SYMBOLIC COMPUTATION OF FUNDAMENTAL SOLUTION MATRICES FOR LINEAR TIME-PERIODIC DYNAMICAL SYSTEMS

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A new technique which employs both *Picard iteration* and expansion in *shifted Chebyshev polynomials* is used to symbolically approximate the fundamental solution matrix for linear time-periodic dynamical systems of arbitrary dimension explicitly as a function of the system parameters and time. As in previous studies, the *integration* and *product operational matrices* associated with the Chebyshev polynomials are employed. However, the need to algebraically solve for the Chebyshev coefficients of the fundamental solution matrix is completely avoided as only matrix multiplications and additions are utilized. Since these coefficients are expressed as homogeneous polynomials of the system parameters, closed form approximations to the true solutions may be obtained. Also, because this method is *not* based on expansion in terms of a small parameter, it can successfully be applied to periodic systems whose internal excitation is strong. Two formulations are proposed. The first is applicable to general time periodic systems while the second approach is useful when the system equations contain a constant matrix. Three different example problems, including a double inverted pendulum subjected to a periodic follower force, are included and CPU times and convergence results are discussed.

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1. INTRODUCTION

The study of dynamical systems governed by a set of ordinary differential equations with periodic coefficients is of great theoretical and practical importance in various fields of science and engineering. These equations generally represent the perturbed dynamics about a steady state periodic motion of the system. In many situations, the linearized perturbed equations may be sufficient for the prediction of stability and therefore the problem reduces to a set of linear differential equations with periodic coefficients. The stability conditions are determined by the requirement that the eigenvalues of the fundamental solution matrix evaluated at the end of the principal period (called the Floquet multipliers) must lie within the unit circle in the complex plane. For various bifurcation studies, it is necessary to generate the non-linear equations of the perturbed dynamics. However, the local bifurcation conditions are once again determined by the nature of the Floquet multipliers of the linearized system. Further, fundamental solution matrices also play an important role in designing feedback controllers for all dynamic systems. If the fundamental solution matrix could be computed as a function of the system parameters, then it would be possible to determine the stability (bifurcation) conditions and the controller gains in a closed form. In this study an attempt has been made toward achieving this goal.

It is well known that exact solutions of periodic systems are possible only in a very limited number of cases (see section 5.1), and, in general such solutions do not exist. Two

common asymptotic methods that have been used in the past to yield an approximate fundamental solution matrix with parametric dependence in closed form include the perturbation method [1] and the averaging technique [2]. These methods are limited in application to systems with weak internal excitation since they are based on expanding the solution in terms of a small parameter that multiplies the time-periodic terms. When the excitation becomes strong, the parameter is not small and the accuracy of the solution is very poor. Moreover, increasing the order of the approximate solution is usually very difficult and does not guarantee uniform convergence to the true solution. A recent approximation technique of Guttalu and Flashner [3, 4] for computing the fundamental solution matrix evaluated at the end of the principal period (called the Floquet transition matrix) by truncated point mappings avoids the restriction of a small periodic parameter but is computationally expensive in terms of CPU time. In addition, the fundamental solution matrix is not obtained in closed form as an explicit function of time.

Another approximation technique which has developed recently involves expanding periodic coefficients in terms of Chebyshev polynomials and was first introduced by Sinha and Chou [5] and Sinha *et al*. [6]. Although these applications were limited to second order scalar equations only, a later study by Sinha and Wu [7] outlined a scheme to obtain an approximate fundamental solution matrix in closed form for a system of second order equations. The approach was based on the idea that the state vector and the periodic system matrix can be expanded in terms of Chebyshev polynomials over the principal period. By employing the *integration* and *product operational matrices* associated with the polynomials, this expansion reduces the original problem to a set of linear algebraic equations for the Chebyshev coefficients of the state vector from which the solution in the interval of one period is obtained. In later studies [8, 9], this method was applied to single-degree-of-freedom problems to symbolically approximate the solution in terms of the system parameters. It was shown that a ten- or twelve-term Chebyshev expansion provides much better accuracy compared to a sixteenth-order perturbation series when the coefficients of the periodic terms are large. However, the Chebyshev coefficients of the solution must be obtained using Cramer's rule, matrix inversion, etc., and therefore this approach is impractical for systems of higher dimension.

In this paper, a new technique which employs both Picard iteration and expansion in shifted Chebyshev polynomials is used to symbolically approximate the fundamental solution matrix for time-periodic dynamical systems of arbitrary dimension explicitly as a function of the system parameters and time. Since this method is *not* based on expansion in terms of a small parameter, it can successfully be applied to periodic systems with strong internal excitations. As in the previously mentioned studies, the integration and product operational matrices associated with the shifted Chebyshev polynomials are employed. However, the need to algebraically solve for the Chebyshev coefficients is completely avoided by employing Picard iterations (sequence of approximations). The Chebyshev coefficients of the resulting fundamental solution matrix are expressed as homogeneous polynomials of the system parameters, thus enabling a closed form approximation to the true solution to be obtained. Two formulations, one applicable to general periodic systems and one for equations which contain a constant system matrix, are outlined and applied to three different example problems including a double inverted pendulum subjected to a periodic follower force. CPU times are included and convergence results are discussed in detail.

2. PICARD ITERATION

The first non-homogeneous differential equation

$$\mathrm{d}x(t)/\mathrm{d}t = f(x(t), t), \qquad x(t_0) = x^0, \tag{1}$$

may be expressed in an equivalent integral form such as

$$x(t) = x^0 + \int_{t_0}^{t} f(x(\tau), \tau) \, \mathrm{d}\tau, \tag{2}$$

where τ is a dummy variable. Assuming an initial approximation $x^{(0)}(t) = x^0$ and inserting it on the right side of equation (2), an approximation $x^{(1)}(t)$ is generated. This is inserted back into the integral to generate the second approximation $x^{(2)}(t)$, etc. This process, called Picard iteration, satisfies the recurrence equation

$$x^{(k+1)}(t) = x^0 + \int_{t_0}^{t} f(x^{(k)}(\tau), \tau) \, \mathrm{d}\tau. \tag{3}$$

and results in a sequence of approximations to the true solution $x(t)$ [10]. If $f(x, t)$ is defined, continuous, and satisfies a Lipschitz condition with Lipschitz constant L in an interval about t_0, then equation (3) may be used to obtain the relationship

$$|x^{(k)}(t) - x^{(k-1)}(t)| \leqslant ML^{k-1}(t - t_0)^k/k!, \tag{4}$$

from which the neighborhood of convergence $h = t - t_0$ is obtained as

$$h \leqslant (1/L) \ln (1 + LB/M), \tag{5}$$

where $|x(t) - x^0| < B$ and $|f(x^0, t)| < M$ [11]. In contrast to asymptotic techniques, all of the system parameters are treated equally in Picard iteration so that the convergence varies radially in the parameter space. This technique is of great theoretical importance in the theory of the existence of solutions of equation (1), but the difficulty of evaluating the integral in equation (2) has made it impractical for general numerical computations even for the special case of linear time-varying equations. It will be seen that this obstacle is circumvented here by expanding the periodic system matrix in Chebyshev polynomials. By employing the associated operational matrices defined in section 3, the successive integrations associated with the Picard iterations are replaced by simple matrix multiplications.

3. SHIFTED CHEBYSHEV POLYNOMIALS AND THE OPERATIONAL MATRICES

The shifted Chebyshev polynomials of the first kind are defined in terms of the standard Chebyshev polynomials of the first kind $T_r(t)$ valid over the interval $[-1, 1]$ by using the change of variable

$$t \rightarrow (t + 1)/2. \tag{6}$$

Thus, the shifted Chebyshev polynomials of the first kind are given by

$$T_r^*(t) = T_r(2t - 1), \qquad 0 \leqslant t \leqslant 1, \tag{7}$$

and are valid over the interval $[0, 1]$. All properties of $T_r^*(t)$ can be deduced from those of $T_r(2t - 1)$. The orthogonality and recurrence relations for these polynomials can be

found in references [12, 13]. If $\mathbf{T}^*(t) = (T_0^*(t)\, T_1^*(t) \cdots T_{m-1}^*(t))^{\mathrm{T}}$ is an $m \times 1$ column vector of the polynomials, then the outer product of two of these vectors is

$\mathbf{T}^*(t)\mathbf{T}^{*\mathrm{T}}(t) =$

$$
\begin{bmatrix}
T_0^*(t) & T_1^*(t) & T_2^*(t) & \cdots & T_{m-1}^*(t) \\
T_1^*(t) & (T_0^*(t) + T_2^*(t))/2 & (T_1^*(t) + T_3^*(t))/2 & \cdots & T_m^*(t) + T_{m-2}^*(t)/2 \\
T_2^*(t) & T_1^*(t) + T_3^*(t)/2 & T_0^*(t) + T_4^*(t)/2 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
T_{m-1}^*(t) & T_m^*(t) + T_{m-2}^*(t)/2 & \cdots & \cdots & T_0^*(t) + T_{2(m-1)}^*(t)/2
\end{bmatrix}.
\tag{8}
$$

Generally, an arbitrary continuous time function $f(t)$ can be approximated by a finite shifted Chebyshev series over the interval $[0, 1]$ where the coefficients of the polynomials can be obtained as shown in [14, 15]. If two such functions are expanded as

$$
f(t) = \sum_{r=0}^{m-1} a_r T_r^*(t), \qquad g(t) = \sum_{r=0}^{m-1} b_r T_r^*(t),
\tag{9}
$$

then

$$
f(t)g(t) = (a_0 a_1 a_2 \cdots a_{m-1})\mathbf{T}^*(t)\mathbf{T}^{*\mathrm{T}}(t)(b_0 b_1 b_2 \cdots b_{m-1})^{\mathrm{T}},
\tag{10}
$$

where a_r and b_r are Chebyshev coefficients of the functions $f(t)$ and $g(t)$, respectively. Using equation (8), equation (10) can be rewritten as

$$
f(t)g(t) = \mathbf{T}^{*\mathrm{T}}(t)\mathbf{Q}_a\mathbf{b},
\tag{11}
$$

where $\mathbf{Q}_a$ is the *product operational matrix* corresponding to $f(t)$ given by

$$
\mathbf{Q}_a =
\begin{bmatrix}
a_0 & a_1/2 & a_2/2 & \cdots & a_{m-1}/2 \\
a_1 & a_0 + a_2/2 & \frac{1}{2}(a_1 + a_3) & \cdots & \frac{1}{2}(a_{m-2} + a_m) \\
a_2 & \frac{1}{2}(a_1 + a_3) & a_0 + a_4/2 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m-1} & \frac{1}{2}(a_{m-2} + a_m) & \cdots & \cdots & a_0 + a_{2(m-1)}/2
\end{bmatrix}
\tag{12}
$$

and $\mathbf{b} = (b_0 b_1 b_2 \ldots b_{m-1})^{\mathrm{T}}$ [7]. The a_r coefficients with $r > m-1$ in equation (12) may be set to zero. The general recursive formula for integration of an $m \times 1$ vector of shifted Chebyshev polynomials of the first kind may be written in vector form as

$$
\int_0^t \int_0^{\tau_0} \underset{(k\ \text{times})}{\cdots} \int_0^{\tau_{k-2}} \mathbf{T}^*(\tau_{k-1})\, \mathrm{d}\tau_{k-1} \cdots \mathrm{d}\tau_1\, \mathrm{d}\tau_0 = \mathbf{G}^k\mathbf{T}^*(t),
\tag{13}
$$

where $\mathbf{G}$ is the $m \times m$ *integration operational matrix* given by

$$\mathbf{G} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots & 0 \\ -1/8 & 0 & 1/8 & 0 & 0 & \cdots & 0 \\ -1/6 & -1/4 & 0 & 1/12 & 0 & \cdots & 0 \\ 1/16 & 0 & -1/8 & 0 & 1/16 & \cdots & \vdots \\ -1/30 & 0 & 0 & -1/12 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 1/4(m-1) \\ (-1)^m/2m(m-2) & 0 & 0 & \cdots & 0 & -1/4(m-2) & 0 \end{bmatrix};$$

(14)

$(\ )^{\mathrm{T}}$ denotes the transpose of the quantity $(\ )$, and all τ_j's are dummy variables [7]. Use of the $\mathbf{G}$ matrix results in a forward difference recurrence procedure in which the $(m + 1)$th term is truncated in order to keep the polynomial vector the same length.

Let the $nm \times n$ *Chebyshev polynomial matrix* be defined as

$$\hat{\mathbf{T}}(t) = \mathbf{I_n} \otimes \mathbf{T}^*(t), \tag{15}$$

where $\otimes$ signifies the Kronecker product defined in Appendix A. Then the recursive formula for integration of this matrix can be written as

$$\int_0^t \int_0^{\tau_0} \underset{k \text{ times)}}{\cdots} \int_0^{\tau_{k-2}} \hat{\mathbf{T}}(\tau_{k-1}) \, \mathrm{d}\tau_{k-1} \cdots \mathrm{d}\tau_1 \, \mathrm{d}\tau_0 = \hat{\mathbf{G}}^k \hat{\mathbf{T}}(t), \tag{16}$$

where $\hat{\mathbf{G}} = \mathbf{I}_n \otimes \mathbf{G}$ is of dimension $nm \times nm$. Also, the product of two periodic matrix functions which have been expanded in Chebyshev polynomials can be expressed as

$$\mathbf{\Theta}(t)\mathbf{\Psi}(t) = \mathbf{A}'\hat{\mathbf{T}}(t)\hat{\mathbf{T}}^{\mathrm{T}}(t)\mathbf{B} = \hat{\mathbf{T}}^{\mathrm{T}}(t)\hat{\mathbf{Q}}_A\mathbf{B}, \tag{17}$$

where

$$\mathbf{A}' = \begin{bmatrix} \mathbf{a}_{11}^{\mathrm{T}} & \cdots & \mathbf{a}_{1n}^{\mathrm{T}} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n1}^{\mathrm{T}} & \cdots & \mathbf{a}_{nn}^{\mathrm{T}} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{b}_{11} & \cdots & \mathbf{b}_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{b}_{n1} & \cdots & \mathbf{b}_{nn} \end{bmatrix} \tag{18}$$

are $n \times nm$ and $nm \times n$, respectively, and $\mathbf{a}_{ij}$ and $\mathbf{b}_{ij}$ are the Chebyshev coefficient vectors in the expansion of $\mathbf{\Theta}_{ij}(t)$ and $\mathbf{\Psi}_{ij}(t)$, respectively. The $nm \times nm$ matrix

$$\hat{\mathbf{Q}}_A = \begin{bmatrix} \mathbf{Q}_{a_{11}} & \cdots & \mathbf{Q}_{a_{1n}} \\ \vdots & \ddots & \vdots \\ \mathbf{Q}_{a_{n1}} & \cdots & \mathbf{Q}_{a_{nn}} \end{bmatrix} \tag{19}$$

consists of an $n \times n$ array of product operational matrices corresponding to the elements of $\mathbf{\Theta}(t)$. (Note that if $\mathbf{\Theta}(t)$ is symmetric then $\mathbf{A}' = \mathbf{A}^{\mathrm{T}}$.) The Chebyshev expansion of the n-dimensional identity matrix is $\mathbf{I}_n = \hat{\mathbf{T}}^{\mathrm{T}}(t)\hat{\mathbf{I}} = \hat{\mathbf{I}}^{\mathrm{T}}\hat{\mathbf{T}}(t)$, where

$$\hat{\mathbf{I}} = \mathbf{I}_n \otimes (1 \quad \underset{(m-1)}{0 \quad \cdots \quad 0})^{\mathrm{T}}$$

is the $nm \times n$ *identity coefficient matrix*.

4. METHOD OF ANALYSIS

4.1. SYSTEM UNDER CONSIDERATION

Consider a system of n linear time-periodic differential equations

$$\dot{\mathbf{x}}(t, \boldsymbol{\alpha}) = \mathbf{A}(t, \boldsymbol{\alpha})\mathbf{x}(t, \boldsymbol{\alpha}), \qquad \mathbf{x}(0, \boldsymbol{\alpha}) = \mathbf{x}^0, \tag{20}$$

where $\mathbf{x}(t, \boldsymbol{\alpha}) \in \mathbb{R}^N$ is the state vector which depends on the time $t \in \mathbb{R}^+$ and a parameter vector $\boldsymbol{\alpha} \in \mathbb{R}^L$ (the derivative is with respect to time), and the $n \times n$ matrix $\mathbf{A}(t, \boldsymbol{\alpha})$ can be written as $\mathbf{A}(t, \boldsymbol{\alpha}) = \mathbf{A}_1(\boldsymbol{\alpha})f_1(t) + \mathbf{A}_2(\boldsymbol{\alpha})f_2(t) + \cdots + \mathbf{A}_r(\boldsymbol{\alpha})f_r(t)$. The functions $f_i(t) = f_i(t + \beta_i)$, $i = 1, \ldots, r$ are periodic with periodic β_i and the $n \times n$ constant matrices $\mathbf{A}_i(\boldsymbol{\alpha})$, $i = 1, \ldots, r$ contain the coefficient of these periodic functions. Assuming that the frequencies are commensurate, the lowest positive number T such that $q_i\beta_i = T$ for positive integers q_i is the principal period of the system matrix $\mathbf{A}(t + T, \boldsymbol{\alpha}) = \mathbf{A}(t, \boldsymbol{\alpha})$. The fundamental solution matrix $\boldsymbol{\Phi}(t, \boldsymbol{\alpha})$ of equation (20) satisfies $\dot{\boldsymbol{\Phi}}(t, \boldsymbol{\alpha}) = \mathbf{A}(t, \boldsymbol{\alpha})\boldsymbol{\Phi}(t, \boldsymbol{\alpha})$; $\boldsymbol{\Phi}(0, \boldsymbol{\alpha}) = \mathbf{I}$ and the solution for the given initial conditions may be expressed as $\mathbf{x}(t, \boldsymbol{\alpha}) = \boldsymbol{\Phi}(t, \boldsymbol{\alpha})\mathbf{x}^0$. In the following, two formulations for finding $\boldsymbol{\Phi}(t, \boldsymbol{\alpha})$ via Picard iteration and Chebyshev expansion are presented.

4.2. THE GENERAL FORMULATION

The first formulation is applicable to general systems of the form of equation (20). An equivalent integral form of equation (20) is

$$\mathbf{x}(t, \boldsymbol{\alpha}) = \mathbf{x}^0 + \int_0^t \mathbf{A}(\tau, \boldsymbol{\alpha})\mathbf{x}(\tau, \boldsymbol{\alpha}) \, \mathrm{d}\tau. \tag{21}$$

As the zeroth approximation, let $\mathbf{x}^{(0)}(t, \boldsymbol{\alpha}) = \mathbf{x}(0, \boldsymbol{\alpha}) = \mathbf{x}^0$. Use of equation (21) then leads to the first approximation

$$\mathbf{x}^{(1)}(t, \boldsymbol{\alpha}) = \mathbf{x}^0 + \int_0^t \mathbf{A}(\tau_0, \boldsymbol{\alpha})\mathbf{x}^{(0)}(\tau_0, \boldsymbol{\alpha}) \, \mathrm{d}\tau_0 = \left[\mathbf{I} + \int_0^t \mathbf{A}(\tau_0, \boldsymbol{\alpha}) \, \mathrm{d}\tau_0\right]\mathbf{x}^0, \tag{22}$$

where τ_0 is a dummy variable. The second approximation is obtained from equation (21) as

$$\mathbf{x}^{(2)}(t, \boldsymbol{\alpha}) = \mathbf{x}^0 + \int_0^t \mathbf{A}(\tau_1, \boldsymbol{\alpha})\mathbf{x}^{(1)}(\tau_1, \boldsymbol{\alpha}) \, \mathrm{d}\tau_1$$

$$= \left[\mathbf{I} + \int_0^t \mathbf{A}(\tau_1, \boldsymbol{\alpha}) \, \mathrm{d}\tau_1 + \int_0^t \mathbf{A}(\tau_1, \boldsymbol{\alpha}) \int_0^{\tau_1} \mathbf{A}(\tau_0, \boldsymbol{\alpha}) \, \mathrm{d}\tau_0 \, \mathrm{d}\tau_1\right]\mathbf{x}^0, \tag{23}$$

where τ_1 is another dummy variable. Further iteration leads to the $(k + 1)$th approximation

$$\mathbf{x}^{(k+1)}(t, \boldsymbol{\alpha}) = \mathbf{x}^0 + \int_0^t \mathbf{A}(\tau_k, \boldsymbol{\alpha})\mathbf{x}^{(k)}(\tau_k, \boldsymbol{\alpha}) \, \mathrm{d}\tau_k$$

$$= \left[\mathbf{I} + \int_0^t \mathbf{A}(\tau_k, \boldsymbol{\alpha}) \, \mathrm{d}\tau_k + \int_0^t \mathbf{A}(\tau_k, \boldsymbol{\alpha}) \int_0^{\tau_k} \mathbf{A}(\tau_{k-1}, \boldsymbol{\alpha}) \, \mathrm{d}\tau_{k-1} \, \mathrm{d}\tau_k \right.$$

$$\left. + \cdots + \int_0^t \mathbf{A}(\tau_k, \boldsymbol{\alpha}) \cdots \int_0^{\tau_1} \mathbf{A}(\tau_0, \boldsymbol{\alpha}) \, \mathrm{d}\tau_0 \cdots \mathrm{d}\tau_k \right] \mathbf{x}^0, \tag{24}$$

where $\tau_0, \ldots, \tau_k$ are all dummy variables. The series of integrals is an approximation to the fundamental matrix $\boldsymbol{\Phi}(t, \boldsymbol{\alpha})$ since it is truncated after a finite number of terms, while the true solution is an infinite series. If $\mathbf{A}(t, \boldsymbol{\alpha}) = \mathbf{A}(\boldsymbol{\alpha})$ is a constant matrix, then this series results in the power series definition of the exponential solution of equation (20), namely

$$\mathbf{x}(t, \boldsymbol{\alpha}) = \mathrm{e}^{\mathbf{A}(\boldsymbol{\alpha})t} \mathbf{x}^0 = [\mathbf{I} + \mathbf{A}(\boldsymbol{\alpha})t + [\mathbf{A}(\boldsymbol{\alpha})t]^2/2! + [\mathbf{A}(\boldsymbol{\alpha})t]^3/3! + \cdots]\mathbf{x}^0. \tag{25}$$

Unfortunately, the symbolic evaluation of the fundamental matrix via equation (24), in general, leads to complicated expressions for $\boldsymbol{\Phi}(t, \boldsymbol{\alpha})$ and, in addition, is not efficient when $r > 1$ due to the necessary repeated integration by parts. Instead, the following approach is taken which results in a more efficient approximation of $\boldsymbol{\Phi}(t, \boldsymbol{\alpha})$. First, the transformation $t = T\tau$ to equation (20) normalizes the system matrix's principal period to one and, after multiplying through by T, the equation is

$$\mathrm{d}\mathbf{x}(\tau, \boldsymbol{\alpha})/\mathrm{d}\tau = \bar{\mathbf{A}}(\tau, \boldsymbol{\alpha})\mathbf{x}(\tau, \boldsymbol{\alpha}), \qquad \bar{\mathbf{A}}(\tau + 1, \boldsymbol{\alpha}) = \bar{\mathbf{A}}(\tau, \boldsymbol{\alpha}), \quad \mathbf{x}(0, \boldsymbol{\alpha}) = \mathbf{x}^0, \tag{26}$$

where

$$\bar{\mathbf{A}}(\tau, \boldsymbol{\alpha}) = \bar{\mathbf{A}}_1(\boldsymbol{\alpha})f_1(\tau) + \bar{\mathbf{A}}_2(\boldsymbol{\alpha})f_2(\tau) + \cdots + \bar{\mathbf{A}}_r(\boldsymbol{\alpha})f_r(\tau), \qquad f_i(\tau) = f_i(\tau + 1),$$

and

$$\bar{\mathbf{A}}_i(\boldsymbol{\alpha}) = T\mathbf{A}_i(\boldsymbol{\alpha}), \quad i = 1, \ldots, r.$$

Next, the Chebyshev polynomial matrix defined in section 2 is used in expanding the normalized system matrix in m shifted Chebyshev polynomials of the first kind as

$$\bar{\mathbf{A}}(\tau, \boldsymbol{\alpha}) = \hat{\mathbf{T}}^{\mathrm{T}}(\tau)\mathbf{D}(\boldsymbol{\alpha}) = \mathbf{D}'(\boldsymbol{\alpha})\hat{\mathbf{T}}(\tau), \tag{27}$$

where the $nm \times n$ (respectively $n \times nm$) Chebyshev coefficient matrix $\mathbf{D}(\boldsymbol{\alpha})$ (respectively $\mathbf{D}'(\boldsymbol{\alpha})$) is defined as

$$\mathbf{D}(\boldsymbol{\alpha}) = \sum_{i=1}^{r} \bar{\mathbf{A}}_i(\boldsymbol{\alpha}) \otimes \mathbf{d}_i, \qquad \mathbf{D}'(\boldsymbol{\alpha}) = \sum_{i=1}^{r} \bar{\mathbf{A}}_i(\boldsymbol{\alpha}) \otimes \mathbf{d}_i^{\mathrm{T}}. \tag{28}$$

The $m \times 1$ column vectors $\mathbf{d}_i$ (respectively $1 \times m$ row vectors $\mathbf{d}_i^{\mathrm{T}}$) contains the known coefficients in the Chebyshev expansion of the 1-periodic functions as

$$f_i(\tau) = \sum_{j=0}^{m-1} \mathrm{d}_{ij} T_j^*(\tau) = \mathbf{T}^{*\mathrm{T}}(\tau)\mathbf{d}_i = \mathbf{d}_i^{\mathrm{T}}\mathbf{T}^*(\tau), \tag{29}$$

where $T_j^*(\tau)$ $(0 \leqslant \tau \leqslant 1)$ are the Chebyshev polynomials. Then, using the integration operational matrix and the identity coefficient matrix defined in section 2, equation (22) can be written as

$$\mathbf{x}^{(1,m)}(\tau, \boldsymbol{\alpha}) = \left[\mathbf{I} + \int_0^\tau \hat{\mathbf{T}}^{\mathrm{T}}(\tau_0)\mathbf{D}(\boldsymbol{\alpha}) \, \mathrm{d}\tau_0 \right] \mathbf{x}^0 = \hat{\mathbf{T}}^{\mathrm{T}}(\tau)[\hat{\mathbf{I}} + \hat{\mathbf{G}}^{\mathrm{T}}\mathbf{D}(\boldsymbol{\alpha})]\mathbf{x}^0, \tag{30}$$

where the superscript $(1, m)$ indicates that the first Picard iteration is approximated by m Chebyshev polynomials. Furthermore, using the product operational matrix also defined in section 2, equation (23) can be written as

$$\mathbf{x}^{(2,m)}(\tau, \boldsymbol{\alpha}) = \left[\hat{\mathbf{T}}^{\mathrm{T}}(\tau)[\hat{\mathbf{I}} + \hat{\mathbf{G}}^{\mathrm{T}}\mathbf{D}(\boldsymbol{\alpha})] + \int_0^\tau \mathbf{D}'(\boldsymbol{\alpha})\hat{\mathbf{T}}(\tau_1)\hat{\mathbf{T}}^{\mathrm{T}}(\tau_1)\hat{\mathbf{G}}^{\mathrm{T}}\mathbf{D}(\boldsymbol{\alpha})\, \mathrm{d}\tau_1 \right]\mathbf{x}^0$$

$$= \left[\hat{\mathbf{T}}^{\mathrm{T}}(\tau)[\hat{\mathbf{I}} + \hat{\mathbf{G}}^{\mathrm{T}}\mathbf{D}(\boldsymbol{\alpha})] + \int_0^\tau \hat{\mathbf{T}}^{\mathrm{T}}(\tau_1)\hat{\mathbf{Q}}_D(\boldsymbol{\alpha})\hat{\mathbf{G}}^{\mathrm{T}}\mathbf{D}(\boldsymbol{\alpha})\, \mathrm{d}\tau_1 \right]\mathbf{x}^0$$

$$= \hat{\mathbf{T}}^{\mathrm{T}}(\tau)[\hat{\mathbf{I}} + \hat{\mathbf{G}}^{\mathrm{T}}\mathbf{D}(\boldsymbol{\alpha}) + \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_D(\boldsymbol{\alpha})\hat{\mathbf{G}}^{\mathrm{T}}\mathbf{D}(\boldsymbol{\alpha})]\mathbf{x}^0, \tag{31}$$

where $\hat{\mathbf{Q}}_D(\boldsymbol{\alpha}) = \sum_{i=1}^r \bar{\mathbf{A}}_i(\boldsymbol{\alpha}) \otimes \mathbf{Q}_{d_i}$ is expressed in terms of the product operational matrices $\mathbf{Q}_{d_i}$ corresponding to the Chebyshev coefficients of the periodic functions $f_i(t)$. Continuing in this way, the approximate fundamental matrix solution of equation (20) over the principal period can be written in terms of the Chebyshev polynomials as

$$\boldsymbol{\Phi}^{(p,m)}(\tau, \boldsymbol{\alpha}) = \hat{\mathbf{T}}^{\mathrm{T}}(\tau)\left[\hat{\mathbf{I}} + \left(\sum_{k=1}^p [\mathbf{L}(\boldsymbol{\alpha})]^{k-1} \right)\mathbf{P}(\boldsymbol{\alpha}) \right] = \hat{\mathbf{T}}^{\mathrm{T}}(\tau)\mathbf{B}(\boldsymbol{\alpha}), \tag{32}$$

where $\mathbf{B}(\boldsymbol{\alpha})$ contains the Chebyshev coefficients of the elements of $\boldsymbol{\Phi}(\tau, \boldsymbol{\alpha})$ and is expressed in terms of $\mathbf{L}(\boldsymbol{\alpha}) = \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_D(\boldsymbol{\alpha})$ and $\mathbf{P}(\boldsymbol{\alpha}) = \hat{\mathbf{G}}^{\mathrm{T}}\mathbf{D}(\boldsymbol{\alpha})$, which are $nm \times nm$ and $nm \times n$, respectively. By selecting a value for p, the number of Picard iterations, this truncated expression gives an approximate solution to any desired degree of accuracy. While this is valid only in the interval $t \in [0, T]$ or $\tau \in [0, 1]$, the solution can be easily extended for $t > T(\tau > 1)$ by utilizing the formula

$$\boldsymbol{\Phi}^{(p,m)}(\tau, \boldsymbol{\alpha}) = \boldsymbol{\Phi}^{(p,m)}(\eta, \boldsymbol{\alpha})[\boldsymbol{\Phi}^{(p,m)}(1, \boldsymbol{\alpha})]^k, \tag{33}$$

where $\tau = k + \eta$, $\eta \in [0, 1]$, $k = 1, 2, \ldots$. The matrix $\boldsymbol{\Phi}^{(p,m)}(1, \boldsymbol{\alpha})$ is the Floquet Transition Matrix (FTM) whose eigenvalues (Floquet multipliers) determine the stability characteristics of the system. While these expressions are in terms of the normalized time, the substitution $\tau = t/T$ yields the results in real time. It should also be noted that, using the special properties of the Kronecker product operation, $\mathbf{L}(\boldsymbol{\alpha})$ and $\mathbf{P}(\boldsymbol{\alpha})$ can be written in a more computationally efficient form as

$$\mathbf{L}(\boldsymbol{\alpha}) = \sum_{i=1}^r \bar{\mathbf{A}}_i(\boldsymbol{\alpha}) \otimes [\mathbf{G}^{\mathrm{T}}\mathbf{Q}_{d_i}], \qquad \mathbf{P}(\boldsymbol{\alpha}) = \sum_{i=1}^r \bar{\mathbf{A}}_i(\boldsymbol{\alpha}) \otimes [\mathbf{G}^{\mathrm{T}}\mathbf{d}_i], \tag{34}$$

in which the amount of effort spent in matrix multiplications is minimized.

Thus the integrations in equation (24) are replaced by the more computationally efficient matrix multiplications in equation (32), and an approximation to the fundamental matrix in terms of the shifted Chebyshev polynomials is made by including a finite number p of Picard iterations and an appropriate number m of Chebyshev polynomials (which determines the sizes of the various matrices). The Chebyshev expansion of the periodic matrix $\bar{\mathbf{A}}(\tau, \boldsymbol{\alpha})$ not only provides the efficient symbolic approximation of the fundamental matrix, but also results in a more compact result than if the actual integrations in equation (24) had been performed. Also, because the solution is approximated via Picard iterations, it is not pre-expanded in a Chebyshev series with unknown coefficients as was done in

earlier studies [7, 8]. This eliminates the need to solve for the Chebyshev coefficient matrix $\mathbf{B}(\boldsymbol{\alpha})$ via Cramer's rule, matrix inversion, etc., and thus permits the symbolic evaluation of this matrix. Therefore an approximate expression for the fundamental solution matrix can be obtained in terms of the system parameter vector $\boldsymbol{\alpha}$ and the normalized time τ via simple matrix multiplications and additions. Three different problems are analyzed using this formulation in section 5.

4.3. ALTERNATE FORMULATION FOR SYSTEMS WITH A CONSTANT MATRIX

Now consider a specific form of equation (20) given by

$$\dot{\mathbf{x}}(t, \boldsymbol{\alpha}) = [\mathbf{A}_1 + \mathbf{A}_t(t, \boldsymbol{\alpha})]\mathbf{x}(t, \boldsymbol{\alpha}), \qquad \mathbf{x}(0, \boldsymbol{\alpha}) = \mathbf{x}^0, \tag{35}$$

where the matrix $\mathbf{A}_1$ is independent of both time and the system parameters and $\mathbf{A}_t(t, \boldsymbol{\alpha}) = \mathbf{A}_2(\boldsymbol{\alpha})f_2(t) + \mathbf{A}_3(\boldsymbol{\alpha})f_3(t) + \cdots + \mathbf{A}_r(\boldsymbol{\alpha})f_r(t)$. The function $f_1(t)$ is unity and has a one-term (in $T_0^*(\tau)$) Chebyshev expansion. The associated product operational matrix is then the identity matrix, and the application of equation (32) to the entire system results in an expression which includes the power series of the exponential matrix solution for the constant matrix $\mathbf{A}_1$ of the form of equation (25). While the general formulation above may provide satisfactory convergence if the elements in $\mathbf{A}_1$ are not very large, the required number of Picard iterations to achieve accuracy may significantly increase as these matrix elements are allowed to increase in magnitude. This is due to the well-known slow convergence rate of the power series solution in equation (25). It is desired, therefore, to take advantage of the closed-form solution of the constant part of equation (35) with the anticipation that a faster convergence would be achieved. For this purpose, the time-periodic term in equation (35) is treated as a forcing term in a non-homogeneous system which has the computable generating solution $e^{A_1 t}$. Use of the superposition integral then results in the integral equation

$$\mathbf{x}(t, \boldsymbol{\alpha}) = e^{A_1 t}\left[\mathbf{x}^0 + \int_0^t e^{-A_1 \tau}\mathbf{A}_t(\tau, \boldsymbol{\alpha})\mathbf{x}(\tau, \boldsymbol{\alpha})\, d\tau\right], \tag{36}$$

which may be iterated similarly to equation (24) to yield the $(k + 1)$th approximation

$$\mathbf{x}^{(k+1)}(t, \boldsymbol{\alpha}) = e^{A_1 t}\left[x^0 + \int_0^t e^{-A_1 \tau_k}\mathbf{A}_t(\tau_k, \boldsymbol{\alpha})\mathbf{x}^{(k)}(\tau_k, \boldsymbol{\alpha})\, d\tau_k\right]$$

$$= e^{A_1 t}\left[\mathbf{I} + \int_0^t e^{-A_1 \tau_1}\mathbf{A}_t(\tau_1, \boldsymbol{\alpha})\, e^{A_1 \tau_1}\, d\tau_1\right.$$

$$\left. + \int_0^t e^{-A_1 \tau_1}\mathbf{A}_t(\tau_1, \boldsymbol{\alpha})\, e^{A_1 \tau_1}\int_0^{\tau_1} e^{-A_1 \tau_0}\mathbf{A}_t(\tau_0, \boldsymbol{\alpha})\, e^{A_1 \tau_0}\, d\tau_0\, d\tau_1 + \cdots\right]\mathbf{x}^0 \tag{37}$$

from which the approximate fundamental solution matrix is obtained. However, as with equation (24), the direct symbolic evaluation of equation (37), in general, leads to complicated expressions and is not efficient for $k > 1$ due to repeated integrations by parts. Instead, the method of expanding in Chebyshev polynomials after normalizing the principal period to identity as in section 4.2 is utilized.

After normalizing, the Chebyshev expansions

$$\bar{\mathbf{A}}_t(\tau, \boldsymbol{\alpha}) = \hat{\mathbf{T}}^T(\tau)\mathbf{D}(\boldsymbol{\alpha}), \qquad e^{\bar{A}_1 \tau} = \hat{\mathbf{T}}^T(\tau)\mathbf{E}, \qquad e^{-\bar{A}_1 \tau} = \hat{\mathbf{T}}^T(\tau)\mathbf{F} \tag{38}$$

are made where $\bar{\mathbf{A}}_1 = T\mathbf{A}_1$ and $\bar{\mathbf{A}}_t(t, \boldsymbol{\alpha}) = T\mathbf{A}_t(\tau, \boldsymbol{\alpha})$ are the normalized system matrices. The $nm \times n$ Chebyshev coefficient matrix $\mathbf{D}(\boldsymbol{\alpha})$ is defined as in equations (28) and (29) while the definitions of $\mathbf{E}$ and $\mathbf{F}$ depend on whether the eigenvalues of $\bar{\mathbf{A}}_1$ are real, imaginary, or complex. If they occur in $n/2$ imaginary pairs $\pm \omega_i^* i$, then the coefficient matrices of $e^{\bar{A}_1\tau} = \sum_{i=1}^{n/2} \mathbf{C}_i \cos(\omega_i\tau) + \mathbf{S}_i \sin(\omega_i\tau)$ and $e^{-\bar{A}_1\tau} = \sum_{i=1}^{n/2} \mathbf{C}_i \cos(\omega_i\tau) - \mathbf{S}_i \sin(\omega_i\tau)$ may be expressed as

$$\mathbf{E} = \sum_{i=1}^{n/2} \mathbf{C}_i \otimes \mathbf{b}_c(\omega_i) + \mathbf{S}_i \otimes \mathbf{b}_s(\omega_i), \qquad \mathbf{F} = \sum_{i=1}^{n/2} \mathbf{C}_i \otimes \mathbf{b}_c(\omega_i) - \mathbf{S}_i \otimes \mathbf{b}_s(\omega_i), \qquad (39)$$

where the $m \times 1$ Chebyshev coefficient column vectors $\mathbf{b}_c(\omega_i)$ and $\mathbf{b}_s(\omega_i)$ may alternately be defined in terms of Bessel functions in the expansions of $\cos(\omega_i\tau) = \mathbf{T}^{*\mathrm{T}}(\tau)\mathbf{b}_c(\omega_i)$ and $\sin(\omega_i\tau) = \mathbf{T}^{*\mathrm{T}}(\tau)\mathbf{b}_s(\omega_i)$ [12, 17]. If the eigenvalues a_i, $i = 1, \ldots, n$ are real, then the coefficient matrices of $e^{\bar{A}_1\tau} = \sum_{i=1}^{n} \mathbf{Z}_i e^{a_i\tau}$ and $e^{-\bar{A}_1\tau} = \sum_{i=1}^{n} \mathbf{Z}_i e^{-a_i\tau}$ are

$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{Z}_i \otimes \mathbf{z}(a_i), \qquad \mathbf{F} = \sum_{i=1}^{n} \mathbf{Z}_i \otimes \mathbf{z}(-a_i), \qquad (40)$$

where $\mathbf{z}(a_i)$ may also be defined in terms of Bessel functions in the expansion of $e^{a_i\tau} = \mathbf{T}^{*\mathrm{T}}(\tau)\mathbf{z}(a_i)$ [17]. If the eigenvalues occur in $n/2$ complex pairs $a_i \pm \omega_i^* i$, then the product operational matrix may be used to yield the Chebyshev coefficient column vectors as

$$e^{a_i\tau} \cos(\omega_i\tau) = \mathbf{z}^{\mathrm{T}}(a_i)\mathbf{T}^*(\tau)\mathbf{T}^{*\mathrm{T}}(\tau)\mathbf{b}_c(\omega_i) = \mathbf{T}^{*\mathrm{T}}(\tau)\mathbf{Q}_{\mathbf{z}(a_i)}\mathbf{b}_c(\omega_i),$$

$$e^{a_i\tau} \sin(\omega_i\tau) = \mathbf{z}^{\mathrm{T}}(a_i)\mathbf{T}^*(\tau)\mathbf{T}^{*\mathrm{T}}(\tau)\mathbf{b}_s(\omega_i) = \mathbf{T}^{*\mathrm{T}}(\tau)\mathbf{Q}_{\mathbf{z}(a_i)}\mathbf{b}_s(\omega_i), \qquad (41)$$

from which the coefficient matrices of $e^{\bar{A}_1\tau} = \sum_{i=1}^{n/2} \mathbf{C}_i^z e^{a_i\tau} \cos(\omega_i\tau) + \mathbf{S}_i^z e^{a_i\tau} \sin(\omega_i\tau)$ and $e^{-\bar{A}_1\tau} = \sum_{i=1}^{n/2} \mathbf{C}_i^z e^{-a_i\tau} \cos(\omega_i\tau) - \mathbf{S}_i^z e^{-a_i\tau} \sin(\omega_i\tau)$ are obtained as

$$\mathbf{E} = \sum_{i=1}^{n/2} \mathbf{C}_i^z \otimes (\mathbf{Q}_{\mathbf{z}(a_i)}\mathbf{b}_c(\omega_i)) + \mathbf{S}_i^z \otimes (\mathbf{Q}_{\mathbf{z}(a_i)}\mathbf{b}_s(\omega_i)),$$

$$\mathbf{F} = \sum_{i=1}^{n/2} \mathbf{C}_i^z \otimes (\mathbf{Q}_{\mathbf{z}(-a_i)}\mathbf{b}_c(\omega_i)) - \mathbf{S}_i^z \otimes (\mathbf{Q}_{\mathbf{z}(-a_i)}\mathbf{b}_s(\omega_i)). \qquad (42)$$

Combinations of these may easily be treated by summing all the appropriate terms in equations (39), (40), and (42). Substituting the expansions of equation (38) into the bracketed portion of equation (37) yields the approximate fundamental solution matrix over the principal period as

$$\boldsymbol{\Phi}^{(p,m)}(\tau, \boldsymbol{\alpha}) = e^{\bar{A}_1\tau}\hat{\mathbf{T}}^{\mathrm{T}}(\tau)\left[\hat{\mathbf{I}} + \left(\sum_{k=1}^{p} [\mathbf{H}(\boldsymbol{\alpha})]^{k-1}\right)\mathbf{K}(\boldsymbol{\alpha})\right] = e^{\bar{A}_1\tau}\hat{\mathbf{T}}^{\mathrm{T}}(\tau)\mathbf{R}(\boldsymbol{\alpha}), \qquad (43)$$

where $\mathbf{H}(\boldsymbol{\alpha}) = \hat{\mathbf{G}}^T\hat{\mathbf{Q}}_F\hat{\mathbf{Q}}_D(\boldsymbol{\alpha})\hat{\mathbf{Q}}_E$ and $\mathbf{K}(\boldsymbol{\alpha}) = \hat{\mathbf{G}}^T\hat{\mathbf{Q}}_F\hat{\mathbf{Q}}_D(\boldsymbol{\alpha})\mathbf{E}$ are generally much less sparse than the corresponding $\mathbf{L}(\boldsymbol{\alpha})$ and $\mathbf{P}(\boldsymbol{\alpha})$ matrices in the general formulation. Unlike the general formulation in which all $\mathbf{A}_i(\boldsymbol{\alpha})$ matrices may include one or more parameters, parametric dependence in the constant $\mathbf{A}_1$ matrix is impractical in this formulation since the product operational matrices $\hat{\mathbf{Q}}_E$ and $\hat{\mathbf{Q}}_F$ in equation (43) cannot be efficiently evaluated in terms of a parameter. Given this restriction, however, the faster convergence of this formulation generally results in the use of much fewer Picard iterations to achieve a desired accuracy as opposed to use of the general formulation via equation (32). Applications of this formulation to a commutative system and to the Mathieu equation are given in section 5.

5. EXAMPLES

5.1. A COMMUTATIVE SYSTEM

Consider the commutative π-periodic system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -1 + \alpha\cos^2 t & 1 - \alpha\sin t\cos t \\ -1 - \alpha\sin t\cos t & -1 + \alpha\sin^2 t \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{44}$$

where α is the system parameter. The exact fundamental solution matrix of the system is [16]

$$\boldsymbol{\Phi}(t, \alpha) = \begin{bmatrix} e^{(\alpha-1)t}\cos t & e^{-t}\sin t \\ -e^{(\alpha-1)t}\sin t & e^{-t}\cos t \end{bmatrix}. \tag{45}$$

It is desired to compare this exact solution with the proposed approximate one. After normalizing the period to one as in equation (26), the new periodic system matrix is $\bar{\mathbf{A}}(\tau, \alpha) = \bar{\mathbf{A}}_1(\alpha) + \bar{\mathbf{A}}_2(\alpha)\cos 2\pi\tau + \bar{\mathbf{A}}_3(\alpha)\sin 2\pi\tau$, where

$$\bar{\mathbf{A}}_1(\alpha) = \pi\begin{bmatrix} -1 + \alpha/2 & 1 \\ -1 & -1 + \alpha/2 \end{bmatrix}, \qquad \bar{\mathbf{A}}_2(\alpha) = \pi\begin{bmatrix} \alpha/2 & 0 \\ 0 & -\alpha/2 \end{bmatrix},$$

$$\bar{\mathbf{A}}_3(\alpha) = \pi\begin{bmatrix} 0 & -\alpha/2 \\ -\alpha/2 & 0 \end{bmatrix}, \tag{46}$$

so that the product operational matrices for $f_1(\tau) = 1$ (which is the identity matrix), $f_2(\tau) = \cos 2\pi\tau$, and $f_3(\tau) = \sin 2\pi\tau$ are utilized.

Appendix B gives the complete first element of the approximated fundamental solution matrix using the general formulation via equation (32) where $p = 14$ and $m = 7$. The computations were performed using MATHEMATICA on a SUN SPARC 20 where the total CPU time was 57·97 seconds. It is seen that the Chebyshev coefficients are homogeneous polynomials of powers of α up to the order $\mathcal{O}(\alpha^{14})$ and that higher order terms possess considerably smaller coefficient magnitudes. This expression was then evaluated for three different values of the parameter α at times spanning the principal period and compared to the exact solution via equation (45). These results are shown in Table 1. To select p and m, the matrix elements of the exact and approximate solutions were plotted over the principal period for all three parameter values. Figure 1 demonstrates the left-to-right convergence of the approximate solution as the number p of iterations is increased ($\alpha = 0·1$, $m = 7$), and Figure 2 demonstrates the more even convergence as the

TABLE 1

The first element of the fundamental solution matrix for the commutative system where $p = 14$ and $m = 7$. α is the system parameter and τ is the normalized time

τ	α		
	0·1	0·5	2·0
0	1·00000[a]	1·00000	1·00000
	1·00006[b]	0·99994	1·00094
$\frac{1}{4}$	0·34874	0·47746	1·55088
	0·34877	0·47754	1·55372
$\frac{1}{2}$	0·00000	0·00000	0·00000
	−0·00021	−0·00016	0·00208
$\frac{3}{4}$	−0·08483	−0·21769	−7·46049
	−0·08543	−0·21772	−7·45842
1	−0·05916	−0·20788	−23·1407
	−0·05908	−0·20797	−23·1621

(a), exact solution; (b), approximate (symbolic) solution.

number m of polynomials is increased ($\alpha = 0\cdot1$, $p = 10$). In either case, however, the result is least accurate when the solution matrix becomes the FTM at $\tau = 1$. The last graph in Figure 2 demonstrates that increasing m beyond a certain value cannot improve the accuracy if p is too low. Since in general one does not know the exact solution, it was found that the best procedure to achieve a desired accuracy is, for a high value of m, to select an appropriate p for a converged FTM, and then decrease m until the joint just before convergence is lost. This should be done at various locations in the parameter region of interest to ensure uniform convergence before the final analytical approximation is obtained. It should be noted that this type of convergence study can be accomplished using
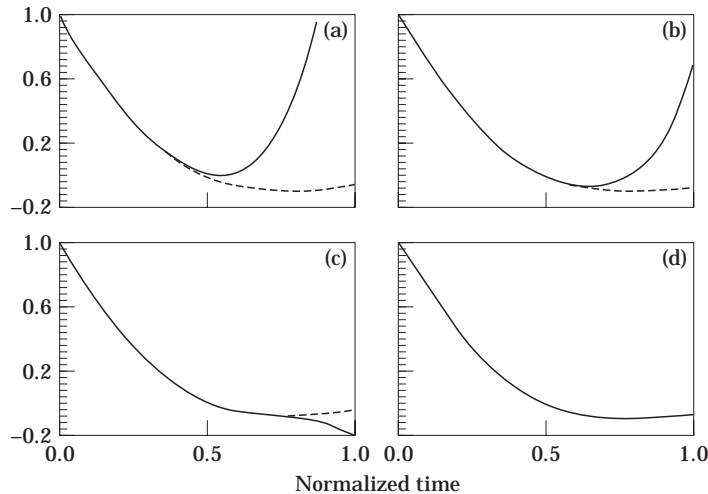

Figure 1. The first element of the exact (dashed) and approximate (solid) fundamental solution matrix for the commutative system plotted over the principal period (versus normalized time) where $\alpha = 0\cdot1$, $m = 7$, and $p =$ (a) 6; (b) 8; (c) 10; (d) 12.
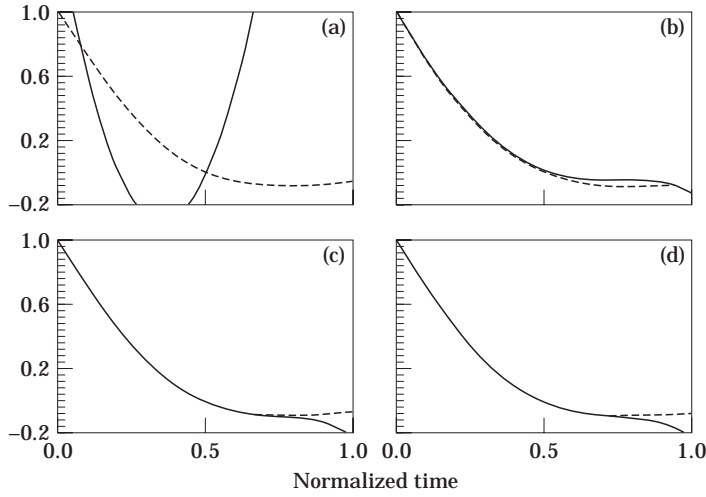
Figure 2. The first element of the exact (dashed) and approximate (solid) fundamental solution matrix for the commutative system plotted over the principal period (versus normalized time) where $\alpha = 0 \cdot 1$, $p = 10$, and $m =$ (a) 3; (b) 5; (c) 7; (d) 9.

a FORTRAN program at a negligible cost compared to obtaining the parameter-dependent analytical approximation using symbolic software.

Finally it should be noted that, although the $\mathbf{A}_1(\alpha)$ matrix includes the system parameter, the alternate formulation can still be utilized in conjunction with the generating solution $\mathbf{\Phi}(\tau, 0)$ by including the parameter-dependent part of $\mathbf{A}_1(\alpha)$ (along with $\mathbf{A}_2(\alpha)$ and $\mathbf{A}_3(\alpha)$) in the $\mathbf{A}_t(t, \alpha)$ matrix in equation (35). The term between the integrals in equation (37) simplifies to

$$\mathbf{\Phi}^{-1}(t, 0)\mathbf{A}_t(t, \alpha)\mathbf{\Phi}(t, 0) = \begin{bmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{bmatrix} \frac{\alpha}{2} \begin{bmatrix} 1 + \cos 2t & -\sin 2t \\ -\sin 2t & 1 - \cos 2t \end{bmatrix}$$

$$\times \begin{bmatrix} e^{-t} \cos t & e^{-t} \sin t \\ -e^{-t} \sin t & e^{-t}\cos t \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}, \tag{47}$$

so that the actual series of integrals in brackets in that equation results in the exponential solution of this matrix, or $\mathrm{diag}\,(\exp(\alpha t), 1)$. The solution then is $\mathbf{\Phi}(t, \alpha) = \mathbf{\Phi}(t, 0)\,\mathrm{diag}\,(\exp(\alpha t), 1)$. Of course, use of the alternate formulation in conjunction with an expansion in m Chebyshev polynomials would yield an approximate solution with $\exp(\alpha t)$ expanded in an $(m - 1)$th degree power series. In this case the alternate formulation is more efficient than the general formulation only for $\alpha < 1$. If the commutative system matrix (with α fixed) were perturbed by another time-periodic parameter-dependent matrix, however, the alternate formulation could be employed such that the complete fundamental matrix solution of equation (45) is used as the generating function. Such a procedure would be much more efficient than would the direct application of the general formulation to a perturbed commutative system.

5.2. MATHIEU EQUATION

The well-known Mathieu equation

$$\ddot{y} + (a + b \cos t)y = 0 \tag{48}$$

is analyzed next where a and b are the system parameters. It should be noted that if the time is normalized in state space form as

$$\dot{\mathbf{x}} = [\bar{\mathbf{A}}_1(a) + \bar{\mathbf{A}}_2(b) \cos 2\pi\tau]\mathbf{x}, \qquad \bar{\mathbf{A}}_1(a) = \begin{bmatrix} 0 & 2\pi \\ -2\pi a & 0 \end{bmatrix}, \qquad \bar{\mathbf{A}}_2(b) = \begin{bmatrix} 0 & 0 \\ -2\pi b & 0 \end{bmatrix}, \quad (49)$$

where $\mathbf{x}^{\mathrm{T}} = (x_1 \; x_2) = (y \; \dot{y})$ and the derivatives are with respect to τ, then the fundamental solution matrix of the original second order system is given by

$$\mathbf{\Phi}^{(p,m)}(\tau, a, b) = \hat{\mathbf{T}}^{\mathrm{T}}(\tau)\bar{\mathbf{B}}(a, b) = \hat{\mathbf{T}}^{\mathrm{T}}(\tau)\begin{bmatrix} (1/2\pi)\mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}\mathbf{B}(a, b)\begin{bmatrix} 2\pi & 0 \\ 0 & 1 \end{bmatrix}. \qquad (50)$$

However, if equation (48) is first normalized and then transformed to the state space form

$$\dot{\mathbf{x}} = [\tilde{\mathbf{A}}_1(a) + \tilde{\mathbf{A}}_2(b) \cos 2\pi\tau]\mathbf{x}, \qquad \tilde{\mathbf{A}}_1(a) = \begin{bmatrix} 0 & 1 \\ -4\pi^2 a & 0 \end{bmatrix}, \qquad \tilde{\mathbf{A}}_2(b) = \begin{bmatrix} 0 & 0 \\ -4\pi^2 b & 0 \end{bmatrix},$$

$$(51)$$

then the fundamental solution matrix is given directly by equation (32). In either case, the product operational matrices for $f_1(\tau) = 1$ and $f_2(\tau) = \cos 2\pi\tau$ are utilized in the approximation.

Appendix C gives selected coefficients of all four elements of the approximated fundamental solution matrix using the general formulation via equation (50) where $p = 24$ and $m = 15$. The computations were again performed using MATHEMATICA on a SUN SPARC 20 where the total CPU time was 15 min 46 s. In Figure 3 the bold line represents the CPU time required for *each* of the 24 iterations. It can be seen that higher iteration steps require increasingly more time since the order of the polynomial coefficients continually increases and that the first iteration requires more time than do the next two because it includes the computation of $\mathbf{L}(a, b)$ and $\mathbf{P}(a, b)$. All of the Chebyshev coefficients here are of order $\mathcal{O}(a^{12}b^{12})$, that is they consist of various powers of $a^i b^j$ up to
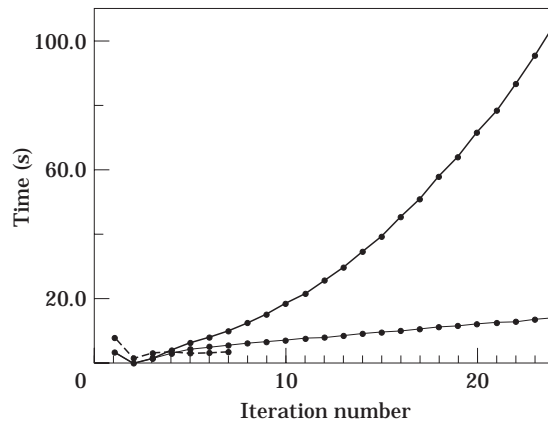

Figure 3. The CPU time (in s) required for *each* of the 24 Picard iterations (*not* the cumulative time after each iteration) in the Mathieu equation for the general formulation with two parameters (bold), general formulation with one parameter (thin), and the alternate formulation with one parameter (dashed) which has equivalent accuracy after 7 iterations.

TABLE 2

The eigenvalues (Floquet multipliers) of (c) the "exact" Runge–Kutta FTM and the approximate FTM via (d) the general formulation where $p = 24$, and $m = 15$ and (e) the constant matrix formulation where $p = 7$ and $m = 17$ for the Mathieu equation with different parameter sets

	b		
a	0·01	0·75	1·5
−0·75	0·00434; 230·754[c]	0·00577; 173·295	0·05810; 17·2120
	0·00433; 230·754[d]	0·00577; 173·297	0·05801; 17·2129
0·0	0·99901 ± 0·04442*i	−8·47371; −0·11801	−34·3546; −0·02911
	0·99901 ± 0·04442*i	−8·47367; −0·11800	−34·3551; −0·02909
0·75	0·66606 ± 0·74590*i	0·36764 ± 0·92997*i	0·41935 ± 0·90782*i
	0·66606 ± 0·74590*i	0·36764 ± 0·92997*i	0·41935 ± 0·90784*i
1·5	0·15804 ± 0·98743*i	0·32108 ± 0·94705*i	1·33050; 0·75160
	0·15829 ± 0·98659*i	0·32169 ± 0·94611*i	1·33493; 0·74892
	0·15804 ± 0·98743*i[e]	0·32101 ± 0·94702*i	1·33124; 0·75052

order $p/2 = 12$ such that $i + j \leqslant 12$, $i = 0, \ldots, 12$, $j = 0, \ldots, 12$. However, only the coefficients of $T_0^*(\tau)$, $T_1^*(\tau)$, and $T_{14}^*(\tau)$ such that $i + j \leqslant 2$ are shown in Appendix C for brevity. It should be noted that some lower order powers of a and b do not appear in the coefficients of higher polynomials such as $T_{14}^*(\tau)$ and that higher order terms possess considerably smaller coefficient magnitudes. This expression was then evaluated for different values of the parameters at the end of the principal period to obtain different Floquet Transition Matrices. The eigenvalues or Floquet multipliers were then computed and compared to those obtained via a fourth order Runge–Kutta (DIVPRK) routine in the IMSL library, considered here as the "exact" solution, where a tolerance of 10^{-8} was used. These results are shown in Table 2. It should be noted that, as explained in section 2, the accuracy in Picard iteration is sensitive to an increase in all parameters unlike asymptotic techniques which depend on small values of the periodic parameter alone. Hence, convergence in the (a, b) plane varies radially from the origin. Convergence in both p and m was studied by plotting the FTM elements versus p while varying m for various parameter sets in the region of interest as described in section 5.1. Convergence plots for two of the parameter sets in Table 2 are shown in Figure 4 (one of the $a = 0·0$ cases to which small-parameter techniques *cannot* be applied) and Figure 5 (at the edge of the converged region in parameter space). It can be seen that the FTM near the origin in parameter space converges faster than does the FTM farther from the origin. Appendix D contains the trace of the FTM which is used to compute the stability boundaries via the relation

$$\operatorname{tr}(\boldsymbol{\Phi}^{(24,15)}(1, a, b)) = \pm 2, \tag{52}$$

where $+2$ and -2 correspond to destabilization via tangent and period doubling routes, respectively, in this Hamiltonian system [17]. Substitution of various values for b into equation (52) yields 12th-degree polynomial equations in a which may be solved to find points on the stability boundaries once the complex roots and those outside the converged region have been eliminated. Figure 6 shows the familiar stability curves computed in this way in the converged parameter region. The curves coincide with those obtained using numerical integration of the FTM. The thin line in Figure 3 represents the CPU time
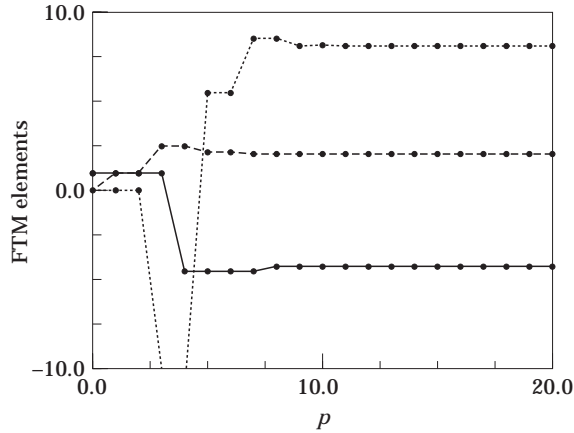
Figure 4. All four FTM elements versus number p of Picard iterations for the Mathieu equation via the general formulation where $a = 0.0$, $b = 0.75$, and $m = 8$. The two diagonal elements (solid line) remain equal.

required for each of the 24 iterations in directly approximating the b-dependent fundamental solution matrix for $a = 1.5$. This approach, which yields the same $\mathcal{O}(b^{12})$ coefficients also obtained from substituting $a = 1.5$ into the expression in Appendix C, requires a total CPU time of 3 min 26 s, a savings of 12 min 20 s over the time required for the 2-parameter approximation.

Equation (48) may also be analyzed using the alternate formulation outlined in section 4.3. As was discussed there, however, a must be fixed at a given value and the fundamental solution matrix obtained in terms of only the periodic parameter b. Since the constant part of the system has the generating solution

$$e^{\tilde{A}_1\tau} = \mathbf{I} \cos \omega\tau + \mathbf{S} \sin \omega\tau, \qquad \mathbf{S} = \begin{bmatrix} 0 & 1/\omega \\ -\omega & 0 \end{bmatrix}, \qquad \omega = 2\pi\sqrt{a}, \qquad (53)$$

using the state space version in equation (51), $\mathbf{H}(b)$ and $\mathbf{K}(b)$ in equation (43) are expressed as

$$\mathbf{H}(b) = (\mathbf{I} \otimes \mathbf{G}^T)(\mathbf{I} \otimes \mathbf{Q}_{b_c} - \mathbf{S} \otimes \mathbf{Q}_{b_s})(\tilde{\mathbf{A}}_2(b) \otimes \mathbf{Q}_{d_2})(\mathbf{I} \otimes \mathbf{Q}_{b_c} + \mathbf{S} \otimes \mathbf{Q}_{b_s})$$
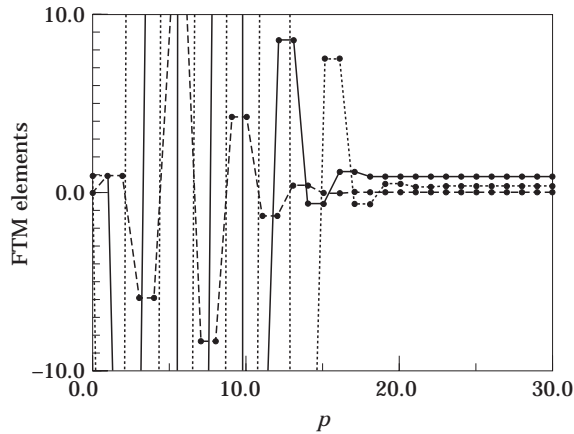

Figure 5. All four FTM elements versus number p of Picard iterations for the Mathieu equation via the general formulation where $a = 1.5$, $b = 1.5$, and $m = 15$. The two diagonal elements (solid line) remain equal.
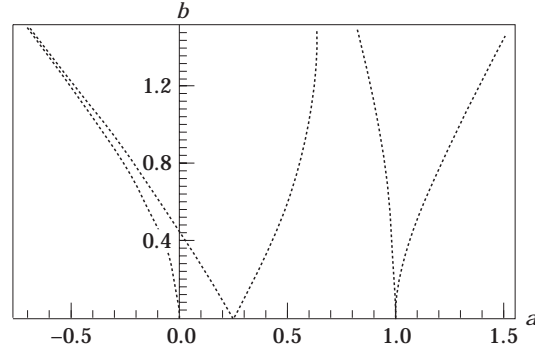
Figure 6. Stability boundaries for the Mathieu equation in the (a, b) parameter plane which result from using the expression in Appendix D generated by application of the general formulation.

$$= \tilde{\mathbf{A}}_2(b) \otimes (\mathbf{G}^{\mathrm{T}} \mathbf{Q}_{b_c} \mathbf{Q}_{d_2} \mathbf{Q}_{b_c}) + (\tilde{\mathbf{A}}_2(b) \mathbf{S} - \mathbf{S} \tilde{\mathbf{A}}_2(b)) \otimes (\mathbf{G}^{\mathrm{T}} \mathbf{Q}_{b_c} \mathbf{Q}_{d_2} \mathbf{Q}_{b_s})$$
$$- (\mathbf{S} \tilde{\mathbf{A}}_2(b) \mathbf{S}) \otimes (\mathbf{G}^{\mathrm{T}} \mathbf{Q}_{b_s} \mathbf{Q}_{d_2} \mathbf{Q}_{b_s}),$$

$$\mathbf{K}(b) = (\mathbf{I} \otimes \mathbf{G}^{\mathrm{T}})(\mathbf{I} \otimes \mathbf{Q}_{b_c} - \mathbf{S} \otimes \mathbf{Q}_{b_s})(\tilde{\mathbf{A}}_2(b) \otimes \mathbf{Q}_{d_2})(\mathbf{I} \otimes \mathbf{b}_c + \mathbf{S} \otimes \mathbf{b}_s)$$

$$= \tilde{\mathbf{A}}_2(b) \otimes (\mathbf{G}^{\mathrm{T}} \mathbf{Q}_{b_c} \mathbf{Q}_{d_2} \mathbf{b}_c) + (\tilde{\mathbf{A}}_2(b) \mathbf{S} - \mathbf{S} \tilde{\mathbf{A}}_2(b)) \otimes (\mathbf{G}^{\mathrm{T}} \mathbf{Q}_{b_c} \mathbf{Q}_{d_2} \mathbf{b}_s)$$
$$- (\mathbf{S} \tilde{\mathbf{A}}_2(b) \mathbf{S}) \otimes (\mathbf{G}^{\mathrm{T}} \mathbf{Q}_{b_s} \mathbf{Q}_{d_2} \mathbf{b}_s), \tag{54}$$

where $\mathbf{d}_2$ contains the Chebyshev coefficients in the expansion of $f_2(\tau) = \cos 2\pi\tau$. Because a must be fixed beforehand, the value of $a = 1.5$ is chosen and the b-dependent fundamental solution matrix is approximated via equation (43) where $p = 7$ and $m = 17$ (see Appendix E). The computations required just 34 s, a savings of 2 min 52 s over the time required for the b-dependent approximation using the general formulation where both results are accurate to within 10^{-2} at $a = b = 1.5$. The dashed line in Figure 3 represents the CPU time required for each of the 7 iterations and is higher at first from the time used in computing $\mathbf{H}(b)$ and $\mathbf{K}(b)$. An additional power in b is gained at each iteration in this formulation so that all of the Chebyshev coefficients are of order $\mathcal{O}(b^p) = \mathcal{O}(b^7)$ and, in general, are more accurate than the $\mathcal{O}(b^{p/2}) = \mathcal{O}(b^{12})$ coefficients of the general formulation
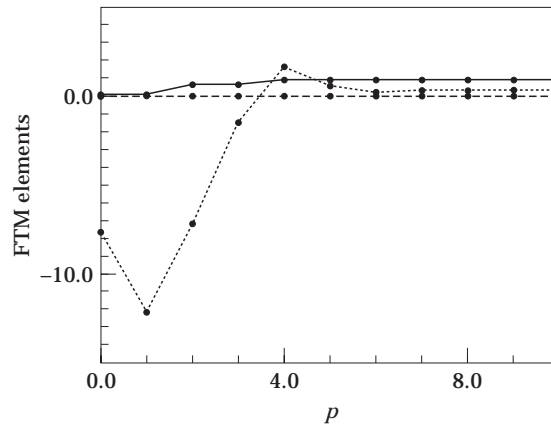

Figure 7. All four FTM elements versus number p of Picard iterations for the Mathieu equation via the alternate formulation where $a = 1.5$, $b = 1.5$, and $m = 17$. The two diagonal elements (solid line) remain equal.
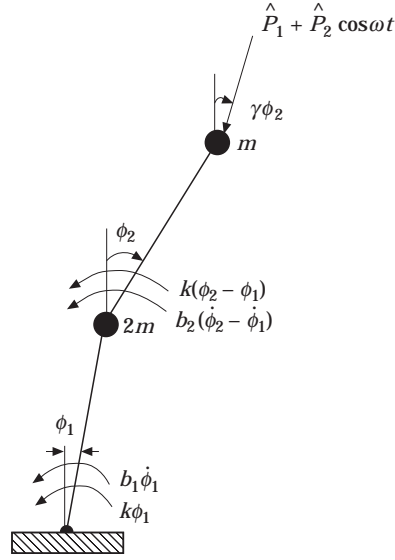
Figure 8. A double inverted pendulum subjected to a periodic follower force.

as can be seen from the corresponding Floquet multipliers in Table 2. The faster convergence is also shown in Figure 7 for the same parameter set as in Figure 5. Thus an improvement over the general formulation in accuracy as well as CPU time is achieved via the alternate formulation provided that a is selected beforehand.

5.3. DOUBLE INVERTED PENDULUM WITH PERIODIC FOLLOWER FORCE

As an example of a higher order system, consider the double inverted pendulum of Figure 8 subjected to a follower force with both constant and periodically varying components. The time-periodic equations of motion for this system (the time-invariant form of which has been treated by Leipholz [18] and Herrmann [19]), may be expressed in the linearized state space form as

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (-3\bar{k}+\bar{p}(t))/2 & \bar{k}-\bar{p}(t)/2 & -B_1/2-B_2 & B_2 \\ (5\bar{k}-\bar{p}(t))/2 & -2\bar{k}+(3/2-\gamma)\bar{p}(t) & B_1/2+2B_2 & -2B_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},
$$

(55)

wjere $(x_1\ x_2\ x_3\ x_4) = (\phi_1\ \phi_2\ \dot{\phi}_1\ \dot{\phi}_2)$ is the state vector, $\bar{k} = k/ml^2$ is the normalized stiffness, $B_1 = b_1/ml^2$ and $B_2 = b_2/ml^2$ are the normalized damping constants, $\bar{p}(t) = (\hat{P}_1 + \hat{P}_2 \cos \omega t)/ml = P_1 + P_2 \cos \omega t$ is the normalized applied load, γ is the load direction parameter, and ω is the internal driving frequency. As with the Mathieu equation, the fundamental solution matrix may be computed after normalizing in either second order or state space form. If the latter is done, then the solution matrix is given by

$$
\mathbf{\Phi}^{(p,m)}(\tau, \boldsymbol{\alpha}) = \hat{\mathbf{T}}^{\mathrm{T}}(\tau)\bar{\mathbf{B}}(\boldsymbol{\alpha}) = \hat{\mathbf{T}}^{\mathrm{T}}(\tau)\begin{bmatrix} (\omega/2\pi)\mathbf{I}_{2m} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{2m} \end{bmatrix}\mathbf{B}(\boldsymbol{\alpha})\begin{bmatrix} (2\pi/\omega)\mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{bmatrix},
$$

(56)

where $\boldsymbol{\alpha}$ includes all the system parameters. In either case, the product operational matrices for $f_1(\tau) = 1$ and $f_2(\tau) = \cos 2\pi\tau$ are utilized after normalizing the period to one.
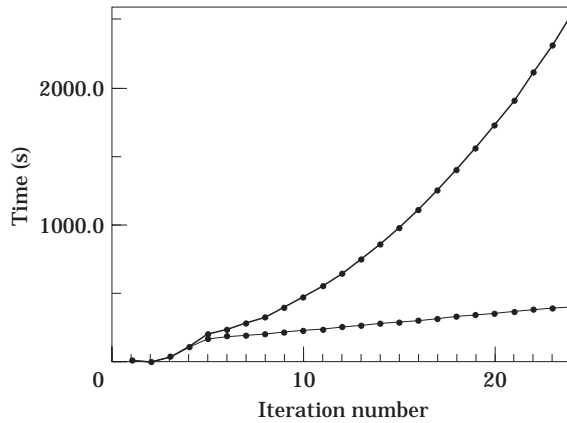
Figure 9. The CPU time (in s) required for *each* of the 24 Picard iterations (*not* the cumulative time after each iteration) in the double inverted pendulum problem for the general formulation with two parameters (bold) and the general formulation with one parameter (thin).

For the fixed parameter set $\bar{k} = 1 \cdot 0$, $B_1 = 0 \cdot 01$, $B_2 = 0 \cdot 01$, $\omega = 2 \cdot 0$, and $\gamma = 1 \cdot 0$, the $P_1 - P_2$ dependent solution may be approximated. Appendix F gives selected coefficients of the leading four elements of the resulting fundamental solution matrix using the general formulation via equation (56) where $p = 24$ and $m = 19$. The computations were again performed using MATHEMATICA on a SUN SPARC 20 where the total CPU time to obtain these coefficients was 6 h 6 min 38 s and the bold line in Figure 9 represents the time required for each of the 24 iterations. All of the Chebyshev coefficients here are of order $\mathcal{O}(P_1^{12}P_2^{12})$, that is they consist of various powers of $P_1^i P_2^j$ up to order $p/2 = 12$ such that $i + j \leqslant 12$, $i = 0, \ldots, 12$, $j = 0, \ldots, 12$. However, only the coefficients of $T_0^*(\tau)$ and $T_1^*(\tau)$ such that $i + j \leqslant 2$ are shown in Appendix F for brevity. The thin line in Figure 9 represents the CPU time required for each of the 24 iterations in directly approximating the P_2-dependent fundamental solution matrix for $P_1 = 1 \cdot 0$. This approach, which yields the same $\mathcal{O}(P_2^{12})$ result also obtained from substituting $P_1 = 1 \cdot 0$ into the expression in Appendix F, requires a total CPU time of 1 h 41 min 50 s, a savings of 4 h 24 min 48 s over the time required for the 2-parameter approximation. It should be noted that the

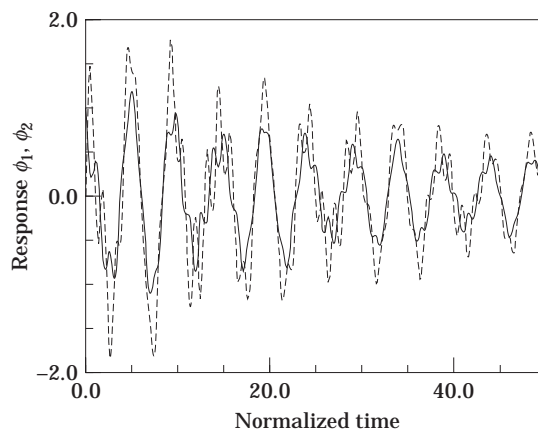

Figure 10. The response $\phi_1(\tau)$ (solid line) and $\phi_2(\tau)$ (dashed line) versus normalized time for the double inverted pendulum with parameter set $\bar{k} = 1 \cdot 0$, $B_1 = 0 \cdot 01$, $B_2 = 0 \cdot 01$, $\omega = 2 \cdot 0$, $\gamma = 1 \cdot 0$, $P_1 = 1 \cdot 0$, $P_2 = 0 \cdot 7$ and initial conditions $(\phi_1 \ \phi_2 \ \dot{\phi}_1 \ \dot{\phi}_2) = (1 \ 0 \ 0 \ 0)$.

alternate formulation of section 4.3 could also be used to obtain a solution in terms of P_2 only, although that is not done here. For $P_1 = 1 \cdot 0$ and $P_2 = 0 \cdot 7$, the response to initial conditions $(x_1 \ x_2 \ x_3 \ x_4) = (1 \ 0 \ 0 \ 0)$ is plotted in Figure 10 versus the normalized time τ over fifty principal periods directly via equation (33). Here it is seen that, in contrast to the truncated point mapping method of Guttalu and Flashner [3, 4], the proposed technique allows the solution to be obtained as an explicit function of time. The Floquet multipliers for this parameter set were computed to be $0 \cdot 2670 \pm 0 \cdot 9418^*i$ and $-0 \cdot 1791 \pm 0 \cdot 9501^*i$ which have absolute values of $0 \cdot 9789$ and $0 \cdot 9669$, respectively. Since their moduli are less than one, the multipliers lie within the unit circle of the complex plane thus indicating asymptotic stability. This can be seen from the decay of the response in Figure 10, which is indistinguishable from that obtained via numerical integration.

6. CONCLUSIONS

A technique for symbolic computation of the fundamental solution matrix for linear time-periodic dynamical systems of arbitrary dimension explicitly as a function of the system parameters and time has been presented. Both Picard iteration and expansion in shifted Chebyshev polynomials with their associated integration and product operational matrices are employed in this technique. It was shown that only matrix multiplications and additions are utilized in the computation of the Chebyshev coefficients of the fundamental solution matrix, which are expressed as homogeneous polynomials of the system parameters. This procedure permits one to obtain a closed form approximation of the fundamental solution matrix for general periodic systems. It was also shown that, unlike traditional perturbation or averaging, the technique is *not* based on expansion in terms of a small parameter and can therefore be successfully applied to periodic systems whose internal excitation is strong. Two formulations were proposed: one applicable to general systems and the other for equations which contain a constant system matrix. The latter formulation restricts the set of parameters that can be used in the solution but converges much faster than the former formulation. Three different example problems, including a double inverted pendulum subjected to a periodic follower force, were included, and results indicated that the proposed method is efficient and the convergence is excellent. It was also shown that the stability boundaries for the Mathieu equation may be obtained from the trace of the parameter-dependent FTM. Possibilities for future work include using the formulations presented here to determine stability and bifurcation conditions as well as controller gains in closed form as a function of the system parameters for higher dimensional systems.

ACKNOWLEDGMENTS

REFERENCES

1. A. H. NAYFEH 1973 *Perturbation Methods*. New York: John Wiley.
2. J. A. SANDERS and F. VERHULST 1985 *Averaging Methods in Nonlinear Dynamical Systems*. New York: Springer-Verlag.
3. R. S. GUTTALU and H. FLASHNER 1995 *Vibration of Nonlinear, Random, and Time-Varying Systems*, proceedings of the Design Engineering Technical Conferences, *v.* 3-A, DE *v.* 84–1, *September* 17–21, *Boston, MA*. An analytical study of stability of periodic systems by Poincaré mappings.

REFERENCES

1. A. H. Nayfeh 1973 *Perturbation Methods*. New York: John Wiley.
2. J. A. Sanders and F. Verhulst 1985 *Averaging Methods in Nonlinear Dynamical Systems*. New York: Springer-Verlag.
3. R. S. Guttalu and H. Flashner 1995 *Vibration of Nonlinear, Random, and Time-Varying Systems, proceedings of the Design Engineering Technical Conferences, v.* 3-A, DE *v.* 84–1, *September* 17–21, *Boston, MA*. An analytical study of stability of periodic systems by Poincaré mappings.
4. R. S. Guttalu and H. Flashner 1996 *Journal of Sound and Vibration* **189**, 33–54. Stability analysis of periodic systems by truncated point mappings.
5. S. C. Sinha and C. C. Chou 1976 *Journal of Sound and Vibration* **49**, 309–326. An approximate analysis of transient response of time-dependent linear systems by orthogonal polynomials.
6. S. C. Sinha, C. C. Chou and H. H. Denman 1979 *Journal of Sound and Vibration* **64**, 515–527. Stability analysis of systems with periodic coefficients: an approximate approach.
7. S. C. Sinha and D.-H. Wu 1991 *Journal of Sound and Vibration* **151**, 91–117. An efficient computational scheme for the analysis of periodic systems.
8. S. C. Sinha, D.-H. Wu, V. Juneja and P. Joseph 1993 *Journal of Vibration and Acoustics* **115**, 96–102. Analysis of dynamic systems with periodically varying parameters via Chebyshev polynomials.
9. S. C. Sinha and V. Juneja 1991 *Proceedings of the 32nd Structures, Structural Dynamics, and Materials Conference, AIAA*-91-1020-*CP, April* 8–10, *Baltimore, MD*, 790–797. An approximate analytical solution for systems with periodic coefficients via symbolic computation.
10. A. Ralston and P. Rabinowitz 1978 *A First Course in Numerical Analysis*. New York: McGraw-Hill.
11. W. E. Milne 1953 *Numerical Solution of Differential Equations*. New York: John Wiley.
12. L. Fox and I. B. Parker 1968 *Chebyshev Polynomials in Numerical Analysis*. Oxford: Oxford University Press.
13. M. A. Snyder 1966 *Chebyshev Methods in Numerical Approximation*. Englewood Cliffs, New Jersey: Prentice-Hall.
14. Y. L. Luke 1969 *The Special Functions and their Approximations*. New York: Academic Press.
15. M. Abramowitz and I. Stegun 1970 *Handbook of Mathematical Functions*. New York: Dover.
16. R. H. Mohler 1991 *Nonlinear Systems, Volume* 1: *Dynamics and Control*. New York: Prentice Hall.
17. J. J. Stoker 1950 *Nonlinear Vibrations in Mechanical and Electrical Systems*. New York: John Wiley.
18. H. Leipholz 1970 *Stability Theory*. New York: Academic Press.
19. G. Herrmann 1971 *Dynamics and Stability of Mechanical Systems with Follower Forces*. NASA contractor report NASA CR-1782, prepared by Stanford University.

APPENDIX A: THE KRONECKER PRODUCT

Consider a 2×2 square matrix $\mathbf{A}$ and an $n \times m$ matrix $\mathbf{B}$. The Kronecker product is

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} \end{bmatrix}. \tag{A.1}$$

The resulting $2n \times 2m$ matrix has dimension $\dim(\mathbf{A} \otimes \mathbf{B}) = (2n, 2m)$. If $\dim(\mathbf{A}) = \dim(\mathbf{C})$ and $\dim(\mathbf{B}) = \dim(\mathbf{D})$, then the relation $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC} \otimes \mathbf{BD})$ holds.

APPENDIX B: APPROXIMATE SOLUTION OF COMMUTATIVE SYSTEM

The following is the complete approximation of the first element of the fundamental solution matrix for the commutative system in section 5.1 obtained using MATHEMATICA. Five significant figures are included.

$\mathbf{\Phi}_{11}^{(14,7)}(\tau, \alpha) =$

$(0 \cdot 24553 + 0 \cdot 0051070 * \alpha - 0 \cdot 060330 * \alpha^{\wedge}2 - 0 \cdot 082741 * \alpha^{\wedge}3 - 0 \cdot 040813 * \alpha^{\wedge}4$

$- 0 \cdot 057310 * \alpha^{\wedge}5 + 0 \cdot 0013716 * \alpha^{\wedge}6 + 0 \cdot 0033286 * \alpha^{\wedge}7 - 0 \cdot 018509 * \alpha^{\wedge}8$

$+ 0 \cdot 0036181 * \alpha^{\wedge}9 + 0 \cdot 0023376 * \alpha^{\wedge}10 - 0 \cdot 0016598 * \alpha^{\wedge}11$

$+ 0 \cdot 00012657 * \alpha^{\wedge}12 + 0 \cdot 000065170 * \alpha^{\wedge}13 - 0 \cdot 000013919 * \alpha^{\wedge}14) * T_0^*(\tau)$

$+ (-0 \cdot 48263 - 0 \cdot 18603 * \alpha - 0 \cdot 14556 * \alpha^{\wedge}2 - 0 \cdot 13693 * \alpha^{\wedge}3 - 0 \cdot 068702 * \alpha^{\wedge}4$

$- 0 \cdot 10213 * \alpha^{\wedge}5 + 0 \cdot 0093078 * \alpha^{\wedge}6 + 0 \cdot 0040990 * \alpha^{\wedge}7 - 0 \cdot 033657 * \alpha^{\wedge}8$

$+ 0 \cdot 0074919 * \alpha^{\wedge}9 + 0 \cdot 0038622 * \alpha^{\wedge}10 - 0 \cdot 0030029 * \alpha^{\wedge}11 + 0 \cdot 00026449 * \alpha^{\wedge}12$

$+ 0 \cdot 00011090 * \alpha^{\wedge}13 - 0 \cdot 000025182 * \alpha^{\wedge}14) * T_1^*(\tau) + (0 \cdot 24042 - 0 \cdot 037125 * \alpha$

$- 0 \cdot 056560 * \alpha^{\wedge}2 - 0 \cdot 048153 * \alpha^{\wedge}3 - 0 \cdot 032410 * \alpha^{\wedge}4 - 0 \cdot 067243 * \alpha^{\wedge}5$

$+ 0 \cdot 018349 * \alpha^{\wedge}6 - 0 \cdot 00027886 * \alpha^{\wedge}7 - 0 \cdot 024197 * \alpha^{\wedge}8 + 0 \cdot 0069384 * \alpha^{\wedge}9$

$+ 0 \cdot 0021546 * \alpha^{\wedge}10 - 0 \cdot 0021739 * \alpha^{\wedge}11 + 0 \cdot 00025315 * \alpha^{\wedge}12$

$+ 0 \cdot 000069647 * \alpha^{\wedge}13 - 0 \cdot 000018914 * \alpha^{\wedge}14) * T_2^*(\tau) + (-0 \cdot 043825$

$+ 0 \cdot 11460 * \alpha + 0 \cdot 054553 * \alpha^{\wedge}2 + 0 \cdot 030591 * \alpha^{\wedge}3 + 0 \cdot 0027666 * \alpha^{\wedge}4$

$- 0 \cdot 029860 * \alpha^{\wedge}5 + 0 \cdot 020138 * \alpha^{\wedge}6 - 0 \cdot 0023430 * \alpha^{\wedge}7 - 0 \cdot 013510 * \alpha^{\wedge}8$

$+ 0 \cdot 0051657 * \alpha^{\wedge}9 + 0 \cdot 00076101 * \alpha^{\wedge}10 - 0 \cdot 00012716 * \alpha^{\wedge}11$

$+ 0 \cdot 00019691 * \alpha^{\wedge}12 + 0 \cdot 000035216 * \alpha^{\wedge}13 - 0 \cdot 000012532 * \alpha^{\wedge}14) * T_3^*(\tau)$

$+ (-0 \cdot 0062832 - 0 \cdot 040524 * \alpha + 0 \cdot 0086368 * \alpha^{\wedge}2 + 0 \cdot 027709 * \alpha^{\wedge}3$

$+ 0 \cdot 010138 * \alpha^{\wedge}4 - 0 \cdot 010369 * \alpha^{\wedge}5 + 0 \cdot 015852 * \alpha^{\wedge}6 - 0 \cdot 0019293 * \alpha^{\wedge}7$

$- 0 \cdot 00064293 * \alpha^{\wedge}8 + 0 \cdot 0033461 * \alpha^{\wedge}9 + 0 \cdot 000075450 * \alpha^{\wedge}10$

$- 0 \cdot 00063279 * \alpha^{\wedge}11 + 0 \cdot 00013029 * \alpha^{\wedge}12 + 0 \cdot 000014169 * \alpha^{\wedge}13$

$- 7 \cdot 1176 * 10^{\wedge} - 6 * \alpha^{\wedge}14) * T_4^*(\tau) + (0 \cdot 0050186 + 0 \cdot 0020560 * \alpha$

$- 0 \cdot 016524 * \alpha^{\wedge}2 + 0 \cdot 0016623 * \alpha^{\wedge}3 + 0 \cdot 0015351 * \alpha^{\wedge}4$

$- 0 \cdot 0045295 * \alpha^{\wedge}5 + 0 \cdot 0084133 * \alpha^{\wedge}6 - 0 \cdot 00099757 * \alpha^{\wedge}7 - 0 \cdot 0025196 * \alpha^{\wedge}8$

$+ 0 \cdot 0017482 * \alpha^{\wedge}9 - 0 \cdot 00013697 * \alpha^{\wedge}10 - 0 \cdot 00023631 * \alpha^{\wedge}11$

$+ 0 \cdot 000066212 * \alpha^{\wedge}12 + 2 \cdot 4179 * 10^{\wedge} - 6 * \alpha^{\wedge}13 - 2 \cdot 8263 * 10^{\wedge}$

$- 6 * \alpha^{\wedge}14) * T_5^*(\tau) + (-0 \cdot 0010099 + 0 \cdot 0025677 * \alpha + 0 \cdot 00054079 * \alpha^{\wedge}2$

$+ 0 \cdot 00040765 * \alpha^{\wedge}3 - 0 \cdot 0021004 * \alpha^{\wedge}4 - 0 \cdot 0027421 * \alpha^{\wedge}5 + 0 \cdot 0034489 * \alpha^{\wedge}6$

$- 0 \cdot 00073006 * \alpha^{\wedge}7 - 0 \cdot 00078287 * \alpha^{\wedge}8 + 0 \cdot 00071068 * \alpha^{\wedge}9$

$- 0 \cdot 00010833 * \alpha^{\wedge}10 - 0 \cdot 000058383 * \alpha^{\wedge}11 + 0 \cdot 000022006 * \alpha^{\wedge}12$

$+ 2 \cdot 7742 * 10^{\wedge} - 7 * \alpha^{\wedge}13 - 8 \cdot 9391 * 10^{\wedge} - 7 * \alpha^{\wedge}14) * T_6^*(\tau).$

APPENDIX C: APPROXIMATE SOLUTION OF MATHIEU EQUATION VIA THE GENERAL FORMULATION

The following contains selected coefficients of all four elements of the approximated fundamental solution matrix from the general formulation via equation (50) for the Mathieu equation in section 5.2 obtained using MATHEMATICA. All of the Chebyshev coefficients are of order $\mathcal{O}(a^{12}b^{12})$; however, only the coefficients of order $\mathcal{O}(a^2b^2)$ in $T_0^*(\tau)$, $T_1^*(\tau)$, and $T_{14}^*(\tau)$ are shown. Five significant figures are included.

$\Phi_{11}^{(24,15)}(\tau, a, b) =$

$$(1 \cdot 0 - 7 \cdot 4022*a - 0 \cdot 69575*b + 17 \cdot 756*a\char`^2 - 0 \cdot 61894*a*b$$

$$- 3 \cdot 1028*b\char`^2 + \cdots)*T_0^*(\tau) + (-9 \cdot 8696*a + 4 \cdot 3651*10\char`^ - 12*b + 28 \cdot 410*a\char`^2$$

$$- 1 \cdot 5007*a*b - 4 \cdot 9348*b\char`^2 + \cdots)*T_1^*(\tau) + \cdots + (1 \cdot 0697*10\char`^ - 8*b$$

$$+ 6 \cdot 1690*10\char`^ - 7*a*b - 0 \cdot 000012710*b\char`^2 + \cdots)*T_{14}^*(\tau),$$

$\Phi_{12}^{(24,15)}(\tau, a, b) =$

$$(0 \cdot 5 - 2 \cdot 0561*a + 0 \cdot 65212*b + 3 \cdot 1962*a\char`^2 - 1 \cdot 5662*a*b$$

$$- 0 \cdot 049695*b\char`^2 + \cdots)*T_0^*(\tau) + (0 \cdot 5 - 3 \cdot 0842*a + 1 \cdot 0760*b + 5 \cdot 3270*a\char`^2$$

$$- 2 \cdot 5872*a*b - 0 \cdot 096485*b\char`^2 + \cdots)*T_1^*(\tau) + \cdots + (5 \cdot 3489*10\char`^ - 9*b$$

$$+ 3 \cdot 2604*10\char`^ - 7*a*b - 6 \cdot 3447*10\char`^ - 6*b\char`^2)*T_{14}^*(\tau),$$

$\Phi_{21}^{(24,15)}(\tau, a, b) =$

$$(-19 \cdot 739*a + 8 \cdot 7302*10\char`^ - 12*b + 81 \cdot 174*a\char`^2 + 8 \cdot 0949*a*b$$

$$- 9 \cdot 8696*b\char`^2)*T_0^*(\tau) + (-19.739*a + 3 \cdot 5765*b + 121 \cdot 76*a\char`^2 + 16 \cdot 762*a*b$$

$$- 12 \cdot 778*b\char`^2)*T_1^*(\tau) + \cdots + (-2 \cdot 6698*10\char`^ - 6*a*b$$

$$+ 1 \cdot 1808*10\char`^ - 18*b\char`^2)*T_{14}^*(\tau),$$

$\Phi_{22}^{(24,15)}(\tau, a, b) =$

$$(1 \cdot 0 - 7 \cdot 4022*a + 1 \cdot 5899*b + 17 \cdot 756*a\char`^2 - 4 \cdot 4657*a*b - 1 \cdot 4570*b\char`^2)*T_0^*(\tau)$$

$$+ (-9 \cdot 8696*a + 1 \cdot 7882*b + 28 \cdot 410*a\char`^2 - 5 \cdot 4746*a*b - 2 \cdot 8129*b\char`^2)*T_1^*(\tau)$$

$$+ \cdots + (1 \cdot 3525*10\char`^ - 7*b + 2 \cdot 1923*10\char`^ - 6*a*b - 0 \cdot 000095334*b\char`^2)*T_{14}^*(\tau).$$

APPENDIX D: TRACE OF FTM FOR MATHIEU EQUATION

The following is the complete trace of order $\mathcal{O}(a^{12}b^{12})$ of the approximated Floquet Transition Matrix (FTM) for the Mathieu equation in section 5.2 obtained using the expression in Appendix C. The stability boundaries are obtained by setting the following expression equal to ± 2. Five significant figures are included.

$\text{tr}\,(\mathbf{\Phi}^{(24,15)}(1, a, b)) =$

$2{\cdot}0 - 39{\cdot}478*a + 129{\cdot}87*a\hat{\ }2 - 170{\cdot}91a\hat{\ }3 + 120{\cdot}48*a\hat{\ }4 - 52{\cdot}852*a\hat{\ }5$

$+ 15{\cdot}807*a\hat{\ }6 - 3{\cdot}4287*a\hat{\ }7 + 0{\cdot}56401*a\hat{\ }8 - 0{\cdot}072765*a\hat{\ }9 + 0{\cdot}0075596*a\hat{\ }10$

$- 0{\cdot}00064593*a\hat{\ }11 + 0{\cdot}000046132*a\hat{\ }12 + 3{\cdot}5792*10\hat{\ } - 9*b + 3{\cdot}4102*10\hat{\ }$

$- 7*a*b + 3{\cdot}4377*10\hat{\ } - 6*a\hat{\ }2*b + 6{\cdot}9151*10\hat{\ } - 6*a\hat{\ }3*b + 8{\cdot}1218*10\hat{\ }$

$- 7*a\hat{\ }4*b - 0{\cdot}000013677*a\hat{\ }5*b - 0{\cdot}000017882*a\hat{\ }6*b$

$+ 0{\cdot}000034434*a\hat{\ }7*b - 8{\cdot}8942*10\hat{\ } - 6*a\hat{\ }8*b - 0{\cdot}000017321*a\hat{\ }9*b$

$+ 0{\cdot}000018431*a\hat{\ }10*b - 8{\cdot}1420*10\hat{\ } - 6*a\hat{\ }11*b - 19{\cdot}739*b\hat{\ }2$

$+ 50{\cdot}921*a*b\hat{\ }2 - 52{\cdot}682*a\hat{\ }2*b\hat{\ }2 + 30{\cdot}247*a\hat{\ }3*b\hat{\ }2 - 11{\cdot}137*a\hat{\ }4*b\hat{\ }2$

$+ 2{\cdot}8625*a\hat{\ }5*b\hat{\ }2 - 0{\cdot}54374*a\hat{\ }6*b\hat{\ }2 + 0{\cdot}080236*a\hat{\ }7*b\hat{\ }2$

$- 0{\cdot}010005*a\hat{\ }8*b\hat{\ }2 + 0{\cdot}0012691*a\hat{\ }9*b\hat{\ }2 - 0{\cdot}00018231*a\hat{\ }10*b\hat{\ }2$

$+ 0{\cdot}00025982*b\hat{\ }3 - 0{\cdot}00041502*a*b\hat{\ }3 + 0{\cdot}0016618*a\hat{\ }2*b\hat{\ }3$

$- 0{\cdot}0046893*a\hat{\ }3*b\hat{\ }3 + 0{\cdot}0059355*a\hat{\ }4*b\hat{\ }3 - 0{\cdot}0036123*a\hat{\ }5*b\hat{\ }3$

$+ 0{\cdot}00043473*a\hat{\ }6*b\hat{\ }3 + 0{\cdot}0010401*a\hat{\ }7*b\hat{\ }3 - 0{\cdot}00096680*a\hat{\ }8*b\hat{\ }3$

$+ 0{\cdot}00048355*a\hat{\ }9*b\hat{\ }3 + 1{\cdot}6271*b\hat{\ }4 - 2{\cdot}2923*a*b\hat{\ }4 + 1{\cdot}4233*a\hat{\ }2*b\hat{\ }4$

$- 0{\cdot}52586*a\hat{\ }3*b\hat{\ }4 + 0{\cdot}13125*a\hat{\ }4*b\hat{\ }4 - 0{\cdot}023634*a\hat{\ }5*b\hat{\ }4$

$+ 0{\cdot}0030291*a\hat{\ }6*b\hat{\ }4 - 0{\cdot}000095092*a\hat{\ }7*b\hat{\ }4 - 0{\cdot}00016207*a\hat{\ }8*b\hat{\ }4$

$- 0{\cdot}00014486*b\hat{\ }5 + 0{\cdot}00094262*a*b\hat{\ }5 - 0{\cdot}0020865*a\hat{\ }2*b\hat{\ }5$

$+ 0{\cdot}0020760*a\hat{\ }3*b\hat{\ }5 - 0{\cdot}00084862*a\hat{\ }4*b\hat{\ }5 - 0{\cdot}00023403*a\hat{\ }5*b\hat{\ }5$

$+ 0{\cdot}00051173*a\hat{\ }6*b\hat{\ }5 - 0{\cdot}00033510*a\hat{\ }7*b\hat{\ }5 - 0{\cdot}018464*b\hat{\ }6$

$+ 0{\cdot}018040*a*b\hat{\ }6 - 0{\cdot}0080620*a\hat{\ }2*b\hat{\ }6 + 0{\cdot}0020871*a\hat{\ }3*b\hat{\ }6$

$- 0{\cdot}00017100*a\hat{\ }4*b\hat{\ }6 - 0{\cdot}00020110*a\hat{\ }5*b\hat{\ }6 + 0{\cdot}00017612*a\hat{\ }6*b\hat{\ }6$

$+ 0{\cdot}000047915*b\hat{\ }7 - 0{\cdot}00014690*a*b\hat{\ }7 + 0{\cdot}00015016*a\hat{\ }2*b\hat{\ }7$

$- 0{\cdot}000041058*a\hat{\ }3*b\hat{\ }7 - 0{\cdot}000045028*a\hat{\ }4*b\hat{\ }7 + 0{\cdot}000053278*a\hat{\ }5*b\hat{\ }7$

$+ 0{\cdot}000060736*b\hat{\ }8 - 0{\cdot}000043842*a*b\hat{\ }8 - 1{\cdot}3738*10\hat{\ } - 6*a\hat{\ }2*b\hat{\ }8$

$+ 0{\cdot}000029818*a\hat{\ }3*b\hat{\ }8 - 0{\cdot}000032847*a\hat{\ }4*b\hat{\ }8 - 2{\cdot}5103*10\hat{\ } - 6*b\hat{\ }9$

$+ 3{\cdot}4706*10\hat{\ } - 6*a*b\hat{\ }9 - 9{\cdot}6823*10\hat{\ } - 7*a\hat{\ }2*b\hat{\ }9 - 1{\cdot}1177*10\hat{\ }$

$- 6*a\hat{\ }3*b\hat{\ }9 + 3{\cdot}1946*10\hat{\ } - 9*b\hat{\ }10 - 5{\cdot}8542*10\hat{\ } - 7*a*b\hat{\ }10 + 1{\cdot}3108*10\hat{\ }$

$- 6*a\hat{\ }2*b\hat{\ }10 + 5{\cdot}7289*10\hat{\ } - 9*b\hat{\ }11 - 7{\cdot}3996*10\hat{\ }$

$- 9*a*b\hat{\ }11 - 7{\cdot}7407*10\hat{\ } - 9*b\hat{\ }12.$

APPENDIX E: APPROXIMATE SOLUTION OF MATHIEU EQUATION VIA THE ALTERNATE FORMULATION

The following contains selected coefficients of the first element of the approximated fundamental solution matrix from the alternate formulation via equation (43) for the Mathieu equation in section 5.2 with $a = 1{\cdot}5$ obtained using MATHEMATICA. All of the Chebyshev coefficients are of order $\mathcal{O}(b^7)$; however, only the coefficients in $T_0^*(\tau)$, $T_1^*(\tau)$, and $T_{16}^*(\tau)$ are shown. Five significant figures are included.

$\Phi_{11}^{(7,17)}(\tau, b) =$

$\{(1 \cdot 0 + 0 \cdot 19130*b - 0 \cdot 15309*b^2 - 0 \cdot 11136*b^3 - 0 \cdot 015859*b^4$

$+ 0 \cdot 0034148*b^5 + 0 \cdot 00049605*b^6 - 0 \cdot 000035975*b^7)*T_0^*(\tau)$

$+ (0 \cdot 085588*b - 0 \cdot 15543*b^2 - 0 \cdot 15754*b^3 - 0 \cdot 028389*b^4 + 0 \cdot 0056275*b^5$

$+ 0 \cdot 00088084*b^6 - 0 \cdot 000071968*b^7)*T_1^*(\tau) + \cdots + (0 \cdot 000070110*b$

$- 0 \cdot 00016223*b^2 + 0 \cdot 000020360*b^3 - 0 \cdot 00026916*b^4$

$+ 0 \cdot 000039529*b^5 + 5 \cdot 8031*10^\wedge - 6*b^6 - 5 \cdot 9949*10^\wedge$

$- 7*b^7)*T_{16}^*(\tau)\}* \cos(7 \cdot 6952*\tau) + \{(-0 \cdot 054348*b + 0 \cdot 20676*b^2$

$+ 0 \cdot 011261*b^3 + 0 \cdot 014069*b^4 + 0 \cdot 000039671*b^5 - 0 \cdot 00046546*b^6$

$- 9 \cdot 6538*10^\wedge - 6*b^7)*T_0^*(\tau) + (0 \cdot 21869*b + 0 \cdot 19093*b^2 + 0 \cdot 023780*b^3$

$+ 0 \cdot 029000*b^4 + 1 \cdot 8713*10^\wedge - 6*b^5 - 0 \cdot 00087213*b^6 - 8 \cdot 7557*10^\wedge$

$- 6*b^7)*T_1^*(\tau) + \cdots + (0 \cdot 00043841*b - 0 \cdot 00057814*b^2 - 0 \cdot 00018025*b^3$

$+ 0 \cdot 000093777*b^4 + 0 \cdot 000038876*b^5 - 1 \cdot 0976*10^\wedge - 6*b^6 - 5 \cdot 4250*10^\wedge$

$- 7*b^7)*T_{16}^*(\tau)\}* \sin(7 \cdot 6952*\tau).$

APPENDIX F: APPROXIMATE SOLUTION OF DOUBLE INVERTED PENDULUM

The following contains selected coefficients of the leading four elements of the approximated fundamental solution matrix from the general formulation via equation (56) for the double inverted pendulum with a periodic follower force in section 5.3 obtained using MATHEMATICA. All of the Chebyshev coefficients are of order $\mathcal{O}(P_1^{12}P_2^{12})$; however, only the coefficients of order $\mathcal{O}(P_1^2 P_2^2)$ in $T_0^*(\tau)$ and $T_1^*(\tau)$ are shown. Five significant figures are included.

$\Phi_{11}^{(24,19)}(\tau, P_1, P_2) =$

$(0 \cdot 51852 - 0 \cdot 10148*P_1 + 0 \cdot 18735*P_2 - 0 \cdot 023678*P_1^2 + 0 \cdot 053361*P_1*P_2$

$- 0 \cdot 021881*P_2^2 + \cdots)*T_0^*(\tau) + (-0 \cdot 28383 - 0 \cdot 22223*P_1 + 0 \cdot 19612*P_2$

$- 0 \cdot 063971*P_1^2 + 0 \cdot 067587*P_1*P_2 - 0 \cdot 032312*P_2^2 + \cdots)*T_1^*(\tau) + \cdots,$

$\Phi_{12}^{(24,19)}(\tau, P_1, P_2) =$

$(0 \cdot 16981 + 0 \cdot 0058477*P_1 - 0 \cdot 13296*P_2 - 0 \cdot 0050470*P_1^2 - 0 \cdot 024209*P_1*P_2$

$+ 0 \cdot 017079*P_2^2 + \cdots)*T_0^*(\tau) + (-0 \cdot 034060 + 0 \cdot 099789*P_1 - 0 \cdot 11882*P_2$

$+ 0 \cdot 019276*P_1^2 - 0 \cdot 021598*P_1*P_2 + 0 \cdot 025314*P_2^2 + \cdots)*T_1^*(\tau) + \cdots,$

$\Phi_{21}^{(24,19)}(\tau, P_1, P_2) =$

$(0 \cdot 42454 + 0 \cdot 14941*P_1 + 0 \cdot 067228*P_2 + 0 \cdot 059353*P_1^2 + 0 \cdot 0016377*P_1*P_2$

$+ 0 \cdot 0074051*P_2^2 + \cdots)*T_0^*(\tau) + (-0 \cdot 085152 + 0 \cdot 23279*P_1 + 0 \cdot 17706*P_2$

$+ 0 \cdot 11556*P_1^2 + 0 \cdot 018990*P_1*P_2 + 0 \cdot 0054839*P_2^2 + \cdots)*T_1^*(\tau) + \cdots,$

$\Phi_{22}^{(24,19)}(\tau, P_1, P_2) =$

$(0 \cdot 43362 - 0 \cdot 15217*P_1 - 0 \cdot 080088*P_2 - 0 \cdot 046025*P_1^2 - 0 \cdot 016773*P_1*P_2$

$- 0 \cdot 0056534*P_2^2 + \cdots)*T_0^*(\tau) + (-0 \cdot 26680 - 0 \cdot 26994*P_1 - 0 \cdot 18122*P_2$

$- 0 \cdot 097363*P_1^2 - 0 \cdot 040087*P_1*P_2 - 0 \cdot 0039017*P_2^2 + \cdots)*T_1^*(\tau) + \cdots.$