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AN ALTERNATIVE HIERARCHICAL FINITE ELEMENT FORMULATION APPLIED TO PLATE VIBRATIONS

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The plate hierarchical finite element in this paper utilizes trigonometric hierarchical shape functions rather than the more usual forms of orthogonal Legendre polynomials. The new hierarchical finite element is formulated in terms of a fixed number of quintic polynomial shape functions plus a variable number of trigonometric hierarchical shape functions. The polynomial shape functions are used to describe the element's nodal degrees of freedom and the trigonometric hierarchical shape functions are used to give additional freedom to the edges and the interior of the element. The numbers of trigonometric hierarchical terms are allowed to vary in both directions of the element's co-ordinate axes. Results are obtained for a number of plates. The results confirm that the solutions always converge from above as the numbers of hierarchical terms are increased and highly accurate values are obtained with the use of a very few hierarchical terms. In comparison with the 36-degree-of-freedom rectangular finite element, the trigonometric hierarchical finite element is found to produce a better accuracy with fewer system degrees of freedom. In comparison with the polynomial hierarchical finite element, the trigonometric hierarchical finite element is found to produce an equivalent accuracy with the same number of system degrees of freedom and fewer numbers of hierarchical terms for a free and a clamped square plate. Furthermore, the trigonometric hierarchical finite element is found to produce a better accuracy with fewer system degrees of freedom and fewer numbers of hierarchical terms for a simply supported square plate and a square plate simply supported on two opposite edges and free on the other two edges.

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1. INTRODUCTION

A hierarchical finite element is formulated for plate vibrations in which trigonometric hierarchical shape functions are used rather than polynomial shape functions which are forms of Legendre orthogonal polynomials. The aim of this paper is to introduce the new plate hierarchical finite element and to investigate its efficiency for a number of plate problems.

The hierarchical finite element method has been applied by Bardell [1, 2] to plate vibrations. This method is formulated in terms of a fixed number of cubic polynomial shape functions plus a variable number of hierarchical shape functions which are forms of Legendre orthogonal polynomials. The cubic polynomial shape functions, if used by themselves, lead to the conforming 16-degree-of-freedom plate rectangular finite element of Bogner *et al*. [3]. This method will be referred to as the polynomial hierarchical finite element method throughout this paper. The hierarchical finite element method has a few major features that make its use desirable for simple uniform structures. The most important feature is that a simple uniform structure is modelled as just one finite element and the number of hierarchical terms is varied. The results can then be obtained to any

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desired degree of accuracy by simply increasing the number of hierarchical terms. The other important feature is that the satisfaction of internal C_0 and/or C_1 continuity along element interfaces is avoided and the problems of stress singularities are overcome. Meirovitch and Baruh [4] and Zhu [5] have shown that the hierarchical finite element method yields a better accuracy than the finite element method for eigenvalue problems of the same order.

The trigonometric hierarchical finite element is formulated in terms of a fixed number of quintic polynomial shape functions plus a variable number of trigonometric hierarchical shape functions. The polynomial shape functions are used to describe the element's nodal degrees of freedom and the trigonometric hierarchical shape functions are used to give additional freedom to the edges and the interior of the element. The polynomial shape functions, if used by themselves, lead to the conforming 36-degree-of-freedom plate rectangular finite element of Bogner *et al*. [3]. The numbers of trigonometric hierarchical terms are allowed to vary in both directions of the element's co-ordinate axes.

Results of frequency calculations by use of the trigonometric hierarchical finite element have been obtained for a number of plates and comparisons have been made with the polynomial hierarchical finite element and the 36-degree-of-freedom rectangular finite element of Bogner *et al*. [3].

2. FORMULATION

2.1. THE SHAPE FUNCTIONS

The co-ordinate system used to define the geometry of a two-node beam element is shown in Figure 1. The x co-ordinate and the non-dimensional ζ co-ordinate are related by (a list of notation is given in the Appendix)

$$\zeta = x/L. \tag{1}$$

The transverse displacement w of the beam element is expressed as

$$w(\zeta) = c_1 + c_2\zeta + c_3\zeta^2 + c_4\zeta^3 + c_5\zeta^4 + c_6\zeta^5 + c_{r+6}\sin\delta_r\zeta, \tag{2}$$

where summation is implied on the index r and

$$\delta_r = r\pi, \qquad r = 1, 2, 3, \ldots . \tag{3}$$

The element's nodal degrees of freedom are the transverse displacement w, the slope $w_{,x}$ and the curvature $w_{,xx}$ at each node. The polynomial terms in the assumed displacement field are used to define the element's nodal degrees of freedom and the trigonometric terms are used to give additional freedom to the interior of the element. Equation (2) can be written in matrix form as

$$w(\zeta) = \mathbf{gc}, \tag{4}$$

where

$$\mathbf{g} = [1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5, \sin\delta_r\zeta] \tag{5}$$

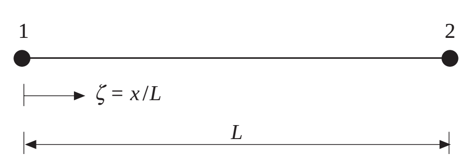


Figure 1. Beam element co-ordinates and dimensions.

and

$$\mathbf{c} = \{c_1, c_2, c_3, c_4, c_5, c_6, c_{r+6}\}^{\mathrm{T}}. \tag{6}$$

The operators $\mathbf{g}$, $L\mathbf{g}_{,x}$ and $L^2\mathbf{g}_{,xx}$ can be evaluated at each node to obtain

$$\mathbf{p} = \mathbf{hc}, \tag{7}$$

where

$$\mathbf{p} = \{w_1, Lw_{1,x}, L^2w_{1,xx}, w_2, Lw_{2,x}, L^2w_{2,xx}, w_{r+6}\}^{\mathrm{T}} \tag{8}$$

and

$$\mathbf{h} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \delta_r \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & (-1)^r\delta_r \\ 0 & 0 & 2 & 6 & 12 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

The vector $\mathbf{c}$ is obtained from equation (7) as

$$\mathbf{c} = \mathbf{h}^{-1}\mathbf{p}, \tag{10}$$

where

$$\mathbf{h}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\delta_r \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ -10 & -6 & -3/2 & 10 & -4 & 1/2 & \delta_r(6+4(-1)^r) \\ 15 & 8 & 3/2 & -15 & 7 & -1 & -\delta_r(8+7(-1)^r) \\ -6 & -3 & -1/2 & 6 & -3 & 1/2 & 3\delta_r(1+(-1)^r) \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

Substituting equation (10) into equation (4) gives the relation

$$w(\zeta) = \mathbf{gh}^{-1}\mathbf{p}. \tag{12}$$

The desired shape functions are therefore given by

$$\mathbf{f} = \mathbf{gh}^{-1}, \tag{13}$$

where

$$\mathbf{f} = [f_1, f_2, f_3, f_4, f_5, f_6, f_{r+6}] \tag{14}$$

and

$$f_1 = 1 - 10\zeta^3 + 15\zeta^4 - 6\zeta^5, \qquad f_2 = \zeta - 6\zeta^3 + 8\zeta^4 - 3\zeta^5, \tag{15, 16}$$

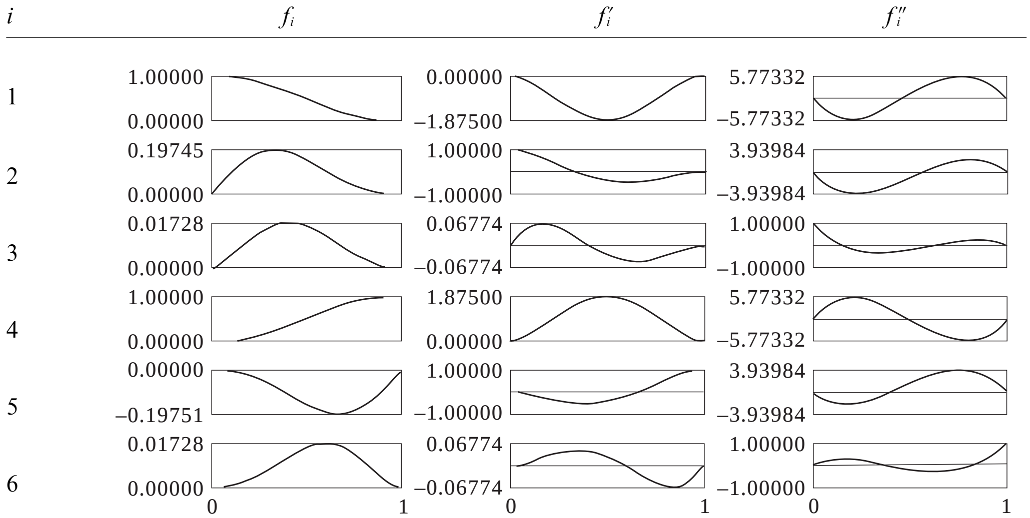
$$f_3 = (1/2)\zeta^2 - (3/2)\zeta^3 + (3/2)\zeta^4 - (1/2)\zeta^5, \qquad f_4 = 10\zeta^3 - 15\zeta^4 + 6\zeta^5, \tag{17, 18}$$

$$f_5 = -4\zeta^3 + 7\zeta^4 - 3\zeta^5, \qquad f_6 = (1/2)\zeta^3 - \zeta^4 + (1/2)\zeta^5, \tag{19, 20}$$

$$f_{r+6} = \delta_r[-\zeta + (6+4(-1)^r)\zeta^3 - (8+7(-1)^r)\zeta^4 + 3(1+(-1)^r)\zeta^5] + \sin\delta_r\zeta. \tag{21}$$

TABLE 1

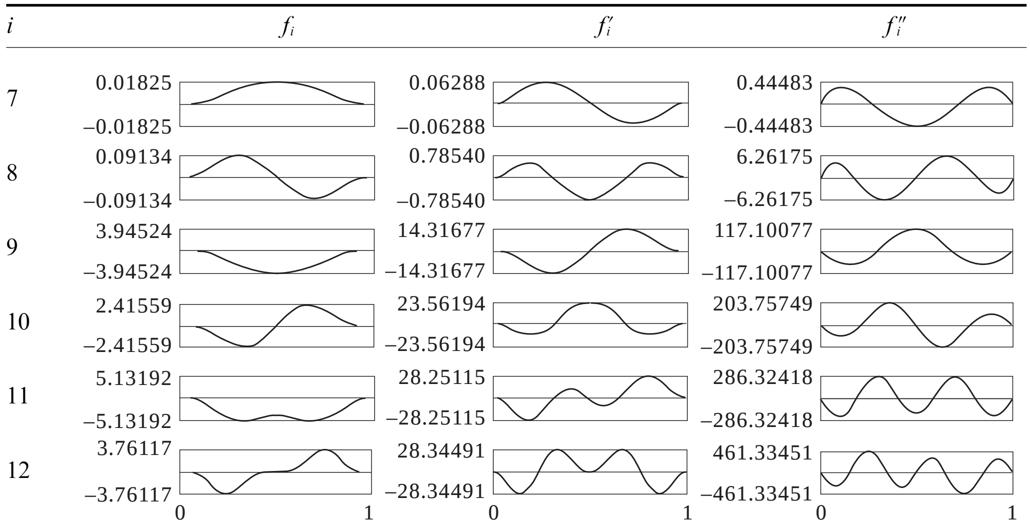
The first six shape functions and their first and second derivatives



The first six shape functions are used in the finite element method. These functions and their first and second derivatives are shown in Table 1. The first six trigonometric hierarchical shape functions f_{r+6} $(r = 1, 2, \ldots, 6)$ and their first and second derivatives are shown in Table 2. These functions possess zero displacement, zero slope, and zero curvature at each node. This feature is highly significant, since these functions only provide

TABLE 2

The first six trigonometric hierarchical shape functions and their first and second derivatives



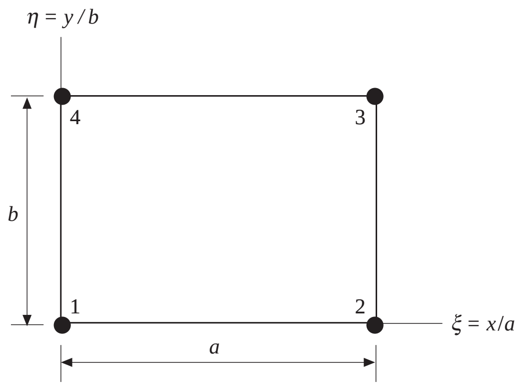


Figure 2. Plate element co-ordinates and dimensions.

additional freedom to the edges and the interior of a rectangular plate element and do not affect the element's nodal degrees of freedom.

2.2. THE PLATE EQUATIONS OF MOTION

The plate is discretized into one hierarchical finite element. The co-ordinate system used to define the geometry of the element is shown in Figure 2. The x and y co-ordinates and the non-dimensional ξ and η co-ordinates are related by

$$\xi = x/a, \qquad \eta = y/b. \tag{22, 23}$$

The transverse displacement w of the plate element is expressed as

$$w(\xi, \eta, t) = \sum_{m=1}^{M+6} \sum_{n=1}^{N+6} q_{mn}(t) f_m(\xi) f_n(\eta). \tag{24}$$

The expressions for the strain energy V and the kinetic energy T of the plate element are

$$V = \frac{D}{2ab} \int_0^1 \int_0^1 \left[\left(\frac{b^2}{a^2}\right)\left(\frac{\partial^2 w}{\partial \xi^2}\right)^2 + \left(\frac{a^2}{b^2}\right)\left(\frac{\partial^2 w}{\partial \eta^2}\right)^2 + 2v\left(\frac{\partial^2 w}{\partial \xi^2}\right)\left(\frac{\partial^2 w}{\partial \eta^2}\right) + 2(1-v)\left(\frac{\partial^2 w}{\partial \xi\, \partial \eta}\right)^2 \right] \mathrm{d}\xi\, \mathrm{d}\eta, \tag{25}$$

$$T = \frac{\rho ab}{2} \int_0^1 \int_0^1 \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}\xi\, \mathrm{d}\eta. \tag{26}$$

Assuming that the transverse motion is harmonic and inserting the expression for the displacement w (equation (24)) into the expressions for the kinetic and potential energy (equations (25) and (26)), and into the known Lagrange equations yields the following equations of motion for undamped free vibration:

$$\sum_{j=1}^{R} (K_{ij} - \omega^2 M_{ij}) q_j = 0, \qquad i = 1, 2, \ldots, R. \tag{27}$$

The element stiffness and mass matrices are expressed as

$$K_{ij} = \frac{D}{ab}\left[\left(\frac{b^2}{a^2}\right)I^{2,2}_{m,k}J^{0,0}_{n,l} + \left(\frac{a^2}{b^2}\right)I^{0,0}_{m,k}J^{2,2}_{n,l} + \nu(I^{2,0}_{m,k}J^{0,2}_{n,l} + I^{0,2}_{m,k}J^{2,0}_{n,l}) + 2(1-\nu)I^{1,1}_{m,k}J^{1,1}_{n,l}\right], \quad (28)$$

$$M_{ij} = \rho ab I^{0,0}_{m,k}J^{0,0}_{n,l}, \quad (29)$$

where the indices m, k, n, and l which represent the numbers of functions used in the assumed displacement field take on the following values

$$m, k = 1, 2, 3, \ldots, M+6, \qquad n, l = 1, 2, 3, \ldots, N+6, \quad (30, 31)$$

and the indices i and j are expressed in terms of the indices m, k, n, and l as

$$i = n + (m-1)(N+6), \qquad j = l + (k-1)(N+6). \quad (32, 33)$$

The order R of the element stiffness and mass matrices is

$$R = (M+6)(N+6). \quad (34)$$

The integrals are defined as

$$I^{\alpha,\beta}_{m,k} = \int_0^1 f^{\alpha}_m(\xi)f^{\beta}_k(\xi)\,\mathrm{d}\xi, \qquad J^{\alpha,\beta}_{n,l} = \int_0^1 f^{\alpha}_n(\eta)f^{\beta}_l(\eta)\,\mathrm{d}\eta, \quad (35, 36)$$

where the indices α and β ($\alpha, \beta = 0, 1, 2$) denote the order of the derivatives.

The values of the integrals in equations (35) and (36) can be easily obtained by using symbolic computing. They can also be obtained to any desired degree of accuracy by using Gaussian quadrature with an appropriate number of integration points for the function in the integrand of each integral. The resulting values of the integrals can then be stocked into a file which is later used by the program that implements the trigonometric hierarchical finite element. This process greatly speeds up the generation of the element stiffness and mass matrices.

Particular boundary conditions can be specified for w, $w_{,x}$, $w_{,y}$, $w_{,xx}$, $w_{,yy}$, $w_{,xy}$, $w_{,xxy}$, $w_{,xyy}$ and $w_{,xxyy}$ on the element's four corners, for w, $w_{,y}$, and $w_{,yy}$ on the element's two edges along the x axis, and for w, $w_{,x}$, and $w_{,xx}$ on the element's two edges along the y axis. Tables 3 and 4 give the most common boundary conditions which can be specified respectively on a corner and on an edge along the x axis (1 and 0 denote respectively a co-ordinate that is restrained and free and S, C, and F denote respectively an edge that is simply supported, clamped, and free). Similar boundary conditions can be specified on an edge along the y axis by interchanging the subscripts for x and y. Assigning the boundary conditions in this way makes it possible to accommodate any combination of corner and edge conditions in the analysis. For each specified boundary condition, the corresponding row and column must be deleted from the element's stiffness and mass matrices. For plates other than completely free ones, this deletion process reduces the order of the element's stiffness and mass matrices. The resulting generalized eigenvalue problem can then be solved using any known technique.

3. NUMERICAL RESULTS

Results of the application of the trigonometric hierarchical finite element to the calculation of the frequency parameter Ω were first obtained for an S–S–S–S square with $\nu = 0{\cdot}3$ and for an S–F–S–F square plate with $\nu = 0{\cdot}16$. Each plate is identified by Leissa's

TABLE 3

Boundary conditions for most common corner conditions

Corner	w	$w_{,x}$	$w_{,y}$	$w_{,xx}$	$w_{,yy}$	$w_{,xy}$	$w_{,xxy}$	$w_{,xyy}$	$w_{,xxyy}$
S └─S	1	1	1	1	1	0	0	0	0
C └─C	1	1	1	1	1	1	1	1	0
F └─F	0	0	0	0	0	0	0	0	0
C └─S	1	1	1	1	1	1	0	1	0
F └─S	1	1	0	1	1	0	0	0	0
C └─F	1	1	1	0	1	1	0	1	0

convention [6]. Thus, the symbolism S–S–S–S indicates that the four edges are simply supported and the symbolism S–F–S–F indicates that two opposite edges are simply supported and the other two edges are free.

In order to see the manner of convergence of the trigonometric hierarchical finite element solutions, each plate is discretized into one element and the number of hierarchical terms $M(=\mathrm{N})$ is varied. An equal number of hierarchical terms is used in both directions because the plate elements are squares. Results for the ten lowest modes of the S–S–S–S plate and the eleven lowest modes of the S–F–S–F plate are shown respectively in Table 5 and Table 6 along with exact solutions. The case $M = N = 0$ corresponds to using one 36-degree-of-freedom rectangular finite element.

Tables 5 and 6 clearly show that rapid convergence from above to the exact values occurs as the number of hierarchical terms is increased from 1 to 4 and highly accurate solutions are obtained despite the use of a very few hierarchical terms. In fact, the trigonometric hierarchical finite element solutions for $M = N = 4$ are in excellent agreement with the exact solutions.

The performance of the trigonometric hierarchical finite element with that of the polynomial hierarchical finite element and that of the 36-degree-of-freedom rectangular finite element of Bogner *et al*. [3] on a total degree-of-freedom basis is also investigated. Results for the ten lowest modes of the S–S–S–S square plate and for the eleven lowest modes of the S–F–S–F square plate are shown respectively in Tables 7 and 8 along with

TABLE 4

Boundary conditions for most common conditions on an edge along x axis

Edge	w	$w_{,y}$	$w_{,yy}$
—S—	1	0	1
—C—	1	1	0
—F—	0	0	0

TABLE 5

Convergence of the ten lowest frequency parameters Ω *of the* S–S–S–S *square plate as a function of the number of trigonometric hierarchical terms* $M(=N)$

$M(=N)$	1	2	3	4	5	6	7	8	9	10
0	19·742	49·491	49·491	79·167	139·599	139·599	164·372	164·372	235·818	235·818
1	19·739	49·491	49·491	79·167	99·890	99·890	129·430	129·430	179·461	179·461
2	19·739	49·348	49·348	78·957	99·890	99·890	129·337	129·337	172·185	172·185
3	19·739	49·348	49·348	78·957	98·696	98·696	128·305	128·305	172·185	172·185
4	19·739	49·348	49·348	78·957	98·696	98·696	128·305	128·305	167·783	167·783
Exact	19·739	49·348	49·348	78·957	98·696	98·696	128·305	128·305	167·783	167·783

TABLE 6

Convergence of the eleven lowest frequency parameters Ω *of the* S–F–S–F *square plate as a function of the number of trigonometric hierarchical terms* $M(=N)$

$M(=N)$	1	2	3	4	5	6	7	8	9	10	11
0	9·812	17·064	38·456	39·544	48·251	74·195	78·868	117·734	139·599	164·372	235·749
1	9·808	17·060	37·953	39·515	48·200	72·998	78·867	89·899	98·918	117·678	125·643
2	9·808	17·060	37·953	39·348	48·049	72·883	76·377	89·899	98·895	113·606	125·643
3	9·808	17·060	37·953	39·348	48·049	72·881	76·377	88·627	97·693	113·606	124·567
4	9·808	17·060	37·953	39·348	48·049	72·881	76·375	88·627	97·691	113·605	124·567
Exact	9·808	17·060	37·953	39·348	48·049	72·881	76·375	88·627	97·691	113·605	124·565

TABLE 7

Comparison of the ten lowest frequency parameters Ω for the S–S–S–S square plate; numbers in parenthesis denote the numbers of system degrees of freedom

Method		1	2	3	4	5	6	7	8	9	10
Trigonometric HFEM	(48)	19·739	49·348	49·348	78·957	98·696	98·696	128·305	128·305	167·783	167·783
Polynomial HFEM	(64)	19·739	49·348	49·348	78·957	98·716	98·716	128·322	128·322	167·987	167·987
FEM	(92)	19·739	49·348	49·348	78·957	98·745	98·745	128·344	128·344	168·085	168·085
Exact		19·739	49·348	49·348	78·957	98·696	98·696	128·305	128·305	167·783	167·783

TABLE 8

Comparison of the eleven lowest frequency parameters Ω for the S–F–S–F *square plate*; *numbers in parenthesis denote the numbers of system degrees of freedom*

Method		1	2	3	4	5	6	7	8	9	10	11
Trigonometric HFEM	(72)	9·808	17·060	37·953	39·348	48·049	72·881	76·375	88·627	97·691	113·605	124·567
Polynomial HFEM	(80)	9·808	17·060	37·953	39·348	48·049	72·881	76·375	88·649	97·712	113·605	124·583
FEM	(112)	9·808	17·060	37·953	39·348	48·050	72·882	76·378	88·679	97·740	113·611	124·608
Exact		9·808	17·060	37·953	39·348	48·049	72·881	76·375	88·627	97·691	113·605	124·565

exact solutions and solutions from the polynomial hierarchical finite element and the 36-degree-of-freedom rectangular finite element. It should be noted that the number of hierarchical terms used in the polynomial hierarchical finite element in this paper refers only to the number of shape functions which are formed from orthogonal Legendre polynomials. The number of hierarchical terms $M(=N)$ used in the trigonometric hierarchical finite element for the S–S–S–S plate is four and the corresponding number of system degrees of freedom is 48. The number of hierarchical terms used in the polynomial hierarchical finite element for the S–S–S–S plate is six and the corresponding number of system degrees of freedom is 64. The number of hierarchical terms $M(=N)$ used in the trigonometric hierarchical finite element for the S–F–S–F plate is four and the corresponding number of system degrees of freedom is 72. The number of hierarchical terms used in the polynomial hierarchical finite element for the S–F–S–F plate is six and the corresponding number of system degrees of freedom is 80. The number of 36-degree-of-freedom rectangular finite elements used in both square plates is nine.

Tables 7 and 8 clearly show that the trigonometric hierarchical finite element solutions are more accurate than the polynomial hierarchical element solutions and the solutions from the 36-degree-of-freedom finite element with fewer system degrees of freedom and fewer hierarchical terms. This is particularly true for the higher modes. For the S–S–S–S plate, Table 7 shows that the trigonometric hierarchical finite element solutions are in excellent agreement with the exact solutions despite the use of about 25% fewer system degrees of freedom than the polynomial hierarchical finite element solutions and about 48% fewer system degrees of freedom than the 36-degree-of-freedom finite element solutions. For the S–F–S–F plate, Table 8 shows that the trigonometric hierarchical finite element solutions are in excellent agreement with the exact solutions despite the use of about 10% fewer system degrees of freedom than the polynomial hierarchical finite element solutions and about 36% fewer system degrees of freedom than the 36-degree-of-freedom finite element solutions.

Additional applications are to F–F–F–F and C–C–C–C square plates with $\nu = 0{\cdot}3$. The symbolism F–F–F–F indicates that the four edges are free and the symbolism C–C–C–C indicates that the four edges are clamped. It appears that there exist no analytical solutions for these two examples and only the frequencies of the lowest four modes are available in the literature [2]. Nevertheless, it is of interest to obtain solutions for a few higher modes, both to examine the performance of the trigonometric hierarchical finite element for rather more complex modes and to provide new frequency values which may be of interest to other investigators. Results for the twelve lowest modes of the F–F–F–F square plate and the C–C–C–C square plate are shown respectively in Tables 9 and 10. Convergence can only be based on the values converged upon by the trigonometric hierarchical finite element method by using eight or more hierarchical terms in the F–F–F–F plate and ten or more hierarchical terms in the C–C–C–C plate. Blanks in Table 10 are in places where there were too few system degrees of freedom to be able to produce these modes. In the case of the F–F–F–F square plate, the generalized eigenvalue problem yields three zero frequency parameters (as expected) which correspond to linear combinations of rigid-body translation in the transverse direction and rigid-body rotations about the symmetry axes. Tables 9 and 10 clearly show that fast convergence from above to the converged values occurs as the number of trigonometric hierarchical terms in the F–F–F–F square plate is increased from 1 to 4 and the number of trigonometric hierarchical terms in the C–C–C–C square plate is increased from 1 to 8.

The performance of the trigonometric hierarchical finite element with that of the polynomial hierarchical finite element and that of the 36-degree-of-freedom finite element on a total degree-of-freedom basis has been also investigated for the F–F–F–F and

TABLE 9

Convergence of the twelve lowest frequency parameters Ω of the F–F–F–F *square plate as a function of the number of trigonometric hierarchical terms M(= N)*

$M(=N)$	1	2	3	4	5	6	7	8	9	10	11	12
0	13·469	19·726	24·541	35·288	35·288	63·020	63·020	66·169	71·208	80·239	111·385	111·385
1	13·469	19·596	24·270	34·808	34·808	62·947	62·947	63·692	71·208	80·239	109·288	109·288
2	13·468	19·596	24·270	34·801	34·801	61·097	61·097	63·692	69·267	77·178	105·523	105·523
3	13·468	19·596	24·270	34·801	34·801	61·095	61·095	63·687	69·267	77·178	105·463	105·463
4	13·468	19·596	24·270	34·801	34·801	61·093	61·093	63·687	69·265	77·172	105·463	105·463
Converged solution	13·468	19·596	24·270	34·801	34·801	61·093	61·093	63·686	69·265	77·172	105·461	105·461

TABLE 10

Convergence of the twelve lowest frequency parameters Ω of the C–C–C–C *square plate as a function of the number of trigonometric hierarchical terms M(= N)*

$M(=N)$	1	2	3	4	5	6	7	8	9	10	11	12
0	36·000	74·297	74·297	108·591	—	—	—	—	—	—	—	—
1	35·990	74·185	74·185	108·591	136·567	137·331	168·260	168·260	223·429	—	—	—
2	35·990	73·410	73·410	108·246	136·567	137·331	168·255	168·255	223·429	225·893	225·893	254·402
3	35·986	73·402	73·402	108·246	131·610	132·237	165·073	165·073	220·136	225·840	225·840	254·402
4	35·986	73·395	73·395	108·220	131·610	132·237	165·038	165·038	210·619	210·619	220·135	242·249
5	35·985	73·395	73·395	108·220	131·582	132·207	165·009	165·009	210·612	210·612	220·046	242·249
6	35·985	73·394	73·394	108·217	131·582	132·207	165·004	165·004	210·527	210·527	220·046	242·160
7	35·985	73·394	73·394	108·217	131·581	132·205	162·002	165·002	210·526	210·526	220·036	242·160
8	35·985	73·394	73·394	108·217	131·581	132·205	165·001	165·001	210·523	210·523	220·035	242·155
Converged solution	35·985	73·394	73·394	108·217	131·581	132·205	165·000	165·000	210·522	210·522	220·033	242·154

TABLE 11

Comparison of the twelve lowest frequency parameters Ω *for the* F–F–F–F *square plate; numbers in parenthesis denote the numbers of system degrees of freedom*

Method		1	2	3	4	5	6	7	8	9	10	11	12
Trigonometric HFEM	(100)	13·468	19·596	24·270	34·801	34·801	61·093	61·093	63·687	69·265	77·172	105·463	105·463
Polynomial HFEM	(100)	13·468	19·596	24·270	34·801	34·801	61·093	61·093	63·687	69·265	77·172	105·463	105·463
FEM	(144)	13·468	19·596	24·270	34·801	34·801	61·095	61·095	63·688	69·269	77·176	105·472	105·472
Converged trigonometric HFE solution		13·468	19·596	24·270	34·801	34·801	61·093	61·093	63·686	69·265	77·172	105·461	105·461

TABLE 12

Comparison of the twelve lowest frequency parameters Ω *for the* C–C–C–C *square plate; numbers in parenthesis denote the numbers of system degrees of freedom*

Method		1	2	3	4	5	6	7	8	9	10	11	12
Trigonometric HFEM	(100)	35·985	73·394	73·394	108·217	131·581	132·205	165·001	165·001	210·523	210·523	220·035	242·155
Polynomial HFEM	(100)	35·985	73·394	73·394	108·217	131·581	132·205	165·000	165·000	210·522	210·522	220·033	242·154
FEM	(121)	35·985	73·395	73·395	108·219	131·583	132·208	165·007	165·007	210·725	210·725	220·044	242·316
Converged trigonometric HFE solution		35·985	73·394	73·394	108·217	131·581	132·205	165·000	165·000	210·522	210·522	220·033	242·154

C–C–C–C square plates. Results for the twelve lowest modes of the F–F–F–F square plate and the C–C–C–C square plate are shown respectively in Table 11 and Table 12 along with the converged trigonometric hierarchical finite element solutions and solutions from the polynomial hierarchical finite element and the 36-degree-of-freedom finite element. The number of hierarchical terms $M(=N)$ used in the trigonometric hierarchical finite element for the F–F–F–F plate is four and the corresponding number of system degrees of freedom is 100. The number of hierarchical terms used in the polynomial hierarchical finite element for the F–F–F–F plate is six and the corresponding number of system degrees of freedom is 100. The number of hierarchical terms $M(=N)$ used in the trigonometric hierarchical finite element for the C–C–C–C plate is eight and the corresponding number of system degrees of freedom is 100. The number of hierarchical terms used in the polynomial hierarchical finite element for the C–C–C–C plate is 10 and the corresponding number of system degrees of freedom is 100. The number of 36-degree-of-freedom rectangular finite elements used in both square plates is nine.

Tables 11 and 12 clearly show that the trigonometric and the polynomial hierarchical finite elements lead to an equivalent accuracy with the same number of system degrees of freedom but the trigonometric hierarchical finite element requires fewer hierarchical terms. Tables 11 and 12 also show that the trigonometric hierarchical finite element solutions are more accurate than the solutions from the 36-degree-of-freedom finite element with fewer system degrees of freedom. All the results confirm that the rate of convergence of the trigonometric hierarchical finite element method in a particular mode is not influenced directly by the number of system degrees of freedom used but is rather influenced by the numbers of hierarchical terms used, the complexity of the mode, and the plate boundary conditions.

The final application is to a rectangular C–C–C–C plate of aspect ratio $b/a=4$ and $\nu=0{\cdot}3$. This example is intended to illustrate the feature of being able to use different numbers of hierarchical terms in different directions. Results for the six lowest frequency parameters Ω as a function of the numbers of hierarchical terms M and N in the x and y directions are shown in Table 13. It can be seen that the rate of convergence is greatly improved as more hierarchical terms are taken in the longer direction (y direction) rather than in the shorter direction (x direction).

4. CONCLUSIONS

A trigonometric hierarchical finite element for plate vibrations has been presented. The element is formulated in terms of a fixed number of quintic polynomial shape functions plus a variable number of trigonometric hierarchical shape functions. The numbers of trigonometric hierarchical terms are allowed to vary in both directions of the element co-ordinate axes.

The results obtained for S–S–S–S, S–F–S–F, F–F–F–F and C–C–C–C square plates have shown that the trigonometric hierarchical finite element solutions always converge from above as the numbers of trigonometric hierarchical terms increase and highly accurate values are obtained despite the use of a very few trigonometric hierarchical terms.

When compared with the 36-degree-of-freedom rectangular finite element, the trigonometric hierarchical finite element was found to yield a better accuracy with fewer system degrees of freedom.

When compared with the polynomial hierarchical finite element, the trigonometric hierarchical finite element was found to yield a better accuracy with fewer system degrees of freedom and fewer hierarchical terms for the S–S–S–S and S–F–S–F square

TABLE 13

Convergence of the six lowest frequency parameters Ω of the C–C–C–C *rectangular plate with aspect ratio $b/a = 4$ as a function of the numbers of trigonometric hierarchical terms M and N in x and y directions*

Ω	M	N: 1	2	3	4	5	6	7	8	9	10	11	12
1	1	22·802	22·802	22·799	22·799	22·799	22·799	22·799	22·799	22·799	22·799	22·799	22·799
	2	22·802	22·802	22·799	22·799	22·799	22·799	22·799	22·799	22·799	22·799	22·799	22·799
	3	22·802	22·802	22·799	22·799	22·799	22·799	22·799	22·799	22·799	22·799	22·799	22·799
2	1	24·155	24·149	24·149	24·144	24·144	24·143	24·143	24·143	24·143	14·143	24·143	24·143
	2	24·155	24·149	24·149	24·144	24·144	24·143	24·143	24·143	24·143	24·143	24·143	24·143
	3	24·155	24·149	24·149	24·144	24·144	24·143	24·143	24·143	24·143	24·143	24·143	24·143
3	1	26·602	26·602	26·576	26·576	26·571	26·571	26·570	26·570	26·570	26·570	26·570	26·570
	2	26·602	26·602	26·576	26·576	26·571	26·571	26·570	26·570	26·570	26·570	26·570	26·569
	3	26·602	26·602	26·576	26·576	26·571	26·571	26·570	26·570	26·570	26·569	26·569	26·569
4	1	63·425	30·543	30·543	30·258	30·258	30·252	30·252	30·251	30·251	30·250	30·250	30·250
	2	62·206	30·543	30·543	30·258	30·258	30·252	30·252	30·251	30·251	30·250	30·250	30·250
	3	62·206	30·543	30·543	30·258	30·258	30·251	30·251	30·250	30·250	30·249	30·249	30·249
5	1	64·942	63·425	36·315	36·315	35·309	35·309	35·299	35·299	35·297	35·297	35·297	35·297
	2	63·817	62·206	36·315	36·315	35·309	35·309	35·299	35·299	35·297	35·297	35·297	35·297
	3	63·817	62·206	36·314	36·314	35·309	35·309	35·298	35·298	35·296	35·296	35·296	35·296
6	1	67·409	64·904	63·415	44·117	44·117	41·774	41·774	41·756	41·756	41·754	41·754	41·753
	2	66·416	63·776	62·195	44·117	44·117	41·774	41·774	41·756	41·756	41·754	41·754	41·753
	3	66·416	63·776	62·195	44·116	44·116	41·772	41·772	41·754	41·754	41·752	41·752	41·751

plates and an equivalent accuracy with the same number of system degrees of freedom but with fewer hierarchical terms for the F–F–F–F and C–C–C–C square plates.

Finally, the results obtained for a C–C–C–C rectangular plate of aspect ratio $b/a = 4$ have shown that the rate of convergence is greatly improved as more trigonometric hierarchical terms are taken in the longer direction rather than in the shorter one.

REFERENCES

1. N. S. Bardell 1992 *Journal of Sound and Vibration* **151**, 263–289. Free vibration analysis of a flat plate using the hierarchical finite element method.
2. N. S. Bardell 1992 *Computers & Structures* **45**, 841–874. The free vibration of skew plates using the hierarchical finite element method.
3. F. K. Bogner, R. L. Fox and L. A. Schmit 1965 *Proceedings of the First Conference on Matrix Methods in Structural Mechanics*, *Ohio*, 397–443. The generation of inter-element compatible stiffness and mass matrices by the use of interpolation formulas.
4. L. Meirovitch and H. Baruh 1983 *International Journal for Numerical Methods in Engineering* **19**, 281–291. On the inclusion principle for the hierarchical finite element method.
5. D. C. Zhu 1986 *Proceedings of the International Conference on Computational Mechanics*, *Tokyo* **I**, 123–128. Development of hierarchical finite element methods at BIAA.
6. A. W. Leissa 1969 *Vibration of plates*, (NASA SP-160). Washington D.C.: U.S. Government Printing Office.

APPENDIX: NOTATION

a	plate element length in the x direction
b	plate element length in the y direction
h	plate thickness
E	modulus of elasticity
v	Poisson's ratio
ρ	mass per unit area
D	flexural rigidity ($= Eh^3/(12(1 - v^2))$)
x, y	plate element co-ordinates
ξ, η	plate element non-dimensional co-ordinates
t	time
w	plate transverse displacement
V	plate element strain energy
T	plate element kinetic energy
K_{ij}	element stiffness matrix
M_{ij}	element mass matrix
q_j	generalized co-ordinate
M	number of hierarchical terms in the element x direction
N	number of hierarchical terms in the element y direction
R	order of the element stiffness and mass matrices
ω	natural frequency
Ω	frequency parameter ($= \omega a^2\sqrt{\rho/D}$)