behaviour is also seen in the other cases.

The first three mode shapes for a fully clamped plate with $\mu = 1{\cdot}0$ and skew angles of 60°, 45° and 30°, respectively, are depicted in Figures 2, 3 and 4.

Finally, we conclude that the Rayleigh–Ritz method is an efficient procedure to compute the first few frequencies and associated mode shapes of plates of various shapes under different combinations of boundary conditions. The mapping of the plate into a standard

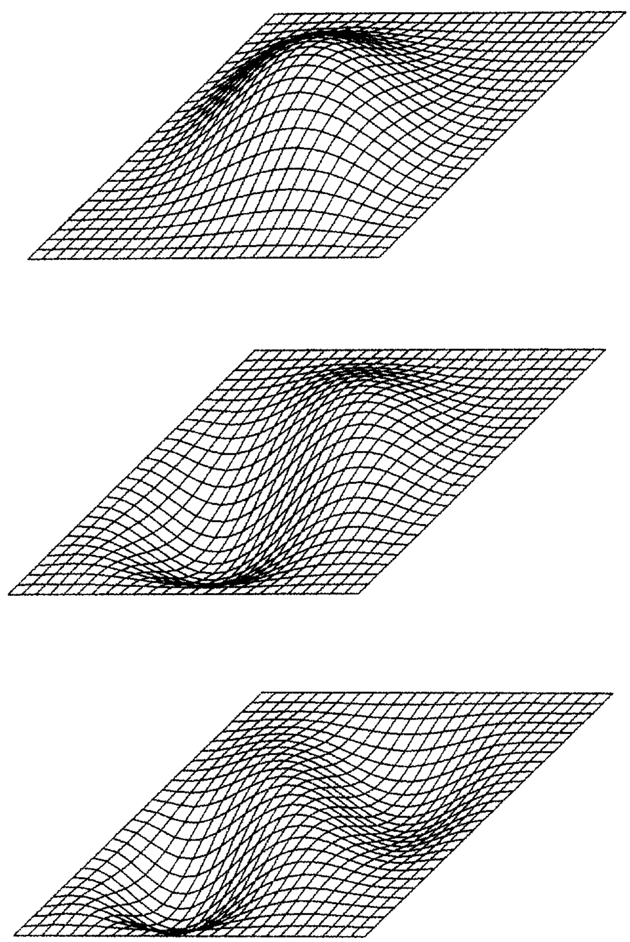

Figure 3. Mode shapes for the first three frequencies for fully clamped plate with $\alpha = \beta = 0{\cdot}5$, $\theta = 45^\circ$ and $\mu = 1{\cdot}0$.

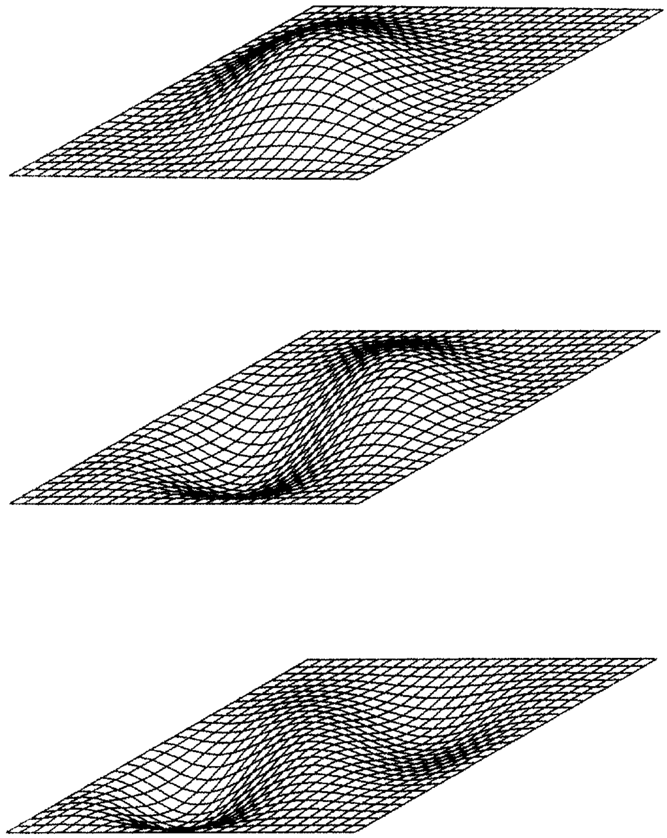

Figure 4. Mode shapes for the first three frequencies for fully clamped plate with $\alpha = \beta = 0{\cdot}5$, $\theta = 30^\circ$ and $\mu = 1{\cdot}0$.

square plate has saved a great deal of computation. A single program can handle a variety of boundary conditions and thickness variations. The procedure is truncated when the desired accuracy is reached. Double precision arithmetic is to be used to avoid numerical instability. Comparison with some known results in special cases confirms that the accuracy is comparable with the best results available.

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