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AXISYMMETRIC VIBRATION OF CIRCULAR AND ANNULAR PLATES WITH ARBITRARILY VARYING THICKNESS

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1. INTRODUCTION

The transverse vibration of circular and annular plates with variable thickness has been studied by several researchers. Prasad *et al*. [1], Luisoni *et al*. [2], and Grossi and Laura [3] have discussed the axisymmetric vibration of the circular plates with linearly varying thickness. Laura and Valerga de Greco [4], and Singh and Saxena [5] have studied circular plates with double linear variable thickness. Singh and Chakraverty [6], Barakat and Baumann [7], and Lenox and Conway [8] have investigated the case of circular plates having parabolic thickness variation. However, most investigations in the references were for some special case of thickness variations and special boundary conditions. Generally speaking, the finite element method has the advantage of extensive applicability. By using this method, any circular plate with variable thickness can be modelled by a series of elements with different thickness and the boundary conditions are easy to treat also. But calculations have proved that the general elements with uniform thickness or linear variable thickness are not very effective and not very accurate for analysing the transverse vibration of circular plates with variable thickness. In many cases a great number of elements are used and poor convergence is obtained.

In this paper annular elements with variable thickness are employed for the analysis of the axisymmetric vibration of circular and annular plates with arbitrarily varying thickness. Provided the function of the plate thickness is known, the thickness of the elements will vary according to the function, so the modelled structure is exactly the same as the actual one. The element matrices are calculated by combining the analytical and the numerical methods. Comparison with available exact results proved that the first five frequencies obtained by this method have more than four significant digits.

2. FINITE ELEMENT METHOD

The vibrating mechanical system being considered is shown in Figure 1. The geometric center of the plate is chosen as the origin of the polar co-ordinate, (r, θ, z), system. The plate has an outer radius a, inner radius b, and a variable thickness $h = h_0 f(r)$, where h_0 is the plate thickness of a certain reference point, $f(r)$ is an arbitrary function of the radial co-ordinate r. The outer edge is elastically restrained against rotation and translation. Φ is the flexibility of the rotational boundary spring and K is the translational spring constant. The whole plate is descretized into N elements. Each element has three nodes (the two ends and the middle point) and each nodal point has two degrees of freedom (W, Ψ) representing the transverse displacement and the slope respectively, shown in Figure 2. For the sake of convenience, the following transformations are introduced to convert the variables into non-dimensional forms

$$X = W/a, \qquad \zeta = z/a, \qquad \xi = r/a$$

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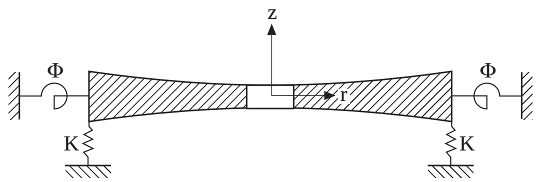

Figure 1. The vibrating mechanical system under study

To calculate the element stiffness matrices and the element mass matrices conveniently by using Gauss quadrature, the domain of the whole element $[\xi_{i-1}, \xi_{i+1}]$ is linearly transformed to the domain $[\eta = -1, \eta = 1]$ by introducing the parameter $\eta = (\xi - \xi_i)/s$. Where the non-dimensional width $s = (a - b)/2Na$. Assume the element displacement is of the following form

$$X = \alpha_1 + \alpha_2\eta + \alpha_3\eta^2 + \alpha_4\eta^3 + \alpha_5\eta^4 + \alpha_6\eta^5 \tag{1}$$

The generalized co-ordinates $\alpha_i (i = 1, 6)$ can be determined by the nodal displacements X_i and the nodal slopes $\Psi_i = X_i'$, where the prime of X denotes the partial derivative of X with respect to ξ. So the interpolation polynomials are obtained and one has

$$X(\eta, \tau) = [\mathbf{N}]^{\mathrm{T}}\{\mathbf{X}^{(e)}\} \tag{2}$$

$\{\mathbf{X}^{(e)}\}$ is the nodal displacement vector defined as

$$\{\mathbf{X}^{(e)}\} = [X_{i-1}, X_{i-1}', X_i, X_i', X_{i+1}, X_{i+1}']^{\mathrm{T}} \tag{3}$$

The strain–displacement relation for a thin plate is

$$\{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{Bmatrix} = -z\begin{Bmatrix} W'' \\ (1/r)W' \end{Bmatrix} \tag{4}$$

where the primes of W denote the partial derivatives of W with respect to r. The linear relation between the strain and stress is assumed to be

$$\{\boldsymbol{\sigma}\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} = \frac{E}{1 - v^2}[\mathbf{C}]\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{Bmatrix} \tag{5}$$

where

$$[\mathbf{C}] = \begin{bmatrix} 1 & v \\ v & 1 \end{bmatrix}$$

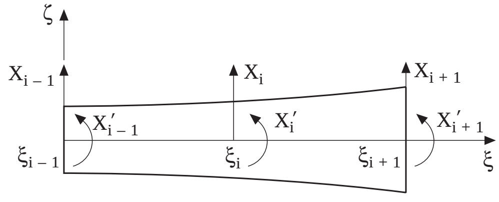

Figure 2. A finite element with variable thickness

The potential energy of the whole system equals

$$U = U_P + U_\Phi + U_K = \frac{1}{2}\int \{\boldsymbol{\sigma}\}^T\{\boldsymbol{\varepsilon}\}\,\mathrm{d}V + \frac{\pi a}{\Phi}(X')^2_{\xi=1} + \pi a^3 K(X)^2_{\xi=1} \tag{6}$$

where U_P, U_Φ, U_K denote the potential energies of the plate, the spring against rotation and the spring against translation, respectively. Similarly, the kinetic energy of the system is

$$T = T_P = \frac{1}{2}\int \rho \dot{W}^2\,\mathrm{d}V \tag{7}$$

where T_P is the kinetic energy of the plate, $\dot{W}$ is the partial derivative of W with respect to time t. Substituting U and T into Lagrange's equations and using equations (2), (4) and (5), one has the equation of motion of the system

$$[\mathbf{M}]\{\ddot{\mathbf{X}}\} + [\mathbf{K}]\{\mathbf{X}\} = 0 \tag{8}$$

The mass matrix $[\mathbf{M}]$ and the stiffness matrix $[\mathbf{K}]$ are obtained by the usual assembly procedure and the element matrices are

$$[\mathbf{M}^{(e)}] = \frac{D_0}{a^2 h_0}\int [\mathbf{N}][\mathbf{N}]^{\mathrm{T}}\,\mathrm{d}V = 2\pi D_0 S\int_{-1}^{1}[\mathbf{N}][\mathbf{N}]^{\mathrm{T}} f(\xi)\xi\,\mathrm{d}\eta \tag{9}$$

$$[\mathbf{K}^{(e)}] = \frac{12 D_0}{s^2 h_0^3}\int \left[\frac{1}{s}N'', \frac{1}{\xi}N'\right][\mathbf{C}]\left[\frac{1}{s}N'', \frac{1}{\xi}N'\right]^{\mathrm{T}}\xi^2\,\mathrm{d}V$$

$$= \frac{2\pi D_0}{s}\int_{-1}^{1}\left[\frac{1}{s}N'', \frac{1}{\xi}N'\right][\mathbf{C}]\left[\frac{1}{s}N'', \frac{1}{\xi}N'\right]^{\mathrm{T}} f^3(\xi)\xi\,\mathrm{d}\eta \tag{10}$$

where the primes denote the derivatives with respect to η. The calculation of the element mass matrices and the element stiffness matrices are performed analytically along the circumference and thickness directions, but numerically along the radial direction. Six point Gauss quadrature is used for the integration. An iteration method is chosen for solving the eigenvalue problem obtained from equation (8).

TABLE 1

Convergence of frequencies of circular plate with simply supported boundary and uniform thickness; $\nu = 0{\cdot}3$

Element number (N)	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
1	4·93515	29·80606	75·90850	–	–	–
2	4·93515	29·72054	74·24724	139·37271	229·01475	363·35686
4	4·93515	29·72001	74·15634	138·32744	222·31920	326·43071
6	4·93515	29·72000	74·15607	138·31844	222·21878	325·88265
8	4·93515	29·72000	74·15606	138·31815	222·21544	325·85201
10	4·93515	29·72000	74·15606	138·31813	222·21511	325·84970
[9]	4·93515	29·72000	74·15605	138·31812	222·21491	324·94466

TABLE 2

Circular plate with simply supported boundary and linear thickness variation: $h = h_0(1 + \alpha\xi)$; $v = 0.3$. *Lower parenthesized values are from reference* [5]

α	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
−0·8	2·57637	15·40097	38·62459	72·16631	116·04000
	(2·5764)	(15·401)	(38·626)	–	–
−0·6	3·25156	19·40902	48·84579	91·47257	147·27308
	(3·2516)	(19·409)	(48·847)	–	–
−0·4	3·83633	23·00497	57·83166	108·24111	174·21376
	(3·8363)	(23·005)	(57·832)	–	–
−0·2	4·39096	26·41477	66·18976	123·69606	198·92450
	(4·3910)	(26·415)	(66·190)	–	–
0·0	4·93515	29·72000	74·15606	138·31813	222·21511
	(4·9351)	(29·720)	(74·156)	–	–
0·2	5·47701	32·95984	81·85122	152·35505	244·50314
	(5·4770)	(32·960)	(81·852)	–	–
0·4	6·02014	36·15611	89·34650	165·95415	266·03714
	(6·0202)	(36·157)	(89·351)	–	–
0·6	6·56616	39·32213	96·68795	179·21129	286·97907
	(6·5662)	(39·324)	(96·701)	–	–
0·8	7·11572	42·46658	103·90726	192·19289	307·44133
	(7·1159)	(42·470)	(103·94)	–	–
1·0	7·66899	45·59535	111·02728	204·94713	327·50562
	(7·6693)	(45·602)	(111·08)	–	–

3. NUMERICAL WORK AND DISCUSSION

To ascertain the characteristics of convergence of this method, a uniform circular plate with a simply supported boundary is chosen for calculation and compared with existing references. The frequency parameters $\Omega_i = \omega_i(\rho h_0 a^4/D_0)^{1/2}$ of the circular plate with uniform

TABLE 3

Annular plate with linear thickness variation: $h = h_0(1 + \alpha\xi)$; $v = 0{\cdot}3$. *Lower parenthesized values are from reference* [10]

α	Boundary condition	Hole size b/a	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
−0·3	C–S	0·3	24·85620	81·12467	169·81045	290·83299	444·16888
			(24·857)	–	–	–	–
−0·3		0·5	47·44856	154·14603	321·97259	550·87172	840·83074
			(47·467)	–	–	–	–
−0·3	C–C	0·3	36·11578	100·13684	196·91384	326·00444	487·41047
			(36·118)	–	–	–	–
−0·3		0·5	68·79539	190·16896	373·28806	617·44029	922·65718
			(68·824)	–	–	–	–
0·3	C–S	0·3	34·92943	119·22132	251·33345	431·49716	659·70949
			(34·908)	–	–	–	–
0·3		0·5	72·01501	241·44484	507·20106	869·58205	1328·58935
			(72·021)	–	–	–	–
0·3	C–C	0·3	54·34590	149·90375	294·02510	486·17811	726·39017
			(54·319)	–	–	–	–
0·3		0·5	109·46483	301·81564	591·75764	978·27030	1461·42278
			(109·477)	–	–	–	–

thickness are shown in Table 1. There ω_i are the angular frequencies, $D_0 = Eh_0^3/12(1-\nu^2)$, is the flexural rigidity of the plate at the reference point, ρ is the density of the plate material, E and ν denote Young's modulus and the Poisson ratio, respectively. From Table 1 it is obvious that the method has very good convergence characteristics. Only one element is needed to obtain six significant digits of the fundamental frequency. By using eight elements, the first five frequencies obtained more than six digits converge and are in good agreement with the values of reference [9].

The convergence of the method for other cases has been tested also. Generally speaking, the lower frequencies converge faster than the higher ones. By using ten elements, except in some extreme cases, the first five frequencies converge to about five significant digits. So the number of elements is fixed as $N = 10$ for all following computations.

Table 2 shows the case of circular plate with linear variable thickness and simply supported boundary. Comparison with reference [5] has been made. For the first three comparable frequencies, there are at least four digits consistent with each other.

Table 4

Circular plate with clamped boundary and double linear thickness variation†. Lower parenthesized values are from reference [5]

α	β_1	β_2	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0·25	−0·5	−0·5	6·15036	27·30022	63·06113	113·26669	177·88499
			(6·1504)	(27·300)	(63·062)	–	–
		0·0	8·97072	35·14633	78·82361	140·00509	218·87699
			(8·9707)	(35·146)	(78·829)	–	–
		0·5	11·82429	42·10190	92·17991	162·39140	253·19873
			(11·824)	(42·104)	(92·213)	–	–
	0·0	−0·5	7·40320	32·06089	73·74515	132·28018	207·47496
			(7·4033)	(32·061)	(73·746)	–	–
		0·0	10·21583	39·77115	89·10414	158·18423	247·00655
			(10·216)	(39·771)	(89·104)	–	–
		0·5	13·06585	46·72655	102·38859	180·33170	280·82948
			(13·066)	(46·727)	(102·39)	–	–
	0·5	−0·5	8·64704	36·76911	84·33698	151·09455	236·59153
			(8·6472)	(36·770)	(84·345)	–	–
		0·0	11·45481	44·38622	99·40194	176·34516	274·94339
			(11·455)	(44·387)	(99·413)	–	–
		0·5	14·30214	51·34801	112·63595	198·27915	308·31039
			(14·302)	(51·349)	(112·64)	–	–
0·5	−0·5	0·0	7·84489	31·47870	71·12663	126·76694	198·27951
			(7·8449)	(31·480)	(71·131)	–	–
		0·5	9·51748	35·02541	78·08897	137·90047	215·31258
			(9·5176)	(35·031)	(78·099)	–	–
	0·0	−0·5	8·51335	35·91994	81·89586	146·23172	229·08052
			(8·5134)	(35·920)	(81·897)	–	–
		0·5	11·90788	43·15254	95·57903	168·49948	262·67303
			(11·908)	(43·153)	(95·580)	–	–
	0·5	−0·5	10·86518	44·45527	99·82141	177·60437	277·22232
			(10·865)	(44·458)	(99·829)	–	–
		0·0	12·58700	48·08992	106·49784	188·56112	293·60242
			(12·587)	(48·092)	(106·50)	–	–

† $h = \begin{cases} h_0(1+\beta_1\xi) & 0 \leqslant \xi \leqslant \alpha \\ h_0[1+\beta_1\alpha+\beta_2(\xi-\alpha)] & \alpha \leqslant \xi \leqslant 1, \end{cases}$ $\quad \nu = 0{\cdot}3$.

Table 3 shows the case of an annular plate with linear variable thickness and clamped inner edge. The outer edge of the plate is simply supported (C–S) or clamped (C–C). Comparison has been made with reference [10]. For all the calculated cases, at least three digits of the fundamental frequencies are the same.

The first five frequency parameters of the circular plate with double linear variable thickness and clamped boundary are listed in Table 4. By comparison with reference [5], that for any combinations of the thickness parameters α, β_1 and β_2, the results obtained in this investigation agree well with those by Singh and Saxena. Three or four significant digits can be determined for the first three frequencies. The case of linear variable thickness occurs when $\beta_1 = \beta_2 = \beta$ and no matter what values α takes, the same results are obtained. So the three cases: (i) $\beta_1 = \beta_2 = -0{\cdot}5$; (ii) $\beta_1 = \beta_2 = 0$; (iii) $\beta_1 = \beta_2 = 0{\cdot}5$, are listed only once under $\alpha = 0{\cdot}25$.

Table 5 is for the case of an annular plate with parabolic variable thickness. It may be noted that the values obtained for most boundary conditions and hole sizes agree well with reference [8]. But for the C–C boundary condition and $b/a = 0{\cdot}1$, the inner edge of the

TABLE 5

Annular plate with parabolic thickness variation: $h = h_0\xi^2$; $\nu = 1/3$. *Lower parenthesized values are from reference* [8]

Boundary condition	Hole size (b/a)	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
F–F	0·1	0·00000	4·20486	9·08725	16·75817	28·10988
		(0·00)	(4·20)	(9·09)	–	–
	0·3	0·00000	4·59348	19·86527	47·54915	88·66421
		(0·00)	(4·59)	(19·87)	–	–
	0·5	0·00000	5·89940	50·76553	133·32232	256·88484
		(0·00)	(5·90)	(50·77)	–	–
	0·7	0·00000	9·79196	179·95138	489·72451	955·60526
		(0·00)	(9·79)	(179·95)	–	–
	0·9	0·00000	31·18839	2019·50110	–	–
		(0·00)	(31·19)	(2019·50)	–	–
F–S	0·1	2·85096	8·55271	15·62301	26·27190	40·52744
		(2·85)	(8·55)	(15·62)	–	–
	0·3	2·99851	17·29690	42·24280	80·27911	131·69072
		(3·00)	(17·30)	(42·24)	–	–
	0·5	3·61347	41·24071	114·52250	228·17319	382·60299
		(3·61)	(41·24)	(114·52)	–	–
	0·7	5·54860	136·73412	409·57751	836·99167	1419·21664
		(5·55)	(136·73)	(409·58)	–	–
	0·9	16·26057	1437·25790	4548·94386	9439·72791	–
		(16·26)	(1437·26)	(4548·95)	–	–
C–C	0·1	8·79749	16·56247	28·14739	43·57505	63·03111
		(8·77)	(16·44)	(27·84)	–	–
	0·3	19·26351	47·13877	88·35806	143·04749	211·28147
		(19·26)	(47·14)	(88·36)	–	–
	0·5	50·06709	132·88355	256·56713	421·13868	626·71458
		(50·97)	(132·88)	(256·57)	–	–
	0·7	179·22217	489·27490	955·28270	1576·16420	2352·12946
		(179·22)	(489·28)	(955·28)	–	–
	0·9	2018·66049	5559·85842	–	–	–
		(2018·76)	(5560·17)	–	–	–

TABLE 6

Circular plate with simply supported boundary and parabolic thickness variation: $h = h_0(1 + \alpha\xi^2)$; $\nu = 0{\cdot}25$. *Lower parenthesized values are from reference* [7]

α	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
−0·8	3·32406	19·14402	48·59039	90·79214	145·78036
	(3·149)	(18·64)	–	–	–
−0·6	3·77091	22·41000	56·48379	105·60273	169·77699
	(3·728)	(22·22)	–	–	–
−0·4	4·14551	25·07938	62·99447	117·74552	189·33203
	(4·135)	(25·03)	–	–	–
−0·2	4·50253	27·45898	68·78141	128·46757	206·51809
	(4·501)	(27·45)	–	–	–
0·0	4·86013	29·66215	74·10139	138·26481	222·16253
	(4·860)	(29·66)	–	–	–
0·2	5·22582	31·74490	79·08692	147·39605	236·69739
	(5·225)	(31·75)	–	–	–
0·4	5·60279	33·73945	83·81720	156·01708	250·38268
	(5·602)	(33·73)	–	–	–
0·6	5·99219	35·66622	88·34412	164·23049	263·38959
	(5·990)	(35·65)	–	–	–
0·8	6·39412	37·53903	92·70388	172·10816	275·83799
	(6·391)	(37·51)	–	–	–

plate is too thin (one percent of the thickness of the outer edge) and the mode shapes change sharply within a small region around the inner edge. In such an extreme case the convergence rate becomes lower and more elements are needed to provide the needed accuracy.

Table 6 shows the case of a circular plate with parabolic variable thickness. Comparison with reference [7] has been made. When the thickness parameter $\alpha > -0{\cdot}4$, the results agree with each other better than when $\alpha < -0{\cdot}4$.

From the above results it is obvious that the finite element method presented here is a very effective and convenient method of calculating the frequencies and mode shapes (not plotted here) of circular plates and annular plates with various boundary conditions and various thickness variations. Assigning values other than 0 and ∞ to Φ and K, one can easily compute the cases of plates with elastically restrained boundaries. As another example of applications, circular plate with simply supported boundary and cubic thickness variation has been calculated and the first five frequencies are listed in Table 7.

TABLE 7

Circular plate with simply supported boundary and cubic thickness variation: $h = h_0(1 + \alpha\xi^3)$; $\nu = 0{\cdot}3$

α	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
−0·8	3·85702	21·63200	54·46845	101·60838	162·97078
−0·6	4·16938	24·35194	60·86754	113·48997	182·20038
−0·4	4·42986	26·41676	65·91026	122·91292	197·42424
−0·2	4·68031	28·16485	70·25556	131·03640	210·51304
0·0	4·93515	29·72000	74·15606	138·31813	222·21511
0·2	5·20079	31·14317	77·74051	144·99563	232·92072
0·4	5·48021	32·46934	81·08517	151·21135	242·86456
0·6	5·77465	33·72067	84·23963	157·05888	252·20104
0·8	6·08439	34·91214	87·23832	162·60363	261·03822

CONCLUSIONS

One concludes from the results that the finite element method described here is an efficient and convenient tool for computing the frequencies and mode shapes for axisymmetric vibration of circular and annular plates with various boundary conditions and various thickness variations. The first five frequencies for some special cases have been listed and compared with the results available in the literature. Comparison has proved that the results obtained have very high accuracy. Part of the results presented here are new and are not available elsewhere.

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