Journal of Sound and Vibration (1997) **206**(3), 287–308

ACOUSTIC EIGENFREQUENCIES IN CONCENTRIC SPHEROIDAL–SPHERICAL CAVITIES

G. C. KOKKORAKIS AND J. A. ROUMELIOTIS

Department of Electrical and Computer Engineering, National Technical University of Athens, Athens 15773, *Greece*

(*Received* 8 *May* 1996, *and in final form* 27 *January* 1997)

The acoustic eigenfrequencies f_{nsm} in concentric spheroidal–spherical cavities are determined for both Dirichlet and Neumann boundary conditions. Two types of cavities are examined, one with spheroidal outer and spherical inner boundary and inversely for the other. The pressure field is expressed in terms of both spherical and spheroidal wave functions, connected with one another by well-known expansion formulas. When the solution is specialized to small values of $h = d/(2R_2)$ where d is the interfocal distance of the spheroidal boundary and R_2 the half length of its rotation axis, exact closed-form expressions are obtained for the coefficients $g^{(2)}_{nsm}$ and $g^{(4)}_{nsm}$ in the resulting relations $f_{nsm}(h) = f_{ns}(0)\,[1 + g^{(2)}_{nsm}h^2 + g^{(4)}_{nsm}h^4 + \mathcal{O}(h^6)]$. Numerical results are given for various values of the parameters.

1. INTRODUCTION

Calculation of eigenfunctions and eigenfrequencies in acoustic cavities is an old problem with numerous applications. The pure analytical solution of such problems is severely limited by the shape of the boundaries. For complicated geometries various numerical techniques are used, as well as shape perturbation methods like the ones described in reference [1] in particular. A special analytical shape perturbation method was used in references [2–5], in order to obtain the acoustic resonance frequencies and the corresponding wave functions in a spherical, a cylindrical and a rectangular cavity containing an eccentric inner small sphere.

In this paper the interior problem in the acoustic cavities, shown in Figures 1 and 2, is solved for both Dirichlet and Neumann boundary conditions. In Figure 1 the inner boundary is spherical with radius R_1, while the outer concentric one is prolate spheroidal with major semi-axis R_2 and interfocal distance d. In Figure 2 the inner boundary is prolate spheroidal with major semi-axis R_2 and interfocal distance d, while the outer concentric one is spherical with radius R_1. Both cavities are perturbations of the concentric spherical one with radii R_1 and R_2. The prolate spheroidal boundaries are the only ones to be considered explicitly, but corresponding formulas for the oblate ones are obtained immediately.

Using well-known expansion formulas between spherical and spheroidal wave functions [6], one is able to obtain an infinite determinantal equation for the evaluation of the eigenfrequencies of the former cavities. In the special case of small $h = d/(2R_2)$, one is led to an exact evaluation, up to the order h^4, for the elements of the infinite determinant and, finally, for the determinant itself. It is then possible to obtain the eigenfrequencies in the form $f_{nsm}(h) = f_{ns}(0)\,[1 + g^{(2)}_{nsm}h^2 + g^{(4)}_{nsm}h^4 + \mathcal{O}(h^6)]$, where the coefficients $g^{(2)}_{nsm}$ and $g^{(4)}_{nsm}$ are independent of h and are given by exact closed-form expressions.

0022–460X/97/380287 + 22 $25.00/0/sv971122

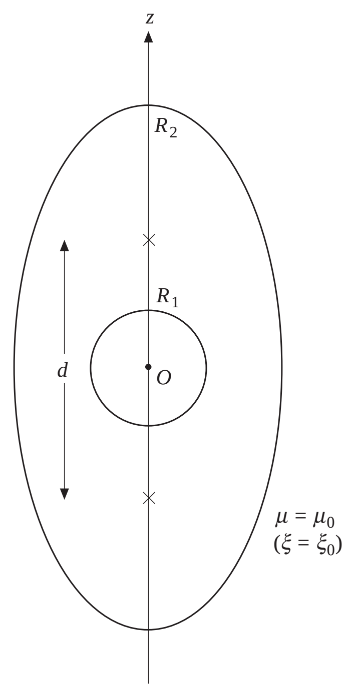

Figure 1. Geometry of the spheroidal–spherical cavity.

The main advantage of such an analytical solution lies in its general validity for all small values of h and for all modes, whereas all numerical techniques require repetition of the evaluation for each different h.

The case of the Dirichlet boundary conditions is examined in section 2, while in section 3 the case of the Neumann boundary conditions is considered. Finally, in section 4, numerical results are given accompanied with discussion and comments.

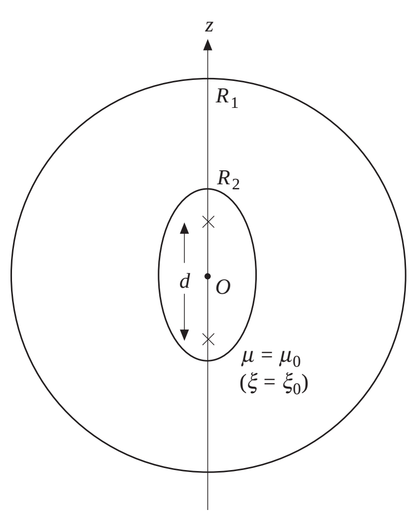

Figure 2. Geometry of the spherical–spheroidal cavity.

2. DIRICHLET BOUNDARY CONDITIONS

The cavities of Figures 1 and 2 are treated simultaneously. Let $\mu = \mu_0$ denote the spheroidal boundary and p the acoustic pressure field inside the cavity. This field satisfies the scalar Helmholtz equation. Its expression satisfying also the homogeneous Dirichlet boundary condition $p = 0$ at the spherical boundary $r = R_1$ is

$$p = \sum_{n=0}^{\infty} \sum_{m=0}^{n} [\mathrm{j}_n (kr) - \mathrm{n}_n (kr)\mathrm{j}_n (x_1)/\mathrm{n}_n (x_1)]\mathrm{P}_n^m(\cos\theta) [A_{nm} \cos m\varphi + B_{nm} \sin m\varphi],$$

$$x_1 = kR_1, \tag{1}$$

where r, θ, φ are the spherical co-ordinates with respect to 0, j_n and n_n are the spherical Bessel functions of the first and second kind, respectively, P_n^m is the associated Legendre function of the first kind and k is the resonant wavenumber.

In order to satisfy the remaining boundary condition $p = 0$ at $\mu = \mu_0$ one expands the spherical wave functions into concentric spheroidal ones by using the formula [6]

$$\mathrm{z}_n^{(\sigma)} (kr)\mathrm{P}_n^m (\cos\theta) = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \sum_{l=m,m+1}^{\infty}{}' \frac{\mathrm{i}^{l-n}}{N_{ml}} d_{n-m}^{ml} \, \mathrm{S}_{ml} (c, \eta)\mathrm{R}_{ml}^{(\sigma)} (c, \xi), \quad c = kd/2, \tag{2}$$

in which $\xi = \cosh\mu$, η are the spheroidal co-ordinates (φ is common in both systems), $\mathrm{z}_n^{(\sigma)}$ ($\sigma = 1$–4) is the spherical Bessel function of any kind, $\mathrm{R}_{ml}^{(\sigma)}$ is the corresponding radial spheroidal function of the same kind, S_{ml} and d_{n-m}^{ml} are the angular spheroidal function of the first kind and its expansion coefficients all defined in Appendix A, while the normalization constant N_{mn} is [6]

$$N_{mn} = 2 \sum_{r=0,1}^{\infty}{}' \frac{(\mathrm{d}_r^{mn})^2 (r+2m)!}{(2r+2m+1)r!}. \tag{3}$$

The prime over the summation symbols in equations (2) and (3) indicates that when $n - m$ is even/odd these summations start with the first/second value of their summation index and continue only with values of the same parity with it.

One substitutes from equation (2) into equation (1) satisfying the boundary condition $p = 0$ at $\mu = \mu_0$ ($\xi = \xi_0$) and uses next the orthogonal properties of the angular spheroidal and the trigonometric functions, to obtain finally the following infinite set of linear homogeneous equations for the expansion coefficients A_{nm} (or B_{nm}):

$$\sum_{n=m,m+1}^{\infty}{}' \alpha_{lnm} A_{nm} = 0, \qquad l \geqslant m, m+1, \tag{4}$$

where

$$\alpha_{lnm} = \frac{2\mathrm{i}^{-n}(n+m)!}{(2n+1)(n-m)!} d_{n-m}^{ml} \left[\mathrm{R}_{ml}^{(1)} (c, \cosh\mu_0) - \mathrm{R}_{ml}^{(2)} (c, \cosh\mu_0) \frac{\mathrm{j}_n (x_1)}{\mathrm{n}_n (x_1)} \right]. \tag{5}$$

In equations (4, 5) l and n are both even or odd, starting with that value of m or $m + 1$, which has the same parity with them. So, the set (4) separates into two distinct subsets, one with l, n even and the other with l, n odd.

Setting $\xi = \cosh \mu$ and $r = l - m \pm 2q$ in the expression (A25) of Appendis A for $\mathrm{R}_{ml}^{(\sigma)}$ and substituting there $d_r^{ml}(c)$ from equation (A3) of the same Appendix, one obtains

$$\mathrm{R}_{ml}^{(\sigma)}(c, \cosh \mu_0) = \frac{(l-m)!}{(l+m)!} \tanh^m \mu_0 \, d_{l-m}^{ml} \left\{ \sum_{q=1}^{\infty} \sum_{s=0}^{\infty} (-1)^q a_{2q,2s}^{ml+} c^{2q+2s} \frac{(l+m+2q)!}{(l-m+2q)!} \right.$$

$$\left. \times \mathrm{z}_{l+2q}^{(\sigma)}(x_2) + \sum_{q=0}^{q_{max}} \sum_{s=0}^{\infty} (-1)^q a_{2q,2s}^{ml-} c^{2q+2s} \frac{(l+m-2q)!}{(l-m-2q)!} \mathrm{z}_{l-2q}^{(\sigma)}(x_2) \right\},$$

$$x_2 = c \cosh \mu_0 = kR_2, \tag{6}$$

where q_{max} is the maximum integer $\leqslant (l-m)/2$.

The summation index s is now replaced by $u = q + s$ in equation (6). By keeping in mind that

$$\sum_{q=1}^{\infty} \sum_{u=q}^{\infty} = \sum_{u=1}^{\infty} \sum_{q=1}^{u} \qquad \text{and} \qquad \sum_{q=0}^{q_{max}} \sum_{u=q}^{\infty} = \sum_{u=0}^{\infty} \sum_{q=0}^{\min(q_{max},u)},$$

one finally finds

$$\mathrm{R}_{ml}^{(\sigma)}(c, \cosh \mu_0) = \frac{(l-m)!}{(l+m)!} \tanh^m \mu_0 \, d_{l-m}^{ml} \left\{ \frac{(l+m)!}{(l-m)!} \mathrm{z}_l^{(\sigma)}(x_2) + \sum_{u=1}^{\infty} c^{2u} \left[\sum_{q=1}^{u} (-1)^q \right. \right.$$

$$\times \frac{(l+m+2q)!}{(l-m+2q)!} \mathrm{z}_{l+2q}^{(\sigma)}(x_2) v_{2u-2q}^{ml+}(2q) a_{2q,0}^{ml+}$$

$$\left. \left. + \sum_{q=1}^{\min(q_{max},u)} (-1)^q \frac{(l+m-2q)!}{(l-m-2q)!} \mathrm{z}_{l-2q}^{(\sigma)}(x_2) v_{2u-2q}^{ml-}(2q) a_{2q,0}^{ml-} \right] \right\}. \tag{7}$$

In equation (7) use has been made of the relations $a_{2q,2u-2q}^{ml\pm} = v_{2u-2q}^{ml\pm}(2q) a_{2q,0}^{ml\pm}$, $v_0^{ml-}(0) = 1$, $a_{0,0}^{ml-} = 1$ and $v_{2u}^{ml-}(0) = 0$ for $u > 0$, from Appendix A.

Setting each one of the two determinants $\Delta(\alpha_{lmn})$ (one with l, n even and the other with l, n odd) of the coefficients α_{lnm} in equation (4) equal to 0, one obtains two determinantal equations for the evaluation of the resonance frequencies. As far as they appear in the same general form, one can treat them simultaneously with the symbol $\Delta(\alpha_{lnm})$. For large values of c, the equation $\Delta(\alpha_{lnm}) = 0$ can be solved by numerical methods only, a procedure with many difficulties, due to the presence of the spheroidal functions. However, for small c an analytical and closed-form solution is possible. One substitutes first from equation (7) into equation (5) and next divides the elements of the lth row of the former determinant by $2(d_{l-m}^{ml})^2 \tanh^m \mu_0$ and the elements of its nth column by $\mathrm{i}^{-n}(n+m)!/[(2n+1)(n-m)!]$. So α_{lnm} is divided by the product of these terms, with no change in the roots of the determinantal equation. The symbol α_{ln} is used for the resulting coefficient, deleting the third subscript m for simplicity. As far as c depends on the unknown resonance wavenumbers k ($c = kd/2$), from now on one can use the parameter $h = d/(2R_2)$, instead of it ($c = x_2 h$). So, for small h, one can set up, to the order h^4,

$$\alpha_{nn} = D_{nn}^{(0)} + D_{nn}^{(2)} h^2 + D_{nn}^{(4)} h^4 + \mathcal{O}(h^6), \qquad h = d/(2R_2), \tag{8}$$

$$\alpha_{ln} = D_{ln}^{(|l-n|)} h^{|l-n|}[1 + \mathcal{O}(h^2)], \qquad l \neq n. \tag{9}$$

In particular, for $h = 0$, it is obvious from equations (8, 9) that $\alpha_{ln}(0) = 0$ for $l \neq n$ and $\alpha_{nn}(0) = D_{nn}^{(0)} \equiv D_{nn}^{0}$. The determinant becomes diagonal and the resonance frequencies are found from the equations $\alpha_{nn}(0) = D_{nn}^{0} = 0$ $(n = 0, 1, 2, \ldots)$, a result independent of m and well known for two concentric spheres with radii R_1 and R_2.

The relations (8, 9) allow a closed-form evaluation of the determinant $\Delta(\alpha_{ln}) = \Delta(\alpha_{lnm})$, up to the order h^4, by the method described in detail in references [7, 8]. So, its development is

$$\Delta(\alpha_{ln}) = P(\alpha_{nn})\left[1 - \sum_{w=m,m+1}^{\infty} \frac{\alpha_{w+2,w}\,\alpha_{w,w+2}}{\alpha_{ww}\alpha_{w+2,w+2}}\right], \qquad n \geqslant m, m+1, \tag{10}$$

$$P(\alpha_{nn}) = \alpha_{ww}\,\alpha_{w+2,w+2}\,\alpha_{w+4,w+4}\ldots, \qquad w = m, m+1. \tag{11}$$

By using equations (8, 9) it is obvious that, up to order h^4,

$$P(\alpha_{nn}) = P(D_{nn}^{0})\left\{1 + h^2 \sum_{w=m,m+1}^{\infty}{}' \frac{D_{ww}^{(2)}}{D_{ww}^{0}} + h^4 \sum_{w=m,m+1}^{\infty}{}' \left[\frac{D_{ww}^{(4)}}{D_{ww}^{0}} + \frac{D_{ww}^{(2)}}{D_{ww}^{0}} \sum_{t=w+2}^{\infty}{}' \frac{D_{tt}^{(2)}}{D_{tt}^{0}}\right] + \mathcal{O}(h^6)\right\}, \tag{12}$$

$$\frac{\alpha_{w+2,w}\,\alpha_{w,w+2}}{\alpha_{ww}\,\alpha_{w+2,w+2}} = \frac{D_{w+2,w}^{(2)}\,D_{w,w+2}^{(2)}}{D_{ww}^{0}\,D_{w+2,w+2}^{0}} h^4. \tag{13}$$

Substituting in equation (10) from equations (12, 13) one obtains

$$\Delta(\alpha_{ln}) = P(D_{nn}^{0})\left\{1 + h^2 \sum_{w=m,m+1}^{\infty}{}' \frac{D_{ww}^{(2)}}{D_{ww}^{0}} + h^4 \sum_{w=m,m+1}^{\infty}{}' \right.$$
$$\left. \times \left[\frac{D_{ww}^{(4)}}{D_{ww}^{0}} + \frac{D_{ww}^{(2)}}{D_{ww}^{0}} \sum_{t=w+2}^{\infty}{}' \frac{D_{tt}^{(2)}}{D_{tt}^{0}} - \frac{D_{w+2,w}^{(2)}\,D_{w,w+2}^{(2)}}{D_{ww}^{0}\,D_{w+2,w+2}^{0}}\right] + \mathcal{O}(h^6)\right\}, \quad n \geqslant m, m+1. \tag{14}$$

Exact expressions for the various Ds appearing in equation (14) are given in Appendix B.

It is evident from equation (14) that by setting $D_{nn}^{0} = \alpha_{nn}(0) = 0$ $(n \geqslant m, m+1)$ yields $\Delta(\alpha_{ln}) \neq 0$; namely, the roots of $\Delta[\alpha_{ln}(0)] = 0$ are not, in general, also roots of the equation $\Delta[\alpha_{ln}(h)] = 0$. Instead, this latter equation requires that

$$1 + h^2 \sum_{w=m,m+1}^{\infty}{}' \frac{D_{ww}^{(2)}}{D_{ww}^{0}} + h^4 \sum_{w=m,m+1}^{\infty}{}' \left[\frac{D_{ww}^{(4)}}{D_{ww}^{0}} + \frac{D_{ww}^{(2)}}{D_{ww}^{0}} \sum_{t=w+2}^{\infty}{}' \frac{D_{tt}^{(2)}}{D_{tt}^{0}} - \frac{D_{w+2,w}^{(2)}\,D_{w,w+2}^{(2)}}{D_{ww}^{0}D_{w+2,w+2}^{0}}\right] = 0. \tag{15}$$

With h small, equation (15) can be satisfied by values of the only varying parameter k that make its denominators D_{nn}^{0} as small as required by the small values of h. In other words, the resonance wavenumbers $k(h)$ correspond one to one and have values near the $k(0) \equiv k^0$ of the concentric spherical cavity. Setting

$$k(h) = k^{(0)} + k^{(2)}h^2 + k^{(4)}h^4 + \mathcal{O}(h^6), \qquad k^{(0)} \equiv k^0, \tag{16}$$

$$x_2(h) = k(h)R_2 = x_2^{(0)} + x_2^{(2)} h^2 + x_2^{(4)} h^4 + \mathcal{O}(h^6), \qquad x_2^{(\rho)} = k^{(\rho)}R_2, \quad \rho = 0, 2, 4, \tag{17}$$

one has

$$D_{nn}^{0}(x_2^0) = 0,$$

$$D_{nn}^{(\rho)}[x_2(h)] = D_{nn}^{(\rho)}(x_2^0) + x_2^{(2)} \frac{\mathrm{d}D_{nn}^{(\rho)}(x_2^0)}{\mathrm{d}x_2} h^2 + \left[x_2^{(4)} \frac{\mathrm{d}D_{nn}^{(\rho)}(x_2^0)}{\mathrm{d}x_2} + \frac{1}{2}\frac{\mathrm{d}^2 D_{nn}^{(\rho)}(x_2^0)}{\mathrm{d}x_2^2}(x_2^{(2)})^2\right]$$

$$\times h^4 + \mathcal{O}(h^6), \quad x_2^0 \equiv x_2^{(0)}, \quad \rho = 0, 2, \quad n \geqslant m, m+1. \tag{18}$$

In equation (18) the relation $x_1 = x_2/\tau$ has been used, where $\tau = R_2/R_1 =$ constant, so x_2 is the only variable.

By retaining only the large terms in equation (15) one finds

$$1 + h^2\left[\frac{D_{nn}^{(2)}}{D_{nn}^0} + \sum_{\substack{w=m,m+1\\ w\neq n}}^{\infty}{}' \frac{D_{ww}^{(2)}}{D_{ww}^0}\right] + h^4\left[\frac{D_{nn}^{(4)}}{D_{nn}^0} + \frac{D_{nn}^{(2)}}{D_{nn}^0}\sum_{\substack{w=m,m+1\\ w\neq n}}^{\infty}{}' \frac{D_{ww}^{(2)}}{D_{ww}^0}\right.$$

$$\left. - \frac{D_{n+2,n}^{(2)}D_{n,n+2}^{(2)}}{D_{nn}^0 D_{n+2,n+2}^0} - \frac{D_{n,n-2}^{(2)}D_{n-2,n}^{(2)}}{D_{n-2,n-2}^0 D_{nn}^0}\right] = 0, \qquad n \geqslant m, m+1. \tag{19}$$

One can now multiply both members of equation (19) by $D_{nn}^0[x_2(h)] \neq 0$ and next substitute there from equation (18), setting the coefficients of h^2 and h^4 equal to zero. So, one finally obtains the following relations for the evaluation of $x_2^{(2)}$ and $x_2^{(4)}$:

$$x_2^{(2)} = -\left[\frac{\mathrm{d}D_{nn}^0(x_2^0)}{\mathrm{d}x_2}\right]^{-1} D_{nn}^{(2)}(x_2^0), \qquad n \geqslant m, m+1, \tag{20}$$

$$x_2^{(4)} = -\left[\frac{\mathrm{d}D_{nn}^0(x_2^0)}{\mathrm{d}x_2}\right]^{-1}\left[\frac{(x_2^{(2)})^2}{2}\frac{\mathrm{d}^2D_{nn}^0(x_2^0)}{\mathrm{d}x_2^2} + x_2^{(2)}\frac{\mathrm{d}D_{nn}^{(2)}(x_2^0)}{\mathrm{d}x_2} + D_{nn}^{(4)}(x_2^0)\right.$$

$$\left. - \frac{D_{n+2,n}^{(2)}(x_2^0)D_{n,n+2}^{(2)}(x_2^0)}{D_{n+2,n+2}^0(x_2^0)} - \frac{D_{n,n-2}^{(2)}(x_2^0)D_{n-2,n}^{(2)}(x_2^0)}{D_{n-2,n-2}^0(x_2^0)}\right], \qquad n \geqslant m, m+1. \tag{21}$$

The two infinite sums of equation (19) do not appear in equation (21) because they have opposite values, as can be easily proved with the use of equation (20).

Formulas (20) and (21) are also valid for the oblate spheroidal boundaries. The only difference in this case is that $D^{(2)}$s change their signs and R_2 is the minor semi-axis of the oblate spheroidal. So, $x_2^{(2)}$ changes its sign, while $x_2^{(4)}$ remains unchanged.

The resonance frequencies for the problem of two concentric spheres with radii R_1 and R_2, used in equations (20) and (21), are given by the equation (equation (B1) in Appendix B) $D_{nn}^0 = 0$, or

$$\mathrm{j}_n(x_1^0)/\mathrm{n}_n(x_1^0) = \mathrm{j}_n(x_2^0)/\mathrm{n}_n(x_2^0), \qquad x_1^0 = x_2^0/\tau, \quad \tau = R_2/R_1. \tag{22}$$

By using equation (22), equations (B1)–(B5) from Appendix B, as well as various recurrence relations and Wronskians for spherical Bessel functions [9] in equations (20) and (21), one finally obtains after lengthy but straightforward calculations the explicit expressions

$$x_2^{(2)} = E\left[1 - \tau\frac{\mathrm{n}_n^2(x_2^0)}{\mathrm{n}_n^2(x_1^0)}\right]^{-1}, \qquad E = x_2^0\frac{n^2+m^2+n-1}{(2n-1)(2n+3)}, \tag{23}$$

$$x_2^{(4)} = \left\{H + \tau\frac{(x_2^0)^2 x_2^{(2)}\mathrm{n}_n(x_2^0)}{2(2n+1)\mathrm{n}_n^2(x_1^0)}\left[\frac{(n+m+1)(n+m+2)}{(2n+3)^2}\mathrm{n}_{n+2}(x_2^0)\right.\right.$$

$$- \frac{(n-m-1)(n-m)}{(2n-1)^2} \mathrm{n}_{n-2}(x_2^0) \Bigg] + (x_2^{(2)})^2 \Bigg[\tau \frac{\mathrm{n}_n(x_2^0)\mathrm{n}_n'(x_2^0)}{\mathrm{n}_n^2(x_1^0)}$$

$$- \frac{\mathrm{n}_n^2(x_2^0)}{\mathrm{n}_n^2(x_1^0)} \left(\frac{1}{x_1^0} + \frac{\mathrm{n}_n'(x_1^0)}{\mathrm{n}_n(x_1^0)} \right) \Bigg] - \frac{\tau^3 \mathrm{n}_n(x_2^0)}{4(2n+1)\mathrm{n}_n(x_1^0)} \Bigg[\frac{[(n+1)^2 - m^2][(n+2)^2 - m^2]}{(2n+3)^2(2n+5)\mathrm{w}_{n+2,n+2}(x_2^0, x_1^0)}$$

$$+ \frac{[(n-1)^2 - m^2](n^2 - m^2)}{(2n-3)(2n-1)^2 \mathrm{w}_{n-2,n-2}(x_2^0, x_1^0)} \Bigg] \Bigg\} \left[1 - \tau \frac{\mathrm{n}_n^2(x_2^0)}{\mathrm{n}_n^2(x_1^0)} \right]^{-1}, \tag{24}$$

where the primes denote derivatives with respect to the argument, while

$$H = \frac{(x_2^{(2)})^2}{x_2^0} + \frac{(n+m+1)(n+m+2)}{2(2n+1)(2n+3)^2} \Bigg\{ x_2^{(2)} [(x_2^0)^2 - (n+1)(2n+3)]$$

$$+ 2(x_2^0)^3 \Bigg[\frac{1 - 4m^2}{(2n-1)(2n+3)(2n+7)}$$

$$+ \frac{(n+m+3)(n+m+4)}{2(2n+5)} \left(\frac{2n+3}{(x_2^0)^2} - \frac{2}{2n+7} \right) \Bigg] \Bigg\}$$

$$- \frac{(n-m-1)(n-m)}{2(2n-1)^2(2n+1)} \Bigg\{ x_2^{(2)} [(x_2^0)^2 - n(2n-1)] - 2(x_2^0)^3 \Bigg[\frac{1 - 4m^2}{(2n-5)(2n-1)(2n+3)}$$

$$- \frac{(n-m-3)(n-m-2)}{8(2n-3)} \left(\frac{2n-1}{(x_2^0)^2} - \frac{2}{2n-5} \right) \Bigg] \Bigg\} \tag{25}$$

and

$$\mathrm{w}_{vv}(x_2^0, x_1^0) = \mathrm{j}_v(x_2^0)\mathrm{n}_v(x_1^0) - \mathrm{n}_v(x_2^0)\mathrm{j}_v(x_1^0). \tag{26}$$

It is evident that equation (17) can be written in the form $x_2(h) = x_2^0[1 + g^{(2)}h^2 + g^{(4)}h^4 + \mathcal{O}(h^6)]$. So, the eigenfrequencies in the cavities of Figures 1 and 2 are given by the expression

$$f_{nsm}(h) = f_{ns}(0)[1 + g_{nsm}^{(2)} h^2 + g_{nsm}^{(4)} h^4 + \mathcal{O}(h^6)], \tag{27}$$

where

$$g_{nsm}^{(2)} = (x_2^{(2)})_{nsm} / (x_2^0)_{ns}, \qquad g_{nsm}^{(4)} = (x_2^{(4)})_{nsm} / (x_2^0)_{ns}. \tag{28}$$

It is clear that one can start the analysis, equivalently, by interchanging j_n and n_n in equation (1). Following next the same procedure as before, one obtains again formulas (23) and (24) with the aforementioned interchanges (formula (22) remains the same). One more difference in equation (24) is that one should also change the minus signs in front of the two fractions containing $\mathrm{w}_{vv}(x_2^0, x_1^0)$ in their denominators into plus signs. This last change is necessary due to the fact that in the numerators of these fractions one has used Wronskians of Bessel functions, which change their signs with the above interchanges.

The former remarks mean also that formulas equivalent to equations (23) and (24) are obtained by replacing there n_v and n_v' by j_v and j_v', respectively, except in w_{vv} which keeps the form (26). This was verified numerically for various values of the parameters. Especially for equation (23) this is also evident from equation (22). These equivalent formulas are the only ones which can be used in the calculations in any case that $\mathrm{n}_n(x_1^0) = 0$ (in this case $\mathrm{j}_n(x_1^0) \neq 0$, while from equation (22) one obtains $\mathrm{n}_n(x_2^0) = 0$ and $\mathrm{j}_n(x_2^0) \neq 0$). In analogy, equations (23) and (24) are the only formulas which can be used in the calculations in any case that $\mathrm{j}_n(x_1^0) = 0$ (in this case $\mathrm{n}_n(x_1^0) \neq 0$, while from equation (22) one has $\mathrm{j}_n(x_2^0) = 0$ and $\mathrm{n}_n(x_2^0) \neq 0$).

By using in equations (23) and (24) the small argument formulas for the various Bessel functions [9] as $R_1 \to 0$ (for the cavity of Figure 1) one obtains, after some manipulation, the expressions for $x_2^{(2)}$ and $x_2^{(4)}$ in the case of a simple spheroidal cavity with major semi-axis R_2 and interfocal distance d (i.e., in the absence of the inner sphere). The same expressions were also obtained by the independent solution, from the beginning, of this last problem and are:

$$x_2^{(2)} = E, \qquad x_2^{(4)} = H. \tag{29}$$

It should be noticed that formulas (29) do not contain any Bessel functions, while x_2^0 there are roots of the equation $\mathrm{j}_n(x_2^0) = 0$.

The results (23)–(25) and (29) can be obtained also by an independent shape perturbation method, in which the pressure field is expressed in terms of spherical wave functions only. Geometrical relations expressing the spheroidal boundary in terms of spherical coordinates are used. In this method there is no need for spheroidal wave functions and the expansion formulas connecting them with the concentric spherical ones. This alternative procedure provides a very convincing check on the results of the present section, but it is not possible to be presented here without making the manuscript extremely lengthy. So, it will be the subject of a forthcoming paper.

3. NEUMANN BOUNDARY CONDITIONS

In this case the expansion for p that corresponds to equation (1) and satisfies the boundary condition $\partial p/\partial r = 0$ at $r = R_1$ is

$$p = \sum_{n=0}^{\infty} \sum_{m=0}^{n} [\mathrm{j}_n(kr) - \mathrm{n}_n(kr)\mathrm{j}'_n(x_1)/\mathrm{n}'_n(x_1)]\mathrm{P}_n^m(\cos\theta)\,[A_{nm}\cos m\varphi + B_{nm}\sin m\varphi]. \tag{30}$$

In order to satisfy the remaining boundary condition $\partial p/\partial\mu = 0$ ($\partial p/\partial\xi = 0$) at $\mu = \mu_0$ ($\xi = \xi_0$) one follows steps identical to those for the Dirichlet case, which lead again to the infinite set (4) with the difference that α_{lnm} is now given by the expression

$$\alpha_{lnm} = \frac{2\mathrm{i}^{-n}(n+m)!}{(2n+1)\,(n-m)!}\, d_{n-m}^{ml} \left[\frac{\partial \mathrm{R}_{ml}^{(1)}(c, \cosh\mu_0)}{\partial\mu} - \frac{\partial \mathrm{R}_{ml}^{(2)}(c, \cosh\mu_0)}{\partial\mu}\frac{\mathrm{j}'_n(x_1)}{\mathrm{n}'_n(x_1)}\right]. \tag{31}$$

The remarks after equation (5) are again valid in this case. From equation (A25) of Appendix A one obtains

$$\frac{\partial \mathrm{R}_{ml}^{(\sigma)}(c, \cosh\mu_0)}{\partial\mu} = \frac{(l-m)!}{(l+m)!}\tanh^{m-1}\mu_0 \sum_{r=0,1}^{\infty}{}' \mathrm{i}^{r+m-l} d_r^{ml}\frac{(r+2m)!}{r!}$$
$$\times\left[\left(x_2 - \frac{c^2}{x_2}\right)\mathrm{z}_{r+m}^{(\sigma)\prime}(x_2) + \frac{c^2}{x_2^2}m\,\mathrm{z}_{r+m}^{(\sigma)}(x_2)\right]. \tag{32}$$

Setting $r = l - m \pm 2q$ in equation (32) and following the same steps as for the Dirichlet case, one obtains, in place of equation (7), the equation

$$\frac{\partial \mathrm{R}_{ml}^{(\sigma)}(c,\cosh\mu_0)}{\partial\mu} = \frac{(l-m)!}{(l+m)!}\tanh^{m-1}\mu_0\, d_{l-m}^{ml}\left\{\frac{(l+m)!}{(l-m)!}\left[\left(x_2 - \frac{c^2}{x_2}\right)\mathrm{z}_l^{(\sigma)\prime}(x_2) + \frac{c^2}{x_2^2}m\mathrm{z}_l^{(\sigma)}(x_2)\right]\right.$$

$$+ \sum_{u=1}^{\infty} c^{2u}\left[\sum_{q=1}^{u}(-1)^q\frac{(l+m+2q)!}{(l-m+2q)!}\left\{\left(x_2 - \frac{c^2}{x_2}\right)\mathrm{z}_{l+2q}^{(\sigma)\prime}(x_2)\right.\right.$$

$$\left. + \frac{c^2}{x_2^2}m\mathrm{z}_{l+2q}^{(\sigma)}(x_2)\right\}v_{2u-2q}^{ml+}(2q)a_{2q,0}^{ml+} + \sum_{q=1}^{\min(q_{max}\, u)}(-1)^q\frac{(l+m-2q)!}{(l-m-2q)!}$$

$$\left.\left.\times\left\{\left(x_2 - \frac{c^2}{x_2}\right)\mathrm{z}_{l-2q}^{(\sigma)\prime}(x_2) + \frac{c^2}{x_2^2}m\mathrm{z}_{l-2q}^{(\sigma)}(x_2)\right\}v_{2u-2q}^{ml-}(2q)a_{2q,0}^{ml-}\right]\right\}. \qquad (33)$$

One substitutes from equation (33) into equation (31) and next divides α_{lnm} by $2(d_{l-m}^{ml})^2\mathrm{i}^{-n}(n+m)!\tanh^{m-1}\mu_0/[(2n+1)(n-m)!]$, in an analogous manner as for the Dirichlet case. The remarks after equation (7) are also valid here. The same is true for equations (8)–(21), but with different expressions for the various expansion coefficients, which are given in Appendix B. In place of equation (22) one now has

$$\mathrm{j}_n'(x_1^0)/\mathrm{n}_n'(x_1^0) = \mathrm{j}_n'(x_2^0)/\mathrm{n}_n'(x_2^0), \qquad x_1^0 = x_2^0/\tau. \qquad (34)$$

By using equation (34), and equations (B6)–(B10) from Appendix B, the recurrence relations and Wronskians for spherical Bessel functions [9] in equations (20) and (21), one can finally obtain after laborious but straightforward calculations the explicit expressions for $x_2^{(2)}$ and $x_2^{(4)}$:

$$x_2^{(2)} = U\left[1 - \tau^3\left(\frac{\mathrm{n}_n'(x_2^0)}{\mathrm{n}_n'(x_1^0)}\right)^2\frac{(x_1^0)^2 - n(n+1)}{(x_2^0)^2 - n(n+1)}\right]^{-1},$$

$$U = x_2^0\left\{\frac{n^2+m^2+n-1}{(2n-1)(2n+3)} - \frac{1}{(2n+1)[(x_2^0)^2 - n(n+1)]}\right.$$

$$\left.\times\left[\frac{(n^2-m^2)(n+1)}{2n-1} - \frac{[(n+1)^2-m^2]n}{2n+3}\right]\right\}, \qquad (35)$$

$$x_2^{(4)} = x_2^{(2)} - \frac{(x_2^{(2)})^2}{x_2^0} + \frac{1}{(x_2^0)^2 - n(n+1)}\left\{Y + \frac{\mathrm{n}_n'(x_2^0)[(x_1^0)^2 - n(n+1)]}{(\mathrm{n}_n'(x_1^0))^2}\right.$$

$$\times\left\{\frac{\tau^3(x_2^0)^2x_2^{(2)}}{2(2n+1)}\left[\frac{(n+m+1)(n+m+2)}{(2n+3)^2}\mathrm{n}_{n+2}'(x_2^0)\right.\right.$$

$$\left. - \frac{(n-m-1)(n-m)}{(2n-1)^2}\mathrm{n}_{n-2}'(x_2^0)\right]$$

$$+ \frac{\tau X_2^0}{(x_1^0)^2}\left[m\, x_2^{(2)}\, \mathrm{n}_n\,(x_2^0) + x_2^0 (x_2^{(2)})^2 \mathrm{n}_n''\,(x_2^0) - x_1^0\,(x_2^{(2)})^2 \mathrm{n}_n'\,(x_2^0)\,\frac{\mathrm{n}_n''\,(x_1^0)}{\mathrm{n}_n'\,(x_1^0)}\right]\Bigg\}$$

$$- \tau x_2^0 \left(\frac{x_2^{(2)}}{x_1^0}\right)^2 \left(\frac{\mathrm{n}_n'\,(x_2^0)}{\mathrm{n}_n'\,(x_1^0)}\right)^2 [(x_1^0)^2 - 2n(n+1)] - \frac{\tau^3 \mathrm{n}_n'\,(x_2^0)}{4(x_1^0)^2(2n+1)\mathrm{n}_n'\,(x_1^0)}$$

$$\times \left[\frac{[(n+1)^2 - m^2]\,[(n+2)^2 - m^2]\,[(x_1^0)^2 - n(n+3)]\,[(x_2^0)^2 - n(n+3)]}{(2n+3)^2(2n+5)\,\mathrm{w}_{n+2,n+2}'\,(x_2^0, x_1^0)}\right.$$

$$\left. + \frac{[(n-1)^2 - m^2]\,(n^2 - m^2)\,[(x_1^0)^2 - (n-2)\,(n+1)]\,[(x_2^0)^2 - (n-2)\,(n+1)]}{(2n-3)\,(2n-1)^2 \mathrm{w}_{n-2,n-2}'\,(x_2^0, x_1^0)}\right]\Bigg\}$$

$$\times \left[1 - \tau^3 \left(\frac{\mathrm{n}_n'\,(x_2^0)}{\mathrm{n}_n'\,(x_1^0)}\right)^2 \frac{(x_1^0)^2 - n(n+1)}{(x_2^0)^2 - n(n+1)}\right]^{-1}, \tag{36}$$

where

$$Y = \frac{(x_2^{(2)})^2}{x_2^0}\,[(x_2^0)^2 - 2n(n+1)] + \frac{(n+m+1)\,(n+m+2)}{2(2n+1)\,(2n+3)^2}$$

$$\times \Bigg\{ x_2^{(2)}\,[(x_2^0)^4 - 3(x_2^0)^2(n+2)^2 + n(n+3)\,(n+4)\,(2n+3)]$$

$$+ \left[(3x_2^{(2)} - x_2^0)\,(2n+3) + \frac{2(1-4m^2)\,(x_2^0)^3}{(2n-1)\,(2n+3)\,(2n+7)}\right]$$

$$\times [(x_2^0)^2 - n(n+3)] - m x_2^0\,[(x_2^0)^2 - n(2n+3)] - \frac{(n+m+3)\,(n+m+4)}{4(2n+5)\,(2n+7)}$$

$$\times [2(x_2^0)^5 - (6n^2 + 30n + 35)\,(x_2^0)^3 + n(n+5)\,(2n+3)\,(2n+7) x_2^0]\Bigg\}$$

$$- \frac{(n-m-1)\,(n-m)}{2(2n-1)^2(2n+1)} \Bigg\{ x_2^{(2)}\,[(x_2^0)^4 - 3(x_2^0)^2(n-1)^2$$

$$+ (n-3)\,(n-2)\,(2n-1)\,(n+1)]$$

$$- \left[(3x_2^{(2)} - x_2^0)\,(2n-1) + \frac{2(1-4m^2)\,(x_2^0)^3}{(2n-5)\,(2n-1)\,(2n+3)}\right] [(x_2^0)^2 - (n-2)\,(n+1)]$$

$$- m x_2^0\,[(x_2^0)^2 - (2n-1)\,(n+1)] - \frac{(n-m-3)\,(n-m-2)}{4(2n-5)\,(2n-3)}$$

$$\times [2(x_2^0)^5 - (6n^2 - 18n + 11)\,(x_2^0)^3 + (n-4)\,(n+1)\,(2n-5)\,(2n-1) x_2^0]\Bigg\} \tag{37}$$

and

$$\mathrm{w}_{vv}'\,(x_2^0, x_1^0) = \mathrm{j}_v'\,(x_2^0)\mathrm{n}_v'\,(x_1^0) - n_v'\,(x_2^0)\mathrm{j}_v'\,(x_1^0). \tag{38}$$

The remarks after equation (28) are also valid in this case, so equations (35) and (36) give the same results if one replaces n_v, n'_v and n''_v by j_v, j'_v and j''_v, respectively, except in w'_{vv} which keeps the form (38). This was verified numerically for various values of the parameters. Especially for equation (35) this is also evident from equation (34). The equivalent formulas are the only ones which can be used in the calculations in any case that $\mathrm{n}'_n(x_1^0) = 0$ (in this case $\mathrm{j}'_n(x_1^0) \neq 0$, while from equation (34) $\mathrm{n}'_n(x_2^0) = 0$ and $\mathrm{j}'_n(x_2^0) \neq 0$). In analogy, equations (35) and (36) are the only formulas which can be used in the calculations in any case that $\mathrm{j}'_n(x_1^0) = 0$ (in this case $\mathrm{n}'_n(x_1^0) \neq 0$, while from equation (34) $\mathrm{j}'_n(x_2^0) = 0$ and $\mathrm{n}'_n(x_2^0) \neq 0$).

Following next the same procedure as that described for the Dirichlet case, one obtains the expressions for $x_2^{(2)}$ and $x_2^{(4)}$ in a simple spheroidal cavity with major semi-axis R_2 and interfocal distance d. These expressions are

$$x_2^{(2)} = U, \qquad x_2^{(4)} = x_2^{(2)} - \frac{(x_2^{(2)})^2}{x_2^0} + \frac{Y}{(x_2^0)^2 - n(n+1)}. \tag{39}$$

Formulas (39) do not contain any Bessel functions, while x_2^0 there are roots of the equation $\mathrm{j}'_n(x_2^0) = 0$.

The same results (35)–(39) are obtained also by the same shape perturbation method referred to after equation (29), providing an excellent check for their validity.

4. NUMERICAL RESULTS AND DISCUSSION

In Tables 1 and 2 the roots $(x_2^0)_{ns}$ $(n = 0\text{–}3, s = 1\text{–}4)$ of equation (22) as well as the corresponding values of $g_{nsm}^{(2)}$ and $g_{nsm}^{(4)}$ are given in the Dirichlet case for $\tau = x_2/x_1 = R_2/R_1 = 1{\cdot}35$, $2{\cdot}0$ (cavity of Figure 1) and $\tau = 0{\cdot}5$, $0{\cdot}7$ (cavity of Figure 2). In Tables 3 and 4 the roots $(x_2^0)_{ns}$ of equation (34) are given and the corresponding values of $g_{nsm}^{(2)}$ and $g_{nsm}^{(4)}$ in the Neumann case, for the same τs as before.

For the oblate spheroidal boundaries the $g^{(2)}$s change their signs, while the $g^{(4)}$s remain the same.

The case $n = 0$ $(m = 0)$ for the Dirichlet problem requires special treatment. In this case equation (22) reduces to $\tan x_1^0 = \tan x_2^0$, with roots $x_2^0 = \tau x_1^0 = x_1^0 \pm s\pi = \pm\tau s\pi/(\tau - 1)$, $s = 1, 2, \ldots,$ where the upper/lower sign corresponds to Figures 1 and 2. So, $\cos x_2^0 = (-1)^s \cos x_1^0$, $\sin x_2^0 = (-1)^s \sin x_1^0$ and $g_{oso}^{(2)} = \tau/[3(\tau - 1)]$ is independent of s, as is easily proved from equation (23) (or its equivalent one with n_0 replaced by j_0) and equation (28), and confirmed by the corresponding results in Tables 1 and 2. In particular when $\pm s/(\tau - 1) = v + 1/2$ (or v), v being an integer, $\cos x_1^0 = \cos x_2^0 = 0$, namely $\mathrm{n}_0(x_1^0) = n_0(x_2^0) = 0$ (or $\sin x_1^0 = \sin x_2^0 = 0$, namely $\mathrm{j}_0(x_1^0) = \mathrm{j}_0(x_2^0) = 0$), so the formulas equivalent to equations (23) and (24) and referred to after equation (28) (or formulas (23, 24)) are the only ones which can be used in the calculations, as is explained there.

On the contrary, it can be seen easily that the case $n = 0$ $(m = 0)$ for the Neumann problem requires no special treatment.

A general remark on the $g_{nsm}^{(2)}$ from Tables 1–4 is that for $s > 2$, their values stabilize and become independent of s and that they, as well as $g_{nsm}^{(4)}$, decrease rapidly with increasing/decreasing τ, for the cavity of Figures 1 and 2. This last result is expected from physical intuition and confirmed by available data for further values of τ. Both these observations agree also with the results of references [7, 8].

The method of the present work can be extended easily for the calculation of higher order terms, like, for example, $g_{nsm}^{(6)}$, in the expansion series of $f_{nsm}(h)$, with respect to h.

TABLE 1

Dirichlet conditions, cavity of Figure 1, $\tau = x_2/x_1 = 1{\cdot}35$ (2·0)

	n	m	s = 1	2	3	4
$(x_2^0)_{ns}$	0		12·11757 (2π)	24·23514 (4π)	36·35271 (6π)	48·47029 (8π)
	1		12·22696 (6·57201)	24·29059 (12·72136)	36·38978 (18·95439)	49·49811 (25·21178)
	2		12·44283 (7·11158)	24·40111 (13·02614)	36·46378 (19·16254)	49·55370 (25·36922)
	3		12·75965 (7·84504)	24·56597 (13·47113)	36·57452 (19·47107)	48·63697 (25·60381)
$g_{nsm}^{(2)}$	0	0	1·28571 (0·66667)	1·28571 (0·66667)	1·28571 (0·66667)	1·28571 (0·66767)
	1	0	0·75963 (0·37609)	1·76838 (0·39289)	0·77008 (0·39672)	0·77060 (0·39813)
		1	1·51926 (0·75218)	1·53675 (0·78578)	1·54017 (0·79344)	1·54120 (0·79626)
	2	0	0·87772 (0·40411)	0·90757 (0·45194)	0·91351 (0·46472)	0·91563 (0·46960)
		1	1·05326 (0·48493)	1·08908 (0·54232)	1·09622 (0·55767)	1·09875 (0·56352)
		2	1·57989 (0·72740)	1·63363 (0·81349)	1·64432 (0·83650)	1·64813 (0·84527)
	3	0	0·86350 (0·36933)	0·92097 (0·44222)	0·93296 (0·46602)	0·93719 (0·47557)
		1	0·94200 (0·40291)	1·00470 (0·48242)	1·01777 (0·50839)	1·02239 (0·51880)
		2	1·17750 (0·50363)	1·25587 (0·60302)	1·27221 (0·63548)	1·27799 (0·64851)
		3	1·57001 (0·67151)	1·67450 (0·80403)	1·69628 (0·84731)	1·70399 (0·86467)
$g_{nsm}^{(4)}$	0	0	−9·55639 (0·19168)	−45·5080 (−1·54597)	−105·4383 (−4·46675)	−189·3598 (−8·55957)
	1	0	−4·19268 (0·17475)	−20·8085 (−0·58878)	−48·5464 (−1·93239)	−87·3736 (−3·82385)
		1	−0·57998 (0·74413)	−11·6380 (0·26347)	−30·1241 (−0·62648)	−55·9952 (−1·88544)
	2	0	9·86410 (0·86511)	34·5571 (2·15138)	75·6305 (4·17271)	133·1015 (6·98391)
		1	−1·09988 (0·43623)	−10·4585 (0·08807)	−26·1693 (−0·65388)	−48·1824 (−1·71900)
		2	1·77139 (0·82112)	−2·84902 (0·72508)	−10·6930 (0·37208)	−21·6957 (−0·15390)
	3	0	4·27583 (0·53486)	12·3799 (1·05780)	25·7251 (1·74059)	44·3768 (2·66268)
		1	2·90572 (0·51198)	6·33954 (0·82022)	11·8826 (1·12658)	19·6054 (1·51724)
		2	0·56766 (0·51754)	−4·72652 (0·43203)	−13·7774 (0·03644)	−26·5035 (−0·56614)
		3	2·57844 (0·77415)	0·34671 (0·86692)	−3·65229 (−0·72596)	−9·33545 (0·47134)

TABLE 2

Dirichlet conditions, cavity of Figure 2, $\tau = x_2/x_1 = 0{\cdot}5\ (0{\cdot}7)$

			s			
	n	m	1	2	3	4
$(x_2^0)_{ns}$	0		π (7·33038)	2π (14·66077)	3π (21·99115)	4π (29·32153)
	1		3·28601 (7·42345)	6·36068 (14·70820)	9·47720 (22·02288)	12·60589 (29·34537)
	2		3·55579 (7·60608)	6·51307 (14·80262)	9·58127 (22·08623)	12·68461 (29·39298)
	3		3·92252 (7·87191)	6·73556 (14·94316)	9·73553 (22·18091)	12·80190 (29·46425)
$g_{nsm}^{(2)}$	0	0	−0·33333 (−0·77778)	−0·33333 (−0·77778)	−0·33333 (−0·77778)	−0·33333 (−0·77778)
	1	0	−0·17609 (−0·45283)	−0·19289 (−0·46303)	−0·19672 (−0·46502)	−0·19813 (−0·46574)
		1	−0·35218 (−0·90565)	−0·38578 (−0·92607)	−0·39344 (−0·93004)	−0·39626 (−0·93149)
	2	0	−0·16601 (−0·50849)	−0·21384 (−0·54275)	−0·22663 (−0·54977)	−0·23150 (−0·55229)
		1	−0·19922 (−0·61019)	−0·25661 (−0·65130)	−0·27195 (−0·65973)	−0·27780 (−0·66274)
		2	−0·29883 (−0·91529)	−0·38491 (−0·97695)	−0·40793 (−0·98959)	−0·41670 (−0·99411)
	3	0	−0·12489 (−0·48026)	−0·19777 (−0·54456)	−0·22158 (−0·55859)	−0·23113 (−0·56367)
		1	−0·13624 (−0·52393)	−0·21575 (−0·59407)	−0·24172 (−0·60937)	−0·25214 (−0·61491)
		2	−0·17030 (−0·65491)	−0·26969 (−0·74259)	−0·30215 (−0·76171)	−0·31517 (−0·76864)
		3	−0·22707 (−0·87321)	−0·35959 (−0·99011)	−0·40287 (−1·01561)	−0·42023 (−1·02485)
$g_{nsm}^{(4)}$	0	0	−0·11435 (−2·61677)	−0·55194 (−11·9018)	−1·28280 (−27·3805)	−2·30624 (−49·0529)
	1	0	−0·07098 (−1·25964)	−0·27654 (−5·55468)	−0·61576 (−12·7198)	−1·08984 (−22·7489)
		1	0·02813 (−0·26966)	−0·11416 (−3·13258)	−0·34216 (−7·91439)	−0·65902 (−14·5925)
	2	0	0·12763 (−2·42596)	0·42135 (8·79319)	0·91916 (19·3987)	1·61910 (34·2454)
		1	−0·00872 (−0·46475)	−0·13365 (−2·89664)	−0·32998 (−6·95777)	−0·60044 (−12·6440)
		2	0·08554 (0·37630)	0·01502 (−0·83665)	−0·08818 (−2·86743)	−0·22561 (−5·71071)
	3	0	0·06275 (0·98268)	0·15389 (3·05656)	0·31092 (6·49914)	0·53593 (11·3184)
		1	0·04852 (0·61273)	0·08122 (1·47711)	0·14192 (2·90357)	0·23311 (4·90090)
		2	0·03511 (−0·00212)	−0·04095 (−1·39939)	−0·16123 (−3·74467)	−0·32078 (−7·02608)
		3	0·11045 (0·62304)	0·07490 (0·01256)	0·01302 (−1·03011)	−0·06207 (−2·48598)

TABLE 3

Neumann conditions, cavity of Figure 1, $\tau = x_2/x_1 = 1{\cdot}35$ (2·0)

			s			
	n	m	1	2	3	4
$(x_2^0)_{ns}$	0		12·22696 (6·57201)	24·29059 (12·72136)	36·38977 (18·95439)	48·49810 (25·21178)
	1		1·61845 (1·84027)	12·34029 (6·91152)	24·34656 (12·88524)	36·42699 (19·06205)
	2		2·80178 (3·15118)	12·56411 (7·55362)	24·45813 (13·20870)	36·50130 (19·27601)
	3		3·95925 (4·38996)	12·89316 (8·43887)	24·62459 (13·68401)	36·61253 (19·59369)
$g_{nsm}^{(2)}$	0	0	1·26605 (0·62682)	1·28063 (0·65482)	1·28344 (0·66121)	1·28443 (0·66355)
	1	0	−0·26836 (−0·06598)	0·73727 (0·33167)	0·76269 (0·38041)	0·76753 (0·39112)
		1	0·43516 (0·40585)	1·49940 (0·70123)	1·53186 (0·77256)	1·53796 (0·78768)
	2	0	0·05092 (0·13226)	0·85465 (0·35031)	0·90195 (0·43783)	0·91104 (0·45875)
		1	0·12691 (0·19940)	1·03034 (0·42628)	1·08362 (0·52760)	1·09383 (0·55156)
		2	0·35488 (0·40085)	1·55742 (0·65417)	1·62863 (0·79689)	1·64218 (0·82998)
	3	0	0·10636 (0·17879)	0·83862 (0·30597)	0·91524 (0·42547)	0·93044 (0·45947)
		1	0·13133 (0·20580)	0·91689 (0·33575)	0·99901 (0·46505)	1·01528 (0·50170)
		2	0·20624 (0·28684)	1·15171 (0·42507)	1·25035 (0·58382)	1·26982 (0·62839)
		3	0·33110 (0·42189)	1·54308 (0·57393)	1·66923 (0·78175)	1·69405 (0·83954)
$g_{nsm}^{(4)}$	0	0	−9·32963 (0·28144)	−45·2651 (−1·43231)	−105·1880 (−4·35012)	−189·0827 (−8·44210)
	1	0	−0·46308 (−0·10236)	−4·06505 (0·25088)	−20·6909 (−0·51955)	−48·4286 (−1·87052)
		1	0·44225 (0·27332)	−0·51491 (0·76858)	−11·5637 (0·30305)	−30·0498 (−0·58811)
	2	0	0·01999 (0·00665)	9·71727 (0·82798)	34·3952 (2·09388)	75·4834 (4·10145)
		1	0·05538 (0·13500)	−1·01054 (0·49006)	−10·3867 (0·14231)	−26·0958 (−0·61203)
		2	0·28172 (0·23731)	1·79518 (0·80756)	−2·81827 (0·74360)	−10·6517 (0·38912)
	3	0	0·02670 (0·02214)	4·24043 (0·52780)	12·3306 (1·06001)	25·6677 (1·72474)
		1	0·05228 (0·07306)	2·90664 (0·51510)	6·32362 (0·83852)	11·8597 (1·12719)
		2	0·12215 (0·18612)	0·63051 (0·52851)	−4·68134 (0·47588)	−13·7373 (0·06591)
		3	0·21568 (0·24224)	2·58774 (0·72257)	0·36253 (0·87766)	−3·63706 (0·73502)

TABLE 4

Neumann conditions, cavity of Figure 2, $\tau = x_2/x_1 = 0{\cdot}5\ (0{\cdot}7)$

	n	m	s = 1	2	3	4
$(x_2^0)_{ns}$	0		3·28601 (7·42345)	6·36068 (14·70820)	9·47720 (22·02289)	12·60589 (29·34537)
	1		0·92013 (1·15806)	3·45576 (7·52119)	6·44262 (14·75627)	9·53103 (22·05482)
	2		1·57559 (2·00378)	3·77681 (7·71337)	6·60435 (14·85199)	9·63801 (22·11855)
	3		2·19498 (2·82951)	4·21944 (7·99392)	6·84200 (14·99449)	9·79684 (22·21382)
$g_{nsm}^{(2)}$	0	0	−0·29349 (−0·75470)	−0·32149 (−0·77172)	−0·32787 (−0·77506)	−0·33022 (−0·77625)
	1	0	0·13904 (0·30508)	−0·11729 (−0·42191)	−0·17574 (−0·45507)	−0·18894 (−0·46148)
		1	0·05762 (0·03646)	−0·30842 (−0·88396)	−0·37490 (−0·92079)	−0·38876 (−0·92775)
	2	0	0·06397 (0·14022)	−0·10313 (−0·47748)	−0·19641 (−0·53515)	−0·21910 (−0·54642)
		1	0·06538 (0·13128)	−0·13602 (−0·58065)	−0·24022 (−0·64428)	−0·26506 (−0·65665)
		2	0·06959 (0·10447)	−0·23469 (−0·89014)	−0·37165 (−0·97169)	−0·40296 (−0·98736)
	3	0	0·03928 (0·10921)	−0·05463 (−0·44652)	−0·17793 (−0·53680)	−0·21356 (−0·55523)
		1	0·04108 (0·11115)	−0·06390 (−0·49038)	−0·19607 (−0·58654)	−0·23394 (−0·60613)
		2	0·04650 (0·11697)	−0·09173 (−0·62197)	−0·25048 (−0·73576)	−0·29505 (−0·75885)
		3	0·05552 (0·12668)	−0·13812 (−0·84127)	−0·34117 (−0·98448)	−0·39692 (−1·01338)
$g_{nsm}^{(4)}$	0	0	−0·11569 (−2·63040)	−0·56181 (−11·9190)	−1·29520 (−27·3965)	−2·31960 (−49·0709)
	1	0	−0·00289 (−0·06644)	−0·04856 (−1·24528)	−0·27277 (−5·55688)	−0·61762 (−12·7246)
		1	0·04072 (0·13034)	0·04317 (−0·26929)	−0·11399 (−3·13702)	−0·34481 (−7·91448)
	2	0	0·00981 (0·07191)	0·16742 (2·46086)	0·44287 (8·81147)	0·93429 (19·4138)
		1	−0·00376 (−0·00442)	0·01760 (−0·44799)	−0·12788 (−2·89623)	−0·33003 (−6·96037)
		2	0·01645 (0·10179)	0·12542 (0·38638)	0·02189 (−0·83770)	−0·08753 (−2·86953)
	3	0	−0·00603 (0·02766)	0·09477 (1·01123)	0·17081 (3·06660)	0·31960 (6·50583)
		1	−0·00733 (0·02138)	0·08159 (0·63994)	0·09622 (1·48477)	0·14836 (2·90781)
		2	−0·01005 (0·01955)	0·07448 (0·02220)	−0·02894 (−1·39674)	−0·15903 (−3·74526)
		3	−0·01066 (0·07320)	0·17062 (0·64635)	0·09107 (0·01415)	0·01638 (−1·03045)

The procedure will be laborious but straightforward. For this purpose further expansion coefficients given in Appendix A will be used.

REFERENCES

1. P. M. MORSE and H. FESHBACH 1953 *Methods of Theoretical Physics, Volume II*. New York: McGraw-Hill.
2. J. A. ROUMELIOTIS, J. D. KANELLOPOULOS and J. G. FIKIORIS 1991 *Journal of the Acoustical Society of America* **90**, 1144–1148. Acoustic resonance frequency shifts in a spherical cavity with an eccentric inner small sphere.
3. J. A. ROUMELIOTIS, J. D. KANELLOPOULOS and J. G. FIKIORIS 1992 *Journal of the Franklin Institute* **329**, 413–427. Acoustic eigenfrequencies of a cylindrical/rectangular cavity, with an eccentric inner small sphere.
4. J. A. ROUMELIOTIS and J. D. KENELLOPOULOS 1992 *Journal of the Franklin Institute* **329**, 727–735. Acoustic eigenfrequencies and modes in a soft-walled spherical cavity with an eccentric inner small sphere.
5. J. A. ROUMELIOTIS 1993 *Journal of the Acoustical Society of America* **93**, 1710–1715. Eigenfrequencies of an acoustic rectangular cavity containing a rigid small sphere.
6. C. FLAMMER 1957 *Spheroidal Wave Functions*. Stanford CA: Stanford University Press.
7. J. D. KANELLOPOULOS and J. G. FIKIORIS 1978 *Journal of the Acoustical Society of America* **64**, 286–297. Acoustic resonant frequencies in an eccentric spherical cavity.
8. J. A. ROUMELIOTIS, A. B. M. SIDDIQUE HOSSAIN and J. G. FIKIORIS 1980 *Radio Science* **15**, 923–937. Cutoff wave numbers of eccentric circular and concentric circular–elliptic metallic wave guides.
9. M. ABRAMOWITZ and I. A. STEGUN 1972 *Handbook of Mathematical Functions*. New York: Dover.
10. J. E. BURKE 1966 *Journal of Mathematics and Physics* **45**, 425–431. Note on spheroidal wave functions.
11. J. MEIXNER and F. W. SCHÄFKE 1954. *Mathieusche Funktionen und Sphäroidfunktionen*. Berlin: Springer-Verlag.
12. D. S. JONES 1964 *The Theory of Electromagnetism*. Oxford: Pergamon Press.

APPENDIX A

In this Appendix, power series expansions for the prolate and the oblate angular spheroidal wave functions of the first kind $S_{mn}^{(1)}(c, \eta)$ and of the second kind $S_{mn}^{(2)}(c, \eta)$, with small arguments c, are derived for general integer values of m and n. The various expansion coefficients can also be used in the evaluation of the radial functions of any kind. The prolate angular spheroidal functions of the first kind are given by an infinite sum of the form [1, 6, 9] (the superscript (1) is omitted as in the main text).

$$S_{mn}(c, \eta) = \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c) P_{m+r}^m(\eta), \tag{A1}$$

where the prime indicates that the summation is over only those values of r having the same parity as $n - m$. The coefficients $d_r^{mn}(c)$ satisfy the following second order recurrence relation [1, 6]:

$$\frac{(2m+r+2)(2m+r+1)c^2}{(2m+2r+3)(2m+2r+5)} d_{r+2}^{mn}(c) + \left[(m+r)(m+r+1) - \lambda_{mn}(c) + \frac{2(m+r)(m+r+1) - 2m^2 - 1}{(2m+2r-1)(2m+2r+3)} c^2\right] d_r^{mn}(c) + \frac{r(r-1)c^2}{(2m+2r-3)(2m+2r-1)} d_{r-2}^{mn}(c) = 0, \qquad r \geqslant 0. \tag{A2}$$

In equation (A2) $\lambda_{mn}(c)$ denotes the eigenvalues and $d_{-r}^{mn} = 0$ for $r > 0$.

When c vanishes, the differential equation satisfied by S_{mn} becomes that satisfied by the associated Legendre functions. In this case $S_{mn} \to P_n^m$, $d_{n-m}^{mn} \to 1$ $(n \geqslant m)$ is the only non-zero coefficient and $\lambda_{mn}(0) = n(n+1)$.

Approximations for S_{mn} valid for small values of c follow by expanding $d_r^{mn}(c)$ in power series in c of the form

$$d_{n-m\pm 2q}^{mn}(c) = [a_{2q,0}^{mn\pm} c^{2q} + a_{2q,2}^{mn\pm} c^{2q+2} + a_{2q,4}^{mn\pm} c^{2q+4} + \cdots] d_{n-m}^{mn}(c),$$

$$n - m \geqslant {}^{0(+)}_{2q(-)}, \qquad q = 0, 1, 2, \ldots, \tag{A3}$$

where $a_{2q,2k}^{mn-} = 0$, $(k \geqslant 0)$ if $0 \leqslant n - m < 2q$.

The expansion for $\lambda_{mn}(c)$ must be

$$\lambda_{mn}(c) = n(n+1) + 1_2^{mn} c^2 + 1_4^{mn} c^4 + 1_6^{mn} c^6 + \cdots. \tag{A4}$$

In what follows the superscripts mn are omitted from the various expansion coefficients, for simplicity.

By substituting from equations (A3) and (A4) into equation (A2) and by equating the coefficients of $c^2, c^4, \ldots$ to zero separately, one obtains, with $r = n - m \geqslant 0$, from the coefficient of c^2 and c^{2k+4}, respectively,

$$l_2 = \frac{2n(n+1) - 2m^2 - 1}{(2n-1)(2n+3)}, \tag{A5}$$

$$\frac{(n+m+1)(n+m+2)}{(2n+3)(2n+5)} a_{2,2k}^{+} + \frac{(n-m-1)(n-m)}{(2n-3)(2n-1)} a_{2,2k}^{-} = l_{2k+4}, \qquad k = 0, 1, 2, \ldots. \tag{A6}$$

Setting now $r = n - m \pm 2q$ $(q \geqslant 1, n - m \geqslant 2q$ for the lower sign) in equation (A2) and using equations (A3) and (A4) one obtains, by equating to zero the coefficients of c^{2q}, c^{2q+2} and c^{2q+2k}, respectively,

$$a_{2q,0}^{\pm} = f^{\pm}(2q-2) a_{2q-2,0}^{\pm}, \qquad a_{0,0}^{\pm} = 1, \qquad q \geqslant 1, \tag{A7}$$

$$a_{2q,2}^{\pm} = f^{\pm}(2q-2) a_{2q-2,2}^{\pm} + g^{\pm}(2q) a_{2q,0}^{\pm}, \qquad a_{0,2}^{\pm} = 0, \qquad q \geqslant 1, \tag{A8}$$

$$a_{2q,2k}^{\pm} = f^{\pm}(2q-2) a_{2q-2,2k}^{\pm} + g^{\pm}(2q) a_{2q,2k-2}^{\pm} + h^{\pm}(2q) \sum_{j=2}^{k} l_{2j}\, a_{2q,2k-2j}^{\pm}$$

$$+ p^{\pm}(2q+2) a_{2q+2,2k-4}^{\pm}, \qquad a_{0,2k}^{\pm} = 0, \quad k \geqslant 2, q \geqslant 1, \tag{A9}$$

where the following rotational substitutions have been made:

$$h^{\pm}(2q) = \pm 1/2q(2n \pm 2q + 1), \tag{A10}$$

$$f^{\pm}(2q-2) = -\frac{[n + 1 \pm (2q - m - 1)]\,[n \pm (2q - m - 1)]}{[2n \pm (4q-3)]\,[2n + 2 \pm (4q-3)]} h^{\pm}(2q), \tag{A11}$$

$$g^{\pm}(2q) = \left[l_2 - \frac{2(n \pm 2q)(n \pm 2q + 1) - 2m^2 - 1}{(2n \pm 4q - 1)(2n \pm 4q + 3)} \right] h^{\pm}(2q)$$

$$= \frac{2(1 - 4m^2)}{(2n-1)(2n+3)(2n \pm 4q - 1)(2n \pm 4q + 3)}, \tag{A12}$$

$$p^{\pm}(2q+2) = -\frac{[n+1\pm(2q+m+1)]\,[n\pm(2q+m+1)]}{[2n\pm(4q+3)]\,[2n+2\pm(4q+3)]}\,h^{\pm}(2q). \tag{A13}$$

From equation (A8) with $q=1$ one obtains

$$a^{\pm}_{2,2} = g^{\pm}(2)a^{\pm}_{2,0} = v^{\pm}_{2}\,(2)a^{\pm}_{2,0}, \qquad v^{\pm}_{2}\,(2) = g^{\pm}(2). \tag{A14}$$

Setting also for $q \geqslant 2$

$$a^{\pm}_{2q,2} = v^{\pm}_{2}\,(2q)a^{\pm}_{2q,0}, \tag{A15}$$

one obtains from equations (A7), (A8) and (A15) that

$$a^{\pm}_{2q,2} = [v^{\pm}_{2}\,(2q-2) + g^{\pm}(2q)]a^{\pm}_{2q,0}. \tag{A16}$$

From equations (A15) and (A16) one finds the relation

$$v^{\pm}_{2}\,(2q) = v^{\pm}_{2}\,(2q-2) + g^{\pm}(2q), \qquad q \geqslant 1, \tag{A17}$$

where $v^{\pm}_{2}\,(0) = 0$ and finally

$$v^{\pm}_{2}\,(2q) = \sum_{\mathrm{i}=1}^{q} g^{\pm}(2\mathrm{i}) = \frac{2q(1-4m^2)}{(2n+1\mp 2)\,(2n+1\mp 2)^2[2n+1\pm(4q+2)]}, \qquad q \geqslant 0. \tag{A18}$$

Setting now for $q \geqslant 1,\ k \geqslant 2$,

$$a^{\pm}_{2q,2k} = v^{\pm}_{2k}\,(2q)a^{\pm}_{2q,0}, \tag{A19}$$

and following the same procedure as before, by using the result $v^{\pm}_{2k}\,(0) = 0$ for $k \geqslant 1$, one finally obtains

$$v^{\pm}_{2k}\,(2q) = \sum_{i=1}^{q}\Bigg[g^{\pm}(2i)v^{\pm}_{2k-2}\,(2i) + h^{\pm}(2i)\sum_{j=2}^{k} l_{2j}\,v^{\pm}_{2k-2j}\,(2i) + p^{\pm}(2i+2) \\ \times v^{\pm}_{2k-4}\,(2i+2)f^{\pm}(2i)\Bigg], \qquad q \geqslant 1,\ k \geqslant 2. \tag{A20}$$

The recurrence relation (A20) can be used for the calculation of $v^{\pm}_{2k}\,(2q)$, and consequently of $a^{\pm}_{2q,2k}$ by using the expressions for $v^{\pm}_{2k-2s}\,(2i)$, $i \geqslant 1,\ 1 \leqslant s \leqslant k$. The coefficients $l_{2j}\,(2 \leqslant j \leqslant k)$ are calculated from equation (A6), by using $a^{\pm}_{2,2j-4}$ from equation (A19) for $q=1$ (for $j \leqslant 5$ they are also found in references [6, 9, 11]).

For large values of k, $v^{\pm}_{2k}\,(2q)$ is obtained only numerically from equation (A20). However, for small values of k, analytical closed-form expressions are obtained, valid for each n, m and q. So, for $k = 2$, 3 one finds after very lengthy manipulation, the expressions

$$v^{+}_{4}\,(2q) = -\frac{(1-4m^2)^2 q}{4(2n-1)\,(2n+3)^4(2n+4q+3)} + \frac{[(n-1)^2-m^2]\,(n^2-m^2)q}{4(2n-3)\,(2n-1)^3(2n+1)^2(q+1)} \\ + \frac{[(n+1)^2-m^2]\,[(n+2)^2-m^2]q}{(2n+1)^2(2n+3)^4(2n+5)\,(2n+2q+3)} \\ + \frac{(9-4m^2)\,(1-4m^2)q}{4(2n-3)\,(2n+1)^2(2n+5)^2(2n+4q+5)}, \qquad q \geqslant 0, \tag{A21a}$$

$$v_4^-(2q) = -\frac{(1-4m^2)^2 q}{4(2n-1)^4(2n+3)(2n-4q-1)} + \frac{[(n+1)^2-m^2][(n+2)^2-m^2]q}{4(2n+1)^2(2n+3)^3(2n+5)(q+1)}$$

$$+ \frac{(n^2-m^2)[(n-1)^2-m^2]q}{(2n-3)(2n-1)^4(2n+1)^2(2n-2q-1)}$$

$$+ \frac{(9-4m^2)(1-4m^2)q}{4(2n-3)^2(2n+1)^2(2n+5)(2n-4q-3)}, \qquad q \geqslant 0, \tag{A21b}$$

$$v_6^+(2q) = \frac{(1-4m^2)[(n-1)^2-m^2](n^2-m^2)}{2(2n-5)(2n-3)(2n-1)^5(2n+1)^2(2n+3)}\frac{q}{q+1}$$

$$+ \left\{\frac{(1-4m^2)^2}{32(2n-1)(2n+3)^6} + \frac{[(n-1)^2-m^2](n^2-m^2)}{2(2n-5)(2n-3)(2n-1)^3(2n+1)^2(2n+3)^2}\right.$$

$$- \frac{[(n+1)^2-m^2][(n+2)^2-m^2]}{(2n+1)^2(2n+3)^6(2n+5)(2n+7)}$$

$$\left. - \frac{(9-4m^2)(1-4m^2)}{48(2n-3)(2n+1)^2(2n+3)^2(2n+5)^2}\right\}$$

$$\times \frac{(1-4m^2)q}{2n+4q+3} + \frac{(1-4m^2)[(n+1)^2-m^2][(n+2)^2-m^2]}{(2n-1)(2n+1)(2n+3)^6(2n+5)}$$

$$\times \left[\frac{2}{(2n+1)(2n+7)} + \frac{1}{2n-1}\right]\frac{q}{2n+2q+3}$$

$$- \frac{(9-4m^2)(1-4m^2)^2}{48(2n-1)(2n+1)^2(2n+3)^2(2n+5)^2}$$

$$\times \frac{q}{2n+4q+5} + \frac{(1-4m^2)(9-4m^2)(25-4m^2)}{96(2n-5)(2n-1)^2(2n+3)^2(2n+7)^2}\frac{q}{2n+4q+7}, \tag{A22a}$$

$$v_6^-(2q) = \frac{(1-4m^2)[(n+1)^2-m^2][(n+2)^2-m^2]}{2(2n-1)(2n+1)^2(2n+3)^5(2n+5)(2n+7)}\frac{q}{q+1}$$

$$+ \left\{\frac{(1-4m^2)^2}{32(2n-1)^6(2n+3)} + \frac{[(n+1)^2-m^2][(n+2)^2-m^2]}{2(2n-1)^2(2n+1)^2(2n+3)^3(2n+5)(2n+7)}\right.$$

$$+ \frac{[(n-1)^2-m^2](n^2-m^2)}{(2n-5)(2n-3)(2n-1)^6(2n+1)^2}$$

$$\left. - \frac{(9-4m^2)(1-4m^2)}{48(2n-3)^2(2n-1)^2(2n+1)^2(2n+5)}\right\}$$

$$\times \frac{(1-4m^2)q}{2n-4q-1} + \frac{(1-4m^2)[(n-1)^2-m^2](n^2-m^2)}{(2n-3)(2n-1)^6(2n+1)(2n+3)}$$

$$\times \left[\frac{2}{(2n-5)(2n+1)} - \frac{1}{2n+3}\right]\frac{q}{2n-2q-1}$$

$$
-\frac{(9-4m^2)\,(1-4m^2)^2}{48(2n-3)^2(2n-1)^2(2n+1)^2(2n+3)}
$$

$$
\times\frac{q}{2n-4q-3}+\frac{(1-4m^2)\,(9-4m^2)\,(25-4m^2)}{96(2n-5)^2(2n-1)^2(2n+3)^2(2n+7)}\frac{q}{2n-4q-5}. \quad \text{(A22b)}
$$

Furthermore from equations (A7), (A10) and (A11) one obtains easily

$$
a^{+}_{2q,0}=(-1)^q\frac{(2n-1)!!(2n+1)!!(n-m+2q)!}{2^q q!(n-m)!(2n+4q-1)!!(2n+2q+1)!!}, \qquad q\geqslant 0, \quad \text{(A23a)}
$$

$$
a^{-}_{2q,0}=\frac{(n+m)!(2n-2q-1)!!(2n-4q+1)!!}{2^q q!(2n-1)!!(2n+1)!!(n+m-2q)!}, \qquad q\geqslant 0, \quad \text{(A23b)}
$$

while from equations (A15), (A18) and (A23) one has the results

$$
a^{+}_{2q,2}=(-1)^q\frac{2q(1-4m^2)\,(2n-1)!!(2n+1)!!(n-m+2q)!(2n+4q+1)}{2^q q!(2n-1)\,(2n+3)^2(n-m)!(2n+2q+1)!!(2n+4q+3)!!}, \qquad q\geqslant 0, \quad \text{(A24a)}
$$

$$
a^{-}_{2q,2}=\frac{2q(1-4m^2)\,(n+m)!(2n-2q-1)!!(2n-4q+1)!!}{2^q q!(2n-1)^2(2n+3)\,(2n-1)!!(2n+1)!!(n+m-2q)!(2n-4q-1)}, \qquad q\geqslant 0. \quad \text{(A24b)}
$$

Finally, from equation (A19) with $k=2$ and 3 one obtains $a^{\pm}_{2q,4}$ and $a^{\pm}_{2q,6}$, by using equations (A21) and (A22), respectively, with equations (A23).

The explicit values of the ds depend on the normalization used.

The various calculated expansion coefficients are also useful for the evaluation of the spheroidal radial functions of any kind $\mathrm{R}^{(\sigma)}_{mn}(c,\xi)$, $\sigma=1$–4, where [1, 6]

$$
\mathrm{R}^{(\sigma)}_{mn}(c,\xi)=\frac{(n-m)!}{(n+m)!}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\sum_{r=0,1}^{\infty}{}' \,\mathrm{i}^{r+m-n}d_r(c)\frac{(2m+r)!}{r!}z^{(\sigma)}_{m+r}(c\xi), \quad \text{(A25)}
$$

as well as for the evaluation of the angular spheroidal functions of the second kind $\mathrm{S}^{(2)}_{mn}(c,\eta)$, given by the expansion [6]

$$
\mathrm{S}^{(2)}_{mn}(c,\eta)=\sum_{r=-2m,-2m+1}^{\infty}{}' \,d_r\,\mathrm{Q}^m_{m+r}(\eta)+\sum_{r=2m+2,2m+1}^{\infty}{}' \,d_{\rho|r}\,\mathrm{P}^m_{r-m-1}(\eta). \quad \text{(A26)}
$$

In equation (A26) Q^m_s are the associated Legendre functions of the second kind. The coefficients d_r, $r\geqslant 0$ are the same as the ones already calculated. The coefficients d_r, $-2m\leqslant r<0$, are given also by the same formulas, with the lower (minus) sign, but with $n-m<2q\leqslant n+m$ now, while $d_{\rho|r}$ ($\rho=$ positive) are given by the limit

$$
d_{\rho|r}=\lim_{\rho\to 0}\frac{d-r+\rho}{\rho}, \qquad r>2m, \quad \text{(A27)}
$$

which can be calculated from equation (A3) with the lower sign. In this case $-r=n-m-2q<-2m$, or $2q>n+m$. So, with $r=2m+1$ $(2q=n+m+1)$ and $r=2m+2$ $(2q=n+m+2)$, respectively, one obtains from equations (A7), (A10) and (A11)

$$
\lim_{\rho\to 0}\frac{a^{-}_{2q-\rho,0}}{\rho}=\frac{1}{(2m-3)\,(2m-1)\,(n-m)\,(n+m+1)}a^{-}_{2q-2,0}=a^{-}_{\rho|2q,0}, \quad \text{(A28)}
$$

$$\lim_{\rho\to 0}\frac{a^{-}_{2q-\rho,0}}{\rho} = -\frac{1}{(2m-1)(2m+1)(n-m-1)(n+m+2)}\,a^{-}_{2q-2,0} = a^{-}_{\rho|2q,0}, \quad \text{(A29)}$$

where $a^{-}_{2q-2,0}$ for $r = 2m+1, 2m+2$ is calculated from equation (A23b) ($2q-2 \leqslant n+m$ in these cases).

Finally, from equations (A7), (A10), (A11) and (A28), (A29) one obtains the expansion coefficients $a^{-}_{\rho|2(q+t),0} = Aa^{-}_{\rho|2q,0}$ where q takes the special values used in equations (A28) and (A29), $t = 0, 1, 2, \dots$, and

$$A = \left\{\begin{array}{ll} (2t)!\dfrac{(2m-1)!!}{(2m-1+4t)!!}\dfrac{(n+m+1)!!}{(n+m+1+2t)!!}\dfrac{(n-m-2-2t)!!}{(n-m-2)!!}, & 2q = n+m+1 \\ (2t+1)!\dfrac{(2m+1)!!}{(2m+1+4t)!!}\dfrac{(n+m+2)!!}{(n+m+2+2t)!!}\dfrac{(n-m-3-2t)!!}{(n-m-3)!!}, & 2q = n+m+2 \end{array}\right\}. \quad \text{(A30)}$$

The coefficients $a^{-}_{\rho|2(q+t),2k}$ can be found from the limits

$$\begin{aligned} a^{-}_{\rho|2(q+t),2k} &= \lim_{\rho\to 0}[a^{-}_{2(q+t)-\rho,2k}/\rho] \\ &= \lim_{\rho\to 0}\{v^{-}_{2k}[2(q+t)-\rho]a^{-}_{2(q+t)-\rho,0}/\rho\} = v^{-}_{2k}[2(q+t)]a^{-}_{\rho|2(q+t),0}, \\ & k \geqslant 1, \qquad t \geqslant 0. \end{aligned} \quad \text{(A31)}$$

Calculation of various expansion coefficients appearing in this Appendix verifies the results given in the literature [6, 9–12], thus confirming the validity of our procedure. For special values of the parameters the relations $(-1)!! = 1$ and $(-2s-1)!! = (-1)^s/(2s-1)!!$ for $s = 0, 1, 2, \dots$, have been used.

The power series expansions for the oblate angular functions are obtained from the corresponding formulas for the prolate ones, simply by replacing c by $-ic$ (equivalently c^2 by $-c^2$), while those for the oblate radial functions are obtained from the corresponding formulas for the prolate ones, simply by replacing c by $-ic$ and ξ by $i\xi$ [6].

APPENDIX B

The expressions for the various Ds appearing in equations (20) and (21) are the following (for the oblate spheroidal boundaries $D^{(2)}$s change their signs and R_2 is the minor semi-axis of the oblate spheroidal):

B.1. DIRICHLET BOUNDARY CONDITIONS

$$D^{0}_{nn} = u_{nn}(x_2, x_1) \quad \text{(B1)}$$

$$D^{(2)}_{nn} = \frac{x_2^2}{2(2n+1)}\left[\frac{(n+m+1)(n+m+2)}{(2n+3)^2}u_{n+2,n}(x_2, x_1) - \frac{(n-m-1)(n-m)}{(2n-1)^2}u_{n-2,n}(x_2, x_1)\right], \quad \text{(B2)}$$

$$D_{nn}^{(4)} = x_2^4 \frac{(n+m+1)(n+m+2)}{(2n+1)(2n+3)^2(2n+7)}\left[\frac{1-4m^2}{(2n-1)(2n+3)^2} u_{n+2,n}(x_2, x_1) + \frac{(n+m+3)(n+m+4)}{8(2n+5)^2} u_{n+4,n}(x_2, x_1)\right] - x_2^4 \frac{(n-m-1)(n-m)}{(2n-5)(2n-1)^2(2n+1)} \times \left[\frac{1-4m^2}{(2n-1)^2(2n+3)} u_{n-2,n}(x_2, x_1) - \frac{(n-m-3)(n-m-2)}{8(2n-3)^2} u_{n-4,n}(x_2, x_1)\right], \tag{B3}$$

$$D_{n+2,n}^{(2)} = x_2^2 \frac{(n+m+1)(n+m+2)}{2(2n+3)^2(2n+5)} u_{n+2,n}(x_2, x_1),$$

$$D_{n,n+2}^{(2)} = -x_2^2 \frac{(n-m+1)(n-m+2)}{2(2n+1)(2n+3)^2} u_{n,n+2}(x_2, x_1), \tag{B4}$$

where

$$u_{vs}(x_2, x_1) = \mathrm{j}_v(x_2) - \mathrm{n}_v(x_2)\mathrm{j}_s(x_1)/\mathrm{n}_s(x_1). \tag{B5}$$

B.2. NEUMANN BOUNDARY CONDITIONS

$$D_{nn}^{0} = x_2\, p_{nn}(x_2, x_1), \tag{B6}$$

$$D_{nn}^{(2)} = -x_2\, p_{nn}(x_2, x_1) + mq_{nn}(x_2, x_1) + x_2^3 \frac{(n+m+1)(n+m+2)}{2(2n+1)(2n+3)^2} p_{n+2,n}(x_2, x_1) - x_2^3 \frac{(n-m-1)(n-m)}{2(2n-1)^2(2n+1)} p_{n-2,n}(x_2, x_1) \tag{B7}$$

$$D_{nn}^{(4)} = x_2^2 \frac{(n+m+1)(n+m+2)}{2(2n+1)(2n+3)^2}\left\{-x_2\, p_{n+2,n}(x_2, x_1) + mq_{n+2,n}(x_2, x_1) + \frac{2x_2^3}{2n+7}\left[\frac{1-4m^2}{(2n-1)(2n+3)^2} p_{n+2,n}(x_2, x_1) + \frac{(n+m+3)(n+m+4)}{8(2n+5)^2} p_{n+4,n}(x_2, x_1)\right]\right\} - x_2^2 \frac{(n-m-1)(n-m)}{2(2n-1)^2(2n+1)}\left\{-x_2\, p_{n-2,n}(x_2, x_1) + mq_{n-2,n}(x_2, x_1) + \frac{2x_2^3}{2n-5} \times \left[\frac{1-4m^2}{(2n-1)^2(2n+3)} p_{n-2,n}(x_2, x_1) - \frac{(n-m-3)(n-m-2)}{8(2n-3)^2} p_{n-4,n}(x_2, x_1)\right]\right\}, \tag{B8}$$

$$D_{n+2,n}^{(2)} = x_2^3 \frac{(n+m+1)(n+m+2)}{2(2n+3)^2(2n+5)} p_{n+2,n}(x_2, x_1),$$

$$D_{n,n+2}^{(2)} = -x_2^3 \frac{(n-m+1)(n-m+2)}{2(2n+1)(2n+3)^2} p_{n,n+2}(x_2, x_1), \tag{B9}$$

where

$$p_{vs}(x_2, x_1) = \mathrm{j}_v'(x_2) - \mathrm{n}_v'(x_2)\mathrm{j}_s'(x_1)/\mathrm{n}_s'(x_1),$$
$$q_{vs}(x_2, x_1) = \mathrm{j}_v(x_2) - \mathrm{n}_v(x_2)\mathrm{j}_s'(x_1)/\mathrm{n}_s'(x_1). \tag{B10}$$