Journal of Sound and Vibration (1997) **206**(2), 275–279

THE VIBRATIONS OF A CANTILEVER BEAM CARRYING A HEAVY TIP MASS WITH ELASTIC SUPPORTS

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(*Received* 24 *June* 1995, *and in final form* 22 *April* 1997)

1. INTRODUCTION

As an application to mast antenna structures, To [1] investigated the vibrations of a cantilever beam with a tip mass whose center of gravity does not coincide with the point of attachment. Maurizi *et al*. [2] derived the exact and analytical eigenfrequency equations for a cantilever beam with a tip mass, having its free and elastically restrained against rotation or translation. Gürgöze and Batan [3] studied the free vibration of an elastically restrained cantilever beam with a tip mass and Liu and Huang [4] also considered such a case but the effects of tip mass center were included. Lau [5] presented the eigenfrequencies of a cantilever tapered beam with an end mass. Laura and Gutierrez [6] investigated an elastically restrained cantilever beam of varying cross-section with a tip mass by using the Rayleigh–Schmit approach. Furthermore, Alvarez *et al*. [7] studied such a beam but added an intermediate elastic support. The purpose of the present note is to derive the exact and analytical expressions for eigenfrequencies and mode shapes of a cantilever beam carrying a heavy tip mass with translational and rotational elastic supports. Some numerical results are given.

2. MATHEMATICAL MODEL

A uniform cantilever beam carrying a heavy tip mass with elastic supports is shown in Figure 1 and the governing equation of motion for small deflection is

$$EI\,\partial^4 y/\partial x^4 + \rho A\,\partial^2 y/\partial t^2 = 0 \tag{1}$$

where E is the Young's modulus, I is the constant second moment of cross-section A and ρ is the density of the uniform beam.

It is assumed that the axial deflection can be neglected, and that the boundary conditions for the beam at the clamped end with a base excitation are

$$\text{at } x = 0, \qquad y = b(t), \qquad \partial y/\partial x = 0 \tag{2, 3}$$

at $x = l$,

$$EI\frac{\partial^2 y}{\partial x^2} = -J\frac{\partial^3 y}{\partial x\,\partial t^2} - K_2\frac{\partial y}{\partial x} - Md_0^2\frac{\partial^3 y}{\partial x\,\partial t^2} - (d_0+d_1)^2K_1\frac{\partial y}{\partial x} - Md_0\frac{\partial^2 y}{\partial t^2} - K_1(d_0+d_1)y,$$

$$EI\frac{\partial^3 y}{\partial x^3} = (d_0+d_1)K_1\frac{\partial y}{\partial x} + Md_0\frac{\partial^3 y}{\partial x\,\partial t^2} + K_1 y + M\frac{\partial^2 y}{\partial t^2}, \tag{4, 5}$$

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where $J = Mr^2$ is the moment of inertia of the attached mass M, d_0 is the distance between the center of gravity of M and the joint of the beam with mass, K_1 and K_2 are the translational and rotational stiffnesses of the elastic supports added to the mass, respectively. d_1 is the distance between the center of gravity of M and the translational elastic support.

If the base motion is harmonic, one has

$$b(t) = B\,\mathrm{e}^{\mathrm{j}\omega t}, \qquad y(x, t) = Y(x, \omega)\,\mathrm{e}^{\mathrm{j}\omega t}, \tag{6, 7}$$

where B and $Y(x, \omega)$ are the amplitudes of the base motion and the beam deflection respectively, ω is the exciting frequency and $\mathrm{j} = \sqrt{-1}$.

Substituting equations (6) and (7) into equations (1–5), one has

$$\mathrm{d}^4 Y/\mathrm{d}x^4 - \lambda^4 Y = 0, \tag{8}$$

$$\text{at } x = 0, \qquad Y = B, \qquad \mathrm{d}Y/\mathrm{d}x = 0, \tag{9, 10}$$

$$\text{at } x = l,$$

$$EI\,\mathrm{d}^2 Y/\mathrm{d}x^2 = [\omega^2 J - K_2 + Md_0^2\omega^2 - (d_0 + d_1)^2 K_1]\,\mathrm{d}Y/\mathrm{d}x + [Md_0\omega^2 - K_1(d_0 + d_1)]Y,$$

$$EI\,\mathrm{d}^3 Y/\mathrm{d}x^3 = [(d_0 + d_1)K_1 - Md_0\omega^2]\,\mathrm{d}Y/\mathrm{d}x + (K_1 - M\omega^2)Y, \tag{11, 12}$$

where

$$\lambda^4 = \rho A\omega^2/EI.$$

3. SOLUTION OF EQUATION

The general solution of equation (8) is

$$Y(x, \omega) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x) + C_3 \cosh(\lambda x) + C_4 \sinh(\lambda x) \tag{13}$$

where C_i ($i = 1, 2, 3, 4$) are the unknown constants.

Substituting equation (13) into equations (9–12) gives

$$Y(x, \omega) = \frac{B}{D(u)}\frac{1}{2}\{F_{1u}\cos(\lambda x) + F_{2u}\cosh(\lambda x) + F_{3u}[\sin(\lambda x) - \sinh(\lambda x)]\}, \tag{14}$$

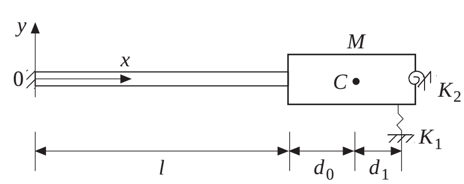

Figure 1. Sketch of a beam carrying a heavy tip mass with elastic supports.

TABLE 1

The first four eigenfrequency parameters u_i ($i = 1, 2, 3, 4$) of a cantilever beam carrying a heavy tip mass with elastic supports, the values in ( ) are from reference [1] *and the values in* [ ] *are from reference* [4]

ϕ	α_0	α_1	α_2	β_1	β_2	u_1	u_2	u_3	u_4
0·5	0·2	−0·2	0·5	1	1	1·4627	2·6146	5·1535	8·1226
0·5	0·2	−0·2	0·5	1	10	1·8177	3·1417	5·1690	8·1237
0·5	0·2	−0·2	0·5	10	0	1·6897	2·6166	5·1582	8·1233
0·5	0·2	−0·2	0·5	10	1	1·8086	2·6691	5·1598	8·1235
0·5	0·2	−0·2	0·5	10	10	2·1155	3·1496	5·1753	8·1246
0·5	0·2	−0·2	0·5	100	10	2·9631	3·4117	5·2473	8·1338
0·5	0·2	0	0·5	0	1	1·3982	2·6094	5·1528	8·1225
0·5	0·2	0	0·5	0	10	1·7731	3·1410	5·1683	8·1237
0·5	0·2	0	0·5	1	1	1·5007	2·6097	5·1531	8·1225
0·5	0·2	0	0·5	1	10	1·8270	3·1412	5·1686	8·1237
0·5	0·2	0	0·5	10	0	1·9351	2·5527	5·1546	8·1229
0·5	0·2	0	0·5	10	1	2·0058	2·6147	5·1562	8·1231
0·5	0·2	0	0·5	10	10	2·1749	3·1440	5·1715	8·1242
0·5	0·2	0	0·5	100	10	3·0893	3·4459	5·2056	8·1297
0·5	0·2	0·2	0·5	1	1	1·5436	2·6105	5·1529	8·1225
0·5	0·2	0·2	0·5	1	10	1·8368	3·1446	5·1684	8·1237
0·5	0·2	0·2	0·5	10	0	2·1363	2·5496	5·1523	8·1227
0·5	0·2	0·2	0·5	10	1	2·1615	2·6281	5·1539	8·1228
0.5	0·2	0·2	0·5	10	10	2·2148	3·1831	5·1691	8·1239
0·5	0·2	0·2	0·5	100	10	2·8779	3·9016	5·1795	8·1266
0·5	0·2	0·2	1	1	1	1·3637	2·1724	5·0411	8·0688
0·5	0·2	0·2	1	1	10	1·7818	2·3904	5·0422	8·0688
0·5	0·2	0·2	1	10	0	1·7661	2·2655	5·0447	8·0693
0·5	0·2	0·2	1	10	1	1·8394	2·2693	5·0448	8·0693
0·5	0·2	0·2	1	10	10	2·1724	2·3915	5·0459	8·0694
0·5	0·2	0·2	1	100	10	2·3893	3·4379	5·0911	8·0749
1	0	0	1	0	1	0·9316 (0·9316)	1·8414 (1·8414)	4·9009 (4·9009)	7·9664 (7·9664)
1	0	0	1	1	1	1·1429 [1·1429]	1·8713 [1·8713]	4·9011	7·9664
1	0·3	−0·3	1	1	0	0·9268	1·9654	4·9312	7·9831
1	0·3	−0·3	1	10	0	1·1460	2·1331	4·9337	7·9834
1	0·3	−0·3	1	100	0	1·3423	3·0367	4·9618	7·9861
1	0·3	−0·3	1	10	10	1·6674	2·2167	4·9343	7·9834
1	0·3	−0·3	1	100	10	1·8600	3·0428	4·9626	7·9861
1	0·3	−0·3	1	100	100	2·8696	3·2672	4·9701	7·9865
1	0·3	0	1	0	1	1·0291	1·9590	4·9309	7·9831
1	0·3	0	1	0	10	1·4641	2·1085	4·9315	7·9831
1	0·3	0	1	1	0	0·9924	1·9572	4·9311	7·9831
1	0·3	0	1	10	0	1·3815	2·0778	4·9329	7·9833
1	0·3	0	1	100	0	1·6512	2·9896	4·9536	7·9854
1	0·3	0	1	10	10	1·7843	2·1558	4·9335	7·9833
1	0·3	0	1	100	10	2·0269	2·9899	4·9542	7·9854
1	0·3	0	1	100	100	2·9814	3·1825	4·9604	7·9859
1	0·3	0·3	1	1	0	1·0692	1·9511	4·9310	7·9831
1	0·3	0·3	1	10	0	1·6098	2·0340	4·9323	7·9832
1	0·3	0·3	1	100	0	1·8460	3·0919	4·9471	7·9848
1	0·3	0·3	1	10	10	1·9105	2·1139	4·9329	7·9833
1	0·3	0·3	1	100	10	2·1037	3·1050	4·9476	7·9849
1	0·3	0·3	1	100	100	2·8372	3·3960	4·9529	7·9853
1	0·4	0	1	0	0	0·8507 (0·8507)	1·9801 (1·9801)	4·9451 (4·9451)	7·9922 (7·9922)
1	0·6	0	1	0	0	0·8105 (0·8105)	2·0454 (2·0454)	4·9782 (4·9782)	8·0150 (8·0150)
2	0·5	0·5	1	1	0	0·9462	1·7718	4·8623	7·9345
2	0·5	0·5	1	10	0	1·5322	1·7984	4·8625	7·9346
2	0·5	0·5	1	100	0	1·7453	2·7797	4·8651	7·9348
2	0·5	0·5	1	10	10	1·6796	1·8877	4·8629	7·9346
2	0·5	0·5	1	100	10	1·8877	2·7944	4·8655	7·9349
2	0·5	0·5	1	100	100	2·4033	3·0137	4·8687	7·9351

where

$$F_{1u} = 1 + \mathrm{cch} - \mathrm{ssh} - 2\Big(\phi u - \beta_1/u^3\Big)\mathrm{sch} - 2(\mathrm{ssh} + \mathrm{cch})\left[\phi u^2\alpha_0 - \frac{\beta_1}{u^2}(\alpha_0 + \alpha_1)\right]$$
$$- 2\left[\phi u^3(\alpha_0^2 + \alpha_2^2) - \frac{\beta_2 + (\alpha_0 + \alpha_1)^2\beta_1}{u}\right]\mathrm{csh} + \left\{\left[\phi u^3(\alpha_0^2 + \alpha_2^2) - \frac{\beta_2 + (\alpha_0 + \alpha_1)^2\beta_1}{u}\right]\right.$$
$$\left.\times\left(\phi u - \frac{\beta_1}{u^3}\right) - \left[\phi u^2\alpha_0 - \frac{\beta_1}{u^2}(\alpha_0 + \alpha_1)\right]^2\right\}(1 - \mathrm{cch} + \mathrm{ssh}),$$

$$F_{2u} = 1 + \mathrm{cch} + \mathrm{ssh} + 2\left(\phi u - \frac{\beta_1}{u^3}\right)\mathrm{csh} + 2(\mathrm{cch} - \mathrm{ssh})\left[\phi u^2\alpha_0 - \frac{\beta_1}{u^2}(\alpha_0 + \alpha_1)\right]$$
$$- 2\left[\phi u^3(\alpha_0^2 + \alpha_2^2) - \frac{\beta_2 + (\alpha_0 + \alpha_1)^2\beta_1}{u}\right]\mathrm{sch} + \left\{\left[\phi u^3(\alpha_0^2 + \alpha_2^2) - \frac{\beta_2 + (\alpha_0 + \alpha_1)^2\beta_1}{u}\right]\right.$$
$$\left.\times\left(\phi u - \frac{\beta_1}{u^3}\right) - \left[\phi u^2\alpha_0 - \frac{\beta_1}{u^2}(\alpha_0 + \alpha_1)\right]^2\right\}(1 - \mathrm{cch} - \mathrm{ssh}),$$

$$F_{3u} = (\mathrm{sch} + \mathrm{csh})\left\{1 - \left[\phi u^3(\alpha_0^2 + \alpha_2^2) - \frac{\beta_2 + (\alpha_0 + \alpha_1)^2\beta_1}{u}\right]\left(\phi u - \frac{\beta_1}{u^3}\right)\right.$$
$$\left. + \left[\phi u^2\alpha_0 - \frac{\beta_1}{u^2}(\alpha_0 + \alpha_1)\right]^2\right\} + 2\left\{\left(\phi u - \frac{\beta_1}{u^3}\right)\mathrm{cch} - \left[\phi u^3(\alpha_0^2 + \alpha_2^2)\right.\right.$$
$$\left.\left. - \frac{\beta_2 + (\alpha_0 + \alpha_1)^2\beta_1}{u}\right]\mathrm{ssh} - \left[\phi u^2\alpha_0 - \frac{\beta_1}{u^2}(\alpha_0 + \alpha_1)\right](\mathrm{sch} - \mathrm{csh})\right\},$$

$$D(u) = 1 + \mathrm{cch} - \left(\phi u - \frac{\beta_1}{u^3}\right)(\mathrm{sch} - \mathrm{csh}) - \left[\phi u^3(\alpha_0^2 + \alpha_2^2) - \frac{\beta_2 + (\alpha_0 + \alpha_1)^2\beta_1}{u}\right]$$
$$\times\left[\mathrm{sch} + \mathrm{csh} - \left(\phi u - \frac{\beta_1}{u^3}\right)(1 - \mathrm{cch})\right] - 2\left[\phi u^2\alpha_0 - \frac{\beta_1}{u^2}(\alpha_0 + \alpha_1)\right]\mathrm{ssh}$$
$$- \left[\phi u^2\alpha_0 - \frac{\beta_1}{u_2}(\alpha_0 + \alpha_1)\right]^2(1 - \mathrm{cch}), \qquad (15\text{–}18)$$

in which,

$$\phi = M/\rho A l, \quad \alpha_0 = d_0/l, \quad \alpha_1 = d_1/l, \quad \alpha_2 = r/l, \quad \beta_1 = K_1 l^3/EI,$$
$$\beta_2 = K_2 l/EI, \quad u = \lambda l, \quad \mathrm{s} = \sin u, \quad \mathrm{c} = \cos u, \quad \mathrm{sh} = \sinh u, \quad \mathrm{ch} = \cosh u \qquad (19)$$

In the above analysis, some investigations obtained are the special cases. For example, when the translational elastic support does not exist or the rotational elastic support does not exist but the translational elastic support is at the end of the cantilever beam, the problem degenerates into that as reported in reference [2]. Furthermore, when both the translational and rotational elastic supports do not exist, the problem degenerates into the one as described in reference [1]. It is evident that the term $-2(\mathrm{ssh}+\mathrm{cch})\phi u^2\alpha_0$ in F_{1u} and the term $2(\mathrm{cch}-\mathrm{ssh})\phi u^2\alpha_0$ in F_{2u} are missing in reference [1].

4. FREQUENCY EQUATION AND MODE SHAPES

When the exciting frequency of the base motion is equal to the eigenfrequencies of the beam, resonance occurs and the amplitude of the beam is infinite. So the eigenfrequencies of the structure can be calculated from relations

$$\omega = (u/l)^2\sqrt{EI/\rho A}, \qquad D(u) = 0 \tag{20, 21}$$

by the method of non-linear algebraic equation search roots, the eigenfrequencies ω_i ($i = 1, 2, 3, \ldots$) which are the ith roots of the frequency equations (20) and (21) can be calculated by means of a computer. By substituting the variable u_i into equation (14) the mode shapes may be expressed as

$$Y_1(x, \omega) = \tfrac{1}{2}\{F_{1i}\cos(u_i x/l) + F_{2i}\cosh(u_i x/l) + F_{3i}[\sin(u_i x/l) - \sinh(u_i x/l)]\},$$
$$i = 1, 2, 3, \ldots, \tag{22}$$

where F_{1i}, F_{2i} and F_{3i} are, respectively, the functions F_{1u}, F_{2u} and F_{3u} with the variable u replaced by u_i.

5. NUMERICAL RESULTS

The first four eigenfrequency parameters u_i ($i = 1, 2, 3, 4$) are tabulated in Table 1 and compared with values from references [1] and [4]. Good agreement is observed.

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