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THE LARGE PERTURBATION INVERSION METHOD OF 2-D MEDIUM PARAMETERS

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The problem of parameter inversion in an inhomogeneous medium was studied and a generalized ray approximation method to inverse the 2-D medium parameters is introduced in this paper. By using referential and perturbational variables, the medium parameters of the wave equation have been rewritten. By using the Green function theory, the integral equation of perturbational parameters was obtained. Based on the local principles of wave function and ray theory in inhomogeneous medium, a generalized ray approximation form of the total wave field is introduced. By defining the medium parameters function, attention has been focused on the Fredholm integral equation of the first kind. By using convolution transforms, the solution of the medium parameters function in 2-D medium was obtained. The perturbations of medium parameters are usually not greater than 20% in the Born weak scattering method. However, in this paper the numerical results show that when the perturbations of the medium parameters are within 50%, this method can effectively inverse its variation.

1. INTRODUCTION

The Born weak scattering approximate theory provides a useful tool to inverse medium parameters in an inhomogeneous medium [1, 2]. This method has the advantage of simplicity in that the total wave field can be replaced by the incident wave field. This assumption requires that the variations of the value of medium parameters must be small enough. By using this method, the inversion of medium parameters is obtained for a case in which the variations of the value of medium parameters are not beyond 20 percent [3].

However, Born weak scattering method cannot be adequately used in the case where the parameter variations are beyond 20%. This "smallness" limitation is too restricted and many real conditions cannot satisfy the requirement of the assumption. In this paper, an approximate solution to the inverse problem of 2-D medium parameters is presented. It is assumed here that the density and velocity of the medium vary only in two dimensions, vertically and laterally. A harmonic concentrated source is set off and the scattering signals are observed in the surface. In the Born weak scattering method, the total wave field in the medium is assumed equivalent to the incident wave field because the scattering wave field is weak. In this paper, this assumption has been discarded and the general scattering inverse problem has been studied. According to a certain incident wave field, the generalized ray approximation form was introduced to the total wave field. This form is derived from the local principles of wave function in an inhomogeneous medium [4] and the ray theory of the forward problem [5]. By introducing amplitude modification and phase delay, the inverse problem is focused on the Fredholm integral equation of the first

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kind of the medium parameters function which is defined in this paper. The parameters inversion in a 1-D medium was described in paper [6] and the stratified technique was adopted. However, integral transforms were used in this paper. The solution of the medium parameters function of the laterally weak variant 2-D medium is obtained and a numerical example is computed successfully.

2. INTEGRAL EQUATION

The elastic wave equation in an inhomogeneous medium is

$$\partial_i(\lambda\, \partial_k u_k) + \partial_j[\mu(\partial_j u_i + \partial_i u_j)] + \rho F_i = \rho \ddot{u}_i, \tag{1}$$

where λ, μ are the Lame' coefficients of the medium, ρ is the density of the medium. u_i is the component of the displacement field, F_i is the component of the body force, ∂_i denotes the partial derivative to the co-ordinate i, $\ddot{u}_i$ denotes the 2-D derivative with respect to time t, and $i, j = 1, 2, 3$.

By assuming zero viscosity, that is $\mu = 0 (\lambda = E)$, the transverse wave does not have to be considered, and if the body force is ignored, also equation (1) can be rewritten as

$$(1/\rho)\, \partial_i(E\theta) = \ddot{u}_i, \tag{2}$$

where $\theta = \partial_i u_i$. Applying ∂_i to both sides of equation (2) and taking $i = 1, 2, 3$, respectively, the equation is obtained as

$$\operatorname{div}\, [(1/\rho)\, \operatorname{grad}\, (E\theta)] = \ddot{\theta}. \tag{3}$$

By defining $\phi = -E\theta$, substituting it for $E\theta$ in equation (3), and using Fourier transformation of $t \sim \omega$, equation (3) has been recast into the acoustic wave equation in an inhomogeneous medium, that is

$$\mathbf{\nabla} \cdot ((1/\rho)\mathbf{\nabla}\phi) + (\omega^2/E)\phi = 0, \tag{4}$$

where $\phi(\vec{r})$ is the total wave field in the medium, $\vec{r}$ denotes the positional vector, ω is the circular frequency, and $\mathbf{\nabla}$ is the 2-D gradient operator.

By using the referential variables ρ_0 and E_0 and the perturbational variables $\delta\rho$ and the δE, the medium parameters ρ and E are presented by

$$\rho = \rho_0 + \delta\rho, \qquad E = E_0 + \delta E, \tag{5}$$

where the referential variables ρ_0 and E_0 are the value of medium parameters in homogeneous medium. The $\delta\rho$ and δE can be considered as the perturbational value of the medium parameters. These non-dimensional variables can be defined as

$$\alpha_1 = \delta\rho/\rho, \qquad \alpha_2 = \delta E/E. \tag{6}$$

Equation (4) can be changed into the Helmholtz equation

$$\mathbf{\nabla}^2\phi + \kappa_0^2\phi = -\delta(\vec{r}) + \mathbf{\nabla} \cdot (\alpha_1 \mathbf{\nabla}\phi) + \kappa_0^2 \alpha_2 \phi, \tag{7}$$

where κ_0 is the wave number, and one has

$$\kappa_0 = \omega/v_0, \qquad v_0 = \sqrt{E_0/\rho_0}. \tag{8}$$

According to the Green's function theory, equation (7) can be transformed into the integral equation which is similar to the result in reference [3], i.e.,

$$\phi^{(s)}(\vec{r}_0) = \int d\vec{r}G(\vec{r}|\vec{r}_0)[\nabla \cdot (\alpha_1 \nabla\phi) + \kappa_0^2\alpha_2\phi], \tag{9}$$

where $\phi^{(s)}(\vec{r}_0)$ is the scattering wave field of the detecting point $\vec{r}_0$, $G(\vec{r}|\vec{r}_0)$ is the Green's function of the homogeneous medium, and

$$G(\vec{r}|\vec{r}_0) = (i/4)H_0^{(1)}(\kappa_0|\vec{r} - \vec{r}_0|). \tag{10}$$

In the 2-D medium $d\vec{r}$ denotes the integral over the 2-D space filled with the medium.

It can be seen that the scattering wave field results from the inhomogeneity of the medium parameters. The perturbations of the medium parameters can be regarded as the sources function produced by the scattering wave field.

3. GENERALIZED RAY APPROXIMATION

The Born weak scattering approximate theory provides a useful tool to inverse medium parameters in an inhomogeneous medium. Based on the small perturbational parameters ($|\alpha_1|, |\alpha_2| \leqslant 20\%$) and the weak scattering assumption, $\phi^{(s)}$ is small enough to ignore, when compared with $\phi^{(i)}$, that is, $\phi = \phi^{(i)} + \phi^{(s)} \approx \phi^{(i)}$. However, when α_1 and α_2 are not small enough ($20\% \leqslant |\alpha_1|, |\alpha_2| \leqslant 50\%$), the assumption is not correct. Based on the analysis of the local principles of wave function in an inhomogeneous medium, the following results can be obtained [4].

(1) The inhomogeneity, measured by the square of the local wave speed gradient, increased the amplitude of waves generated by a concentrated source of fixed strength.

(2) The inhomogeneity decreased spatial oscillation, that is, a medium with large gradients of wave speed can be expected to have longer spatial wavelengths than a medium that is more uniform.

According to the above results and forward results of the ray theory [5], amplitude modification and phase delay are introduced. The form of the total wave field can be assumed as:

$$\phi(\vec{r}) = (1 + \alpha_3)^{-1/2} \exp[i\kappa_0(1 + \alpha_3)r], \tag{11}$$

where r is the module of the vector $\vec{r}$ and α_3 is the variation of the value of the wave number and

$$\alpha_3 = \delta\kappa/\kappa_0 = (\kappa - \kappa_0)/\kappa_0, \tag{12}$$

where α_1, α_2 and α_3 have the following relation

$$(1 + \alpha_3)^2 = (1 - \alpha_2)/(1 - \alpha_1). \tag{13}$$

Substituting equations (10) and (11) into equation (9), the integral equation is obtained

$$\phi^{(s)}(\vec{r}_0) = \int d\vec{r}H_0^{(1)}(\sigma)\alpha(\vec{r}), \tag{14}$$

where

$$\sigma = \kappa_0|\vec{r} - \vec{r}_0|. \tag{15}$$

$\alpha(\vec{r})$ is defined as the medium parameters function in this paper. It has the form in the 2-D medium of

$$\begin{aligned}\alpha(\vec{r}) = {} & (\mathrm{i}/4)\exp[\mathrm{i}k_0(1+\alpha_3)r]\cdot\{k_0^2[\alpha_2(1+\alpha_3)^{-1/2} - \alpha_1(1+\alpha_3)^{3/2}] \\ & + \mathrm{i}\kappa_0(\alpha_1/r)(1+\alpha_3)^{-1/2} + \mathrm{i}\kappa_0(1/r)(1+\alpha_3)^{1/2}(\vec{r}\cdot\nabla\alpha_1) \\ & + \alpha_1[\mathrm{i}\kappa_0(1/r)(1+\alpha_3)^{-1/2} - 2\kappa_0^2(1+\alpha_3)^{1/2}](\vec{r}\cdot\nabla\alpha_3) \\ & + \alpha_1[\tfrac{3}{4}(1+\alpha_3)^{-5/2} - k_0^2r^2(1+\alpha_3)^{-1/2} - \mathrm{i}k_0r(1+\alpha_3)^{-3/2}](\nabla\alpha_3\cdot\nabla\alpha_3) \\ & + [\mathrm{i}k_0r(1+\alpha_3)^{-1/2} - \tfrac{1}{2}(1+\alpha_3)^{-3/2}](\nabla\alpha_1\cdot\nabla\alpha_3) \\ & + \alpha_1[\mathrm{i}k_0r(1+\alpha_3)^{-1/2} - \tfrac{1}{2}(1+\alpha_3)^{-3/2}](\nabla\cdot\nabla\alpha_3)\}\end{aligned} \tag{16}$$

Thus, the objective of inversion has concentrated on the solution of the Fredholm integral equation of the first kind of the medium parameters function, where the Hankel function can be considered as its core.

4. THE SOLUTION OF THE MEDIUM PARAMETERS FUNCTION

When the detection points are all located in the line $z_0 = 0$, equation (15) is simplified to

$$\sigma = \kappa_0\sqrt{(x+x_0)^2 + z^2}. \tag{17}$$

The form of equation (14) allows construction of the $x - \mathrm{D}$ convolution. By using the method described in reference [3], the solution of the medium parameters function can be written as

$$\alpha(x,z) = \frac{1}{\pi^2}\int_{-\infty}^{+\infty}\mathrm{d}\kappa_1\int_{-\infty}^{+\infty}\mathrm{d}\kappa_3\int_{-\infty}^{+\infty}\mathrm{d}x_0\kappa_3\phi^{(s)}(x_0)\exp[2\mathrm{i}\kappa_1(x-x_0) - 2\mathrm{i}\kappa_3 z]. \tag{18}$$

By the use of stratified technology, the inversions of the longitudinal medium parameters have been performed in reference [6]. However, there is lateral variation in real distribution of underground medium parameters and the geophysical medium models are usually laterally weak variants. The laterally weak variant medium models are more precise than the longitudinal medium model in simulating the real distribution of underground medium parameters. In this paper, the integral transforms are used to solve for the solution of the medium parameters function and the solution is shown in equation (18). In the laterally weak variant medium model, the laterally derivative terms are assumed small enough to be ignored for the lateral variation is weak. The longitudinal medium parameters are assumed to vary slowly, so the longitudinal two-rank derivative terms and the two-order terms of the longitudinal derivative terms of medium parameters can be ignored. During the inversion of the medium parameters the effect of these terms is small enough to be ignored compared with the effect of the geodesic and the calculative errors. Therefore, the medium parameters function is simplified and there are only the perturbational terms and the longitudinally one-rank derivative terms in the medium parameters function, that is

$$\alpha(x,z) = (\mathrm{i}/4)\exp[\mathrm{i}\kappa_0(1+\alpha_3)r]\{\kappa_0^2[\alpha_2(1+\alpha_3)^{-1/2} - \alpha_1(1+\alpha_3)^{3/2}]$$

$$+ i\kappa_0 z(\partial\alpha_1/\partial z)(1/r)(1+\alpha_3)^{1/2} + \alpha_1 z(\partial\alpha_3/\partial z)[i\kappa_0(1/r)(1+\alpha_3)^{-1/2}$$
$$- 2\kappa_0^2(1+\alpha_3)^{1/2}]\}. \tag{19}$$

Connecting with equations (13) and (19), the variations of the values of the medium parameters can be decomposed.

To take account of a laterally weak variant medium model, the density model and velocity model are shown in Figures 1 and 2, respectively. The values of the medium parameters outside the grids take the referential variables. The dimensions of the grids are $80 \times 80\ \text{m}^2$ and $60 \times 80\ \text{m}^2$, respectively. The values of density and velocity are constant in every grid for simplicity of calculation. During the inversion of the medium parameters the referential variables take the values

$$\rho_0 = 1000\ \text{kg/m}^3, \qquad v_0 = 1500\ \text{m/s}. \tag{20}$$

The detective points are located along the line ($z_0 = 0$) beginning at the origin, and with a 2 m interval. The total is 128, necessary for the precision of the integral transformation. By using the detected information, that is, the scattering wave field, and using equations (13) and (19), the values of the medium parameters are obtained. By using same grids of the primal medium model and the inverse values of the medium parameters, the values of every grid can be reconstructed. The reconstructed distributions of density and velocity are shown in Figures 3 and 4, respectively.

5. CONCLUSION AND DISCUSSION

The linearized method was adopted for the Born weak scattering approximation and the total wave field was replaced by the incident wave field. It is required that the perturbations must vary within 20%. Compared with the Born weak scattering

1000	1010	1020
1200	1220	1240
1400	1430	1460
1600	1640	1680

Figure 1. Density distribution model.

1500	1540	1580	1620
1650	1710	1770	1830
1800	1840	1880	1920
2100	2150	2200	2300

Figure 2. Velocity distribution model.

995	1006	1018
1183	1215	1231
1407	1421	1443
1587	1598	1602

Figure 3. Reconstructed distribution of the density.

1489	1536	1567	1611
1633	1689	1756	1837
1788	1873	1898	1974
1997	2078	2135	2198

Figure 4. Reconstructed distribution of the velocity.

approximation within 20% perturbation, the effect of the perturbational variations on the total wave field is considered in the above-mentioned method. Therefore, this method can obtain more precise results than the Born weak scattering approximation method with less than 20% perturbation. Moreover, the objective of this new method is to inverse medium parameters within 50% perturbation. In the numerical example, the perturbational values of the density and the wave velocity are about 50%. From the inverse results it can be shown that when the changes of the inhomogeneous medium parameters are large, the new method presented in this paper can effectively inverse its variation.

It can be shown from reference [6] and the above-mentioned discussions that an effective method which can inverse the inhomogeneous medium parameters has been proposed. Based on the local principles of wave function in an inhomogeneous medium, amplitude modification and phase delay have been introduced, and the effect of the perturbational variations on the total wave fields is considered. By defining the medium parameters function in this paper attention can be focused on its Fredholm integral equation of the first kind. This makes the inverse problem more explicit in inverse objective. Nevertheless, the new method fails to inverse the medium parameters the perturbational variations of which are greater than 50%.

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