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A HYPERSINGULAR INTEGRAL FORMULATION FOR ACOUSTIC RADIATION IN MOVING FLOWS

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This paper presents a hypersingular integral equation for acoustic radiation in a subsonic uniform flow. The work is motivated by the need for a normal-derivative integral equation to be used in the Burton and Miller method for overcoming the non-uniqueness difficulty in the boundary integral formulation. Although the non-uniqueness difficulty in the conventional Helmholtz integral formulation has been well studied before, it is shown in this paper that this difficulty becomes more severe in the presence of a mean flow. A generalized normal-derivative operator is defined to derive the hypersingular integral equation. Regularization of the hypersingular kernels is performed to render the integral equation numerically integrable. Theoretical derivation is first given for a general three-dimensional formulation. The resulting hypersingular integral equation is then reduced to the axisymmetric case for numerical implementation. Numerical examples at relatively high frequencies and different Mach numbers are given to verify the formulation.

1. INTRODUCTION

Numerical simulation of sound radiation and scattering from structures submerged in a non-uniform flow field is an important topic in aeroacoustics. One application example in industry is the prediction of noise radiating from turbofan inlets of commercial aircraft engines in steady flight. So far, such numerical simulation has been performed by using the finite element method (FEM) [1–4] or the coupled finite element/boundary element method (FEM/BEM) [5, 6]. When the FEM is used alone, the numerical discretization is truncated somewhere in the far field, and the Sommerfeld radiation condition is approximated by using some kinds of infinite elements or the wave envelope elements [1–4].

On the other hand, the coupled FEM/BEM uses the FEM only in the non-uniform flow region. The Sommerfeld radiation condition is automatically satisfied when the BEM is attached to the FEM to model the acoustic radiation into the infinite, uniform flow region. The coupling between the FEM and the BEM is achieved by converting the BEM model into a radiation admittance matrix to be used as a boundary condition on the exterior boundary of the FEM model [6]. It should be noted that the admittance matrix created by the BEM is fully populated, and hence, represents a non-local boundary condition. Although theoretically the admittance matrix can better represent the radiation condition, it increases the bandwidth of the original FEM matrix. A variable bandwidth matrix solver (or skyline solver) may be used to reduce the memory storage and to speed up the solution process. An important feature of the coupled FEM/BEM is that the FEM discretization is confined to the non-uniform flow region only, which can be determined relatively easily once the flow field is known. For a slender structure, the non-uniform flow is confined in

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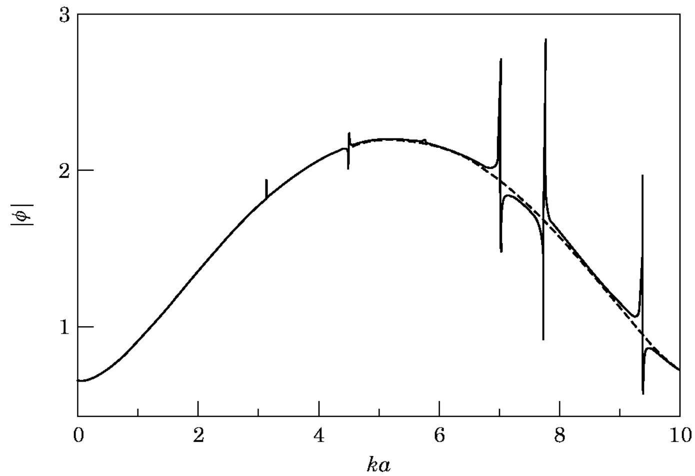

Figure 1. The non-uniqueness difficulty at $M = 0$.

a small region. Under such circumstances, the FEM discretization can be reduced to a minimum extent.

It is well known that the BEM based on the conventional Helmholtz integral equation fails to yield a unique solution at certain characteristic frequencies [7, 8]. This problem becomes much more severe in the presence of a mean flow. To demonstrate this difficulty, a BEM mesh for an imaginary sphere of radius $r = a$ submerged in a uniform flow is constructed by putting two out-of-phase point sources inside the sphere to create an analytical dipole solution. The velocity boundary condition on the imaginary sphere is generated by differentiating the dipole solution. Then the BEM solution at any nodal point on the boundary can be compared to the analytical dipole solution. Figure 1 shows such a comparison under the no-flow condition (i.e., Mach number $M = 0$). The solid line in the figure represents the BEM solution at one point on the sphere surface and the dash line represents the corresponding analytical solution. As expected, the BEM solution fails to yield a unique solution at $ka = \pi$, 4·493, 5·763, 2π . . . etc., where k is the wavenumber. To see the effect of the mean flow, the Mach number is raised to $M = 0{\cdot}5$ and the result

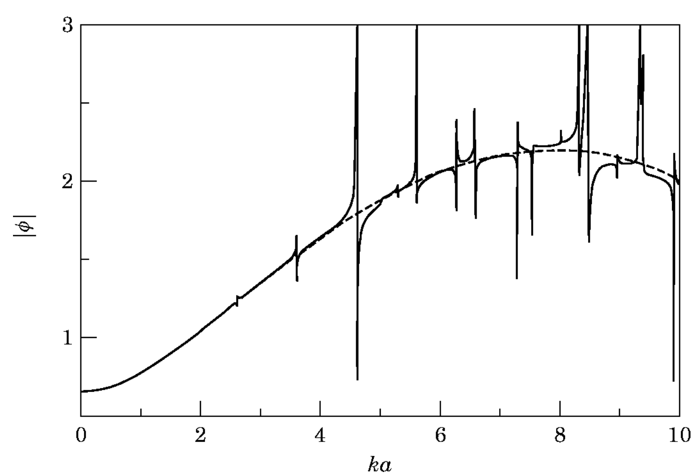

Figure 2. The non-uniqueness difficulty at $M = 0{\cdot}5$.

is shown in Figure 2. Comparing the two figures, one can easily see that Figure 2 has a lot more characteristic frequencies than Figure 1 in the same frequency range. As pointed out in reference [6], the frequency of the acoustic solution in a mean flow seems to be amplified by a factor of $1/(1-M^2)$. More severe non-uniqueness phenomena should be expected as the Mach number goes up.

To overcome the non-uniqueness difficulty in the presence of a mean flow, the theoretically robust formulation by Burton and Miller [8] is chosen. The original Burton and Miller method uses a linear combination of the Helmholtz integral equation and its normal derivative integral equation. Although each individual integral equation has its own characteristic frequencies, a linear combination of the two integral equations does yield a unique solution at all frequencies if the coupling constant is a pure imaginary number [8]. In the presence of a mean flow, the Helmholtz integral equation is replaced by a direct boundary integral equation developed recently by Wu and Lee [9]. A generalized normal derivative operator is then applied to this integral equation. It should be noted that the resulting generalized normal derivative integral equation is hypersingular. Regularization of the hypersingular integral is done by following the procedure originally developed by Krishnasamy *et al*. [10]. In this paper, a general three-dimensional formulation is first presented. The three-dimensional formulation is then reduced to the axisymmetric case for numerical implementation. Because the hypersingular integral equation requires the C^1 continuity condition at the collocation point, the Burton and Miller method is applied to the mid node of each isoparametric quadratic element only. This "reduced" version of the Burton and Miller method has been shown to be effective by Ingber and Hickox [11]. Several numerical examples at different Mach numbers and frequencies are given to verify the formulation.

2. THREE-DIMENSIONAL FORMULATION

The governing differential equation for steady-state linear acoustics in a uniform flow is [12, 13]

$$\nabla^2\phi + k^2\phi - 2\mathrm{i}kM\,\partial\phi/\partial x_1 - M^2\,\partial^2\phi/\partial x_1^2 = 0, \tag{1}$$

where ϕ is the velocity potential, k is the wavenumber, M is the Mach number of the uniform flow, $\mathrm{i}=\sqrt{-1}$, and the uniform flow is assumed to be in the positive x_1 direction. In equation (1), the $\mathrm{e}^{+\mathrm{i}\omega t}$ convention is adopted, where ω is the angular frequency. For a radiation problem in an infinite domain Ω, the "Helmholtz-type" boundary integral equations for equation (1) are given as follows [9]:

For $P\in\Omega$,

$$4\pi\phi(\mathbf{P}) = \int_S G\left(\frac{\partial\phi}{\partial n} - 2\mathrm{i}kM\phi n_1 - M^2\frac{\partial\phi}{\partial x_1}n_1\right)\mathrm{d}S - \int_S\left(\frac{\partial G}{\partial n} - M^2\frac{\partial G}{\partial x_1}n_1\right)\phi\,\mathrm{d}S. \tag{2}$$

For $P\in S$,

$$[4\pi - C^0(\mathbf{P})]\phi(\mathbf{P}) = \int_S G\left(\frac{\partial\phi}{\partial n} - 2\mathrm{i}kM\phi n_1 - M^2\frac{\partial\phi}{\partial x_1}n_1\right)\mathrm{d}S - \int_S\left(\frac{\partial G}{\partial n} - M^2\frac{\partial G}{\partial x_1}n_1\right)\phi\,\mathrm{d}S. \tag{3}$$

In both equations, **P** is the collocation point, S is the boundary surface, G is the Green's function derived from the adjoint operator of equation (1), **n** is the unit normal vector on S directing away from the acoustic domain, n_1 is the x_1 component of the vector **n**, and $C^0(\mathbf{P})$ is a coefficient that depends on the location of **P**. The explicit expressions for G and $C^0(\mathbf{P})$ are

$$G = \frac{\mathrm{e}^{-\mathrm{i}k\{\sqrt{(x_1-x_1^{\mathrm{P}})^2+(1-M^2)[(x_2-x_2^{\mathrm{P}})^2+(x_3-x_3^{\mathrm{P}})^2]}+M(x_1-x_1^{\mathrm{P}})\}/(1-M^2)}}{\sqrt{(x_1-x_1^{\mathrm{P}})^2+(1-M^2)[(x_2-x_2^{\mathrm{P}})^2+(x_3-x_3^{\mathrm{P}})^2]}}, \tag{4}$$

and

$$C^0(\mathbf{P}) = \int_S \left(\frac{\partial G_0}{\partial n} - M^2 \frac{\partial G_0}{\partial x_1} n_1\right) \mathrm{d}S, \tag{5}$$

respectively, where the co-ordinates without a superscript are the co-ordinates of any surface point **Q**, and those with a superscript $^{\mathrm{P}}$ are the co-ordinates of the collocation point **P**. The meaning of the Green's function (derived from the adjoint operator) is the solution due to a point source of strength 4π at **P** in a uniform flow moving in the negative x_1 direction, although the flow of the physical problem is indeed in the positive x_1 direction. In equation (5), G_0 is the "static" Green's function, that is,

$$G_0 = G|_{k=0} = \frac{1}{\sqrt{(x_1-x_1^{\mathrm{P}})^2+(1-M^2)[(x_2-x_2^{\mathrm{P}})^2+(x_3-x_3^{\mathrm{P}})^2]}}. \tag{6}$$

It is noticed that the first integral in equation (2) contains only the weakly singular kernel G, and hence, produces no jump when **P** is approaching the boundary S from Ω. The second integral, however, does produce a jump. The jump of a singular integral is the contribution due to integration over an infinitesimal area on S surrounding **P** when **P** is approaching the boundary. Comparing equations (2) and (3), one can easily construct a jump theorem as follows:

Jump theorem: For any smooth density function σ defined on S, the jump of

$$\int_S \left(\frac{\partial G}{\partial n} - M^2 \frac{\partial G}{\partial x_1} n_1\right)\sigma \,\mathrm{d}S$$

as **P** approaches S from Ω is $-C^0(\mathbf{P})\sigma(\mathbf{P})$.

Since the derivatives of G with respect to the co-ordinates of **P** are simply the negatives of the corresponding derivatives of G with respect to the co-ordinates of Q, one has the following corollary from the jump theorem.

Corollary: For any smooth density function σ defined on S, the jump of

$$\int_S \left(\frac{\partial G}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial G}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\sigma \,\mathrm{d}S$$

as **P** approaches S from Ω is $C^0(\mathbf{P})\sigma(\mathbf{P})$.

To derive the normal derivative integral to be used in the Burton and Miller formulation, a generalized normal-derivative operator is first defined, $(\partial/\partial n^{\mathrm{P}} - M^2(\partial/\partial x_1^{\mathrm{P}})n_1^{\mathrm{P}})$, and this operator is applied to equation (2) (where **P** is still in the domain Ω). Doing so yields

$$4\pi\left(\frac{\partial \phi}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial \phi}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\Bigg|_{\mathbf{P}\in\Omega} = \int_S \left(\frac{\partial G}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial G}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\left(\frac{\partial \phi}{\partial n} - 2\mathrm{i}kM\phi n_1 - M^2 \frac{\partial \phi}{\partial x_1} n_1\right) \mathrm{d}S$$

$$-\int_S \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\left(\frac{\partial G}{\partial n} - M^2 \frac{\partial G}{\partial x_1} n_1\right)\phi \, \mathrm{d}S. \tag{7}$$

One then lets **P** approach the surface S and applies the jump corollary to the first integral of equation (7). For $\mathbf{P} \in S$, the "normal-derivative" integral equation becomes

$$[4\pi - C^0(\mathbf{P})]\left[\frac{\partial \phi}{\partial n^{\mathrm{P}}}(\mathbf{P}) - M^2 \frac{\partial \phi}{\partial x_1^{\mathrm{P}}}(\mathbf{P}) n_1^{\mathrm{P}}\right] + C^0(\mathrm{P})[2\mathrm{i}kMn_1^{\mathrm{P}}\phi(\mathrm{P})]$$

$$= \int_S \left(\frac{\partial G}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial G}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\left(\frac{\partial \phi}{\partial n} - 2\mathrm{i}kM\phi n_1 - M^2 \frac{\partial \phi}{\partial x_1} n_1\right) \mathrm{d}S$$

$$- \,\not{\!\!\int}_S \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\left(\frac{\partial G}{\partial n} - M^2 \frac{\partial G}{\partial x_1} n_1\right)\phi \, \mathrm{d}S \tag{8}$$

where $\not{\!\!\int}_S$ represents a hypersingular integral that should be interpreted only in the Hadmard finite-part sense. Notice that the singularity of the first integral of equation (8) is in the same order as that of the second integral of equation (3). It has been shown in reference [9] that such a singularity is of the order of $1/r$ only (not $1/r^2$), where r is the distance between **P** and Q. Numerical integration of the $1/r$ singularity can be easily done by a simple polar co-ordinate transformation. The kernel of the second integral of equation (8) (the hypersingular integral), however, is in the order of $1/r^3$ and requires special treatment. To do this, one breaks the surface S into two regions: the singular region ΔS and the non-singular region $S - \Delta S$. The singular region ΔS is a small (but finite) region that contains the singular point **P**, and the non-singular region $S - \Delta S$ represents the rest of the boundary surface. Now attention is focused on the hypersingular integral over ΔS. To regularize this integral, one first subtracts the static Green's function G_0 from the Green's function G and then adds it back. This procedure has been routinely used as the first step in regularizing hypersingular integral equations (Krishnasamy *et al*. [10], Chien *et al*. [14]). The hypersingular integral over ΔS becomes

$$\not{\!\!\int}_{\Delta S} \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\left(\frac{\partial G}{\partial n} - M^2 \frac{\partial G}{\partial x_1} n_1\right)\phi \, \mathrm{d}S$$

$$= \int_{\Delta S} \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\left(\frac{\partial (G - G_0)}{\partial n} - M^2 \frac{\partial (G - G_0)}{\partial x_1} n_1\right)\phi \, \mathrm{d}S$$

$$+ \,\not{\!\!\int}_{\Delta S} \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\left(\frac{\partial G_0}{\partial n} - M^2 \frac{\partial G_0}{\partial x_1} n_1\right)\phi \, \mathrm{d}S. \tag{9}$$

Due to cancellation of the singularity between G_0 and G, the first integral on the right side of equation (9) is non-singular. Although the last integral is still hypersingular, the singularity is now passed to the "static" Green's function G_0, which has a much simpler form than G. By subtracting and adding back the first two terms of the Taylor series expansion of ϕ about point **P**, the last integral of equation (9) becomes

$$\begin{aligned}
&\fint_{\Delta S}\left(\frac{\partial}{\partial n^{\mathrm{P}}}-M^2\frac{\partial}{\partial x_1^{\mathrm{P}}}n_1^{\mathrm{P}}\right)\left(\frac{\partial G_0}{\partial n}-M^2\frac{\partial G_0}{\partial x_1}n_1\right)\phi\,\mathrm{d}S\\
&\quad=\int_{\Delta S}\left(\frac{\partial}{\partial n^{\mathrm{P}}}-M^2\frac{\partial}{\partial x_1^{\mathrm{P}}}n_1^{\mathrm{P}}\right)\left(\frac{\partial G_0}{\partial n}-M^2\frac{\partial G_0}{\partial x_1}n_1\right)\left\{\phi-\phi(\mathbf{P})-\frac{\partial\phi}{\partial x_k}(\mathbf{P})[x_k-x_k(\mathbf{P})]\right\}\mathrm{d}S\\
&\quad+\phi(\mathbf{P})\fint_{\Delta S}\left(\frac{\partial}{\partial n^{\mathrm{P}}}-M^2\frac{\partial}{\partial x_1^{\mathrm{P}}}n_1^{\mathrm{P}}\right)\left(\frac{\partial G_0}{\partial n}-M^2\frac{\partial G_0}{\partial x_1}n_1\right)\mathrm{d}S\\
&\quad+\fint_{\Delta S}\left(\frac{\partial}{\partial n^{\mathrm{P}}}-M^2\frac{\partial}{\partial x_1^{\mathrm{P}}}n_1^{\mathrm{P}}\right)\left(\frac{\partial G_0}{\partial n}-M^2\frac{\partial G_0}{\partial x_1}n_1\right)\left\{\frac{\partial\phi}{\partial x_k}(\mathbf{P})[x_k-x_k(\mathbf{P})]\right\}\mathrm{d}S,
\end{aligned}\tag{10}$$

where $\fint$ represents a Cauchy principal value (CPV) integral. One should notice that this step does require the C^1 continuity condition on the variable ϕ at the collocation point because the first order derivative of ϕ at $\mathbf{P}$ is being used. With the subtraction of the first two Taylor series expansion terms, the integrand of the first integral on the right side of equation (10) is in the order of $1/r$, which is only weakly singular. Regularization of the other two integrals on the right side of equation (10) requires the use of Stokes' theorem [10] and this procedure is given in Appendix A. With the results of Appendix A, equation (10) becomes

$$\begin{aligned}
&\fint_{\Delta S}\left(\frac{\partial}{\partial n^{\mathrm{P}}}-M^2\frac{\partial}{\partial x_1^{\mathrm{P}}}n_1^{\mathrm{P}}\right)\left(\frac{\partial G_0}{\partial n}-M^2\frac{\partial G_0}{\partial x_1}n_1\right)\phi\,\mathrm{d}S\\
&\quad=\int_{\Delta S}\left(\frac{\partial}{\partial n^{\mathrm{P}}}-M^2\frac{\partial}{\partial x_1^{\mathrm{P}}}n_1^{\mathrm{P}}\right)\left(\frac{\partial G_0}{\partial n}-M^2\frac{\partial G_0}{\partial x_1}n_1\right)\left\{\phi-\phi(\mathbf{P})-\frac{\partial\phi}{\partial x_k}(\mathbf{P})[x_k-x_k(\mathbf{P})]\right\}\mathrm{d}S\\
&\quad+\frac{\partial\phi}{\partial x_i}(\mathbf{P})\beta_i\int_{\Delta S}\left(\frac{\partial G_0}{\partial n^{\mathrm{P}}}-M^2\frac{\partial G_0}{\partial x_1^{\mathrm{P}}}n_1^{\mathrm{P}}\right)n_i\,\mathrm{d}S\\
&\quad+\left[\frac{\partial\phi}{\partial n}(\mathbf{P})-M^2\frac{\partial\phi}{\partial x_1}(\mathbf{P})n_1^{\mathrm{P}}\right]\int_{\Delta S}\left(\frac{\partial G_0}{\partial n}-M^2\frac{\partial G_0}{\partial x_1}n_1\right)\mathrm{d}S\\
&\quad-\phi(\mathbf{P})\beta_j\beta_i n_j^{\mathrm{P}}\varepsilon_{rij}\oint_C\frac{\partial G_0}{\partial x_i}\,\mathrm{d}x_r-\frac{\partial\phi}{\partial x_k}(\mathbf{P})\beta_j\beta_i n_j^{\mathrm{P}}\varepsilon_{rij}\oint_C\frac{\partial G_0}{\partial x_i}[x_k-x_k(\mathbf{P})]\,\mathrm{d}x_r\\
&\quad-\frac{\partial\phi}{\partial x_i}(\mathbf{P})\beta_j\beta_i n_j^{\mathrm{P}}\varepsilon_{rji}\oint_C G_0\,\mathrm{d}x_r,
\end{aligned}\tag{11}$$

where $\beta_1=1-M^2$, $\beta_2=\beta_3=1$, ε_{rij} is the alternating symbol, C is the contour along the edge of ΔS, and the summation convention is used for repeated indices.

Substituting equation (11) into equation (9) and then equation (9) into equation (8), one finally has the regularized "normal derivative" integral equation for acoustic radiation in a mean flow:

$$
\begin{aligned}
&[4\pi - C^0(\mathbf{P})]\left[\frac{\partial \phi}{\partial n}(\mathbf{P}) - M^2 \frac{\partial \phi}{\partial x_1}(\mathbf{P}) n_1^{\mathrm{P}}\right] + C^0(\mathbf{P})[2\mathrm{i}kMn_1^{\mathrm{P}}\phi(\mathbf{P})] \\
&\quad = \int_S \left(\frac{\partial G}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial G}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\left(\frac{\partial \phi}{\partial n} - 2\mathrm{i}kM\phi n_1 - M^2 \frac{\partial \phi}{\partial x_1} n_1\right) \mathrm{d}S \\
&\quad - \int_{S-\Delta S} \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\left(\frac{\partial G}{\partial n} - M^2 \frac{\partial G}{\partial x_1} n_1\right)\phi \, \mathrm{d}S \\
&\quad - \int_{\Delta S} \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\left(\frac{\partial (G-G_0)}{\partial n} - M^2 \frac{\partial (G-G_0)}{\partial x_1} n_1\right)\phi \, \mathrm{d}S \\
&\quad - \int_{\Delta S} \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right)\left(\frac{\partial G_0}{\partial n} - M^2 \frac{\partial G_0}{\partial x_1} n_1\right)\left\{\phi - \phi(\mathbf{P}) - \frac{\partial \phi}{\partial x_k}(\mathbf{P})[x_k - x_k(\mathbf{P})]\right\} \mathrm{d}S \\
&\quad - \frac{\partial \phi}{\partial x_i}(\mathbf{P})\beta_i \int_{\Delta S} \left(\frac{\partial G_0}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial G_0}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right) n_i \, \mathrm{d}S \\
&\quad - \left[\frac{\partial \phi}{\partial n}(\mathbf{P}) - M^2 \frac{\partial \phi}{\partial x_1}(\mathbf{P}) n_1^{\mathrm{P}}\right] \int_{\Delta S} \left(\frac{\partial G_0}{\partial n} - M^2 \frac{\partial G_0}{\partial x_1} n_1\right) \mathrm{d}S \\
&\quad + \phi(\mathbf{P})\beta_j\beta_i n_j^{\mathrm{P}}\varepsilon_{rij} \oint_C \frac{\partial G_0}{\partial x_i}\,\mathrm{d}x_r + \frac{\partial \phi}{\partial x_k}(\mathbf{P})\beta_j\beta_i n_j^{\mathrm{P}}\varepsilon_{rij} \oint_C \frac{\partial G_0}{\partial x_i}[x_k - x_k(\mathbf{P})]\,\mathrm{d}x_r \\
&\quad + \frac{\partial \phi}{\partial x_i}(\mathbf{P})\beta_j\beta_i n_j^{\mathrm{P}}\varepsilon_{rji} \oint_C G_0\,\mathrm{d}x_r .
\end{aligned} \tag{12}
$$

It is to be noticed that all the integrals in equation (12) are either regular or at most weakly singular.

3. AXISYMMETRIC FORMULATION

For an axisymmetric problem, one sets up a cylindrical co-ordinate system (ρ, θ, z) and lets the mean flow be in the z direction. One also lets $\theta^{\mathrm{P}} = 0$ so that the generator of the geometry lies in the xz-plane, although integration still has to be carried out over a true three-dimensional boundary surface. The differential surface area in the integral equation is evaluated by $\mathrm{d}S = \rho\,\mathrm{d}\theta\,\mathrm{d}\Gamma$, where Γ is the generator of the boundary surface S. One then evaluates the following two inner products for an axisymmetric problem:

$$\partial\phi/\partial x_k(\mathbf{P})[x_k - x_k(\mathbf{P})] = \partial\phi/\partial z(\mathbf{P})[z - z^{\mathrm{P}}] + \partial\phi/\partial\rho(\mathbf{P})[\rho\cos\theta - \rho^{\mathrm{P}}] \tag{13}$$

and

$$\partial\phi/\partial x_i(\mathbf{P}) n_i = \partial\phi/\partial z(\mathbf{P}) n_z + \partial\phi/\partial\rho(\mathbf{P}) n_\rho \cos\theta. \tag{14}$$

These two inner products are used in the fourth and the fifth integrals of equation (12), respectively. Substitute equations (13) and (14) into equation (12) and define three new Green's functions:

$$\tilde{G} = \int_0^{2\pi} G \, \mathrm{d}\theta, \qquad \tilde{G}_0 = \int_0^{2\pi} G_0 \, \mathrm{d}\theta, \qquad \tilde{G}_1 = \int_0^{2\pi} G_0 \cos\theta \, \mathrm{d}\theta. \tag{15–17}$$

Equation (12) then becomes

$$[4\pi - C^0(\mathbf{P})]\left[\frac{\partial \phi}{\partial n}(\mathbf{P}) - M^2 \frac{\partial \phi}{\partial z}(\mathbf{P}) n_z^{\mathrm{P}}\right] + C^0(\mathbf{P})[2\mathrm{i}kMn_z^p \phi(\mathbf{P})]$$

$$= \int_\Gamma \left(\frac{\partial \tilde{G}}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial \tilde{G}}{\partial z^{\mathrm{P}}} n_z^{\mathrm{P}}\right)\left(\frac{\partial \phi}{\partial n} - 2\mathrm{i}kM\phi n_z - M^2 \frac{\partial \phi}{\partial z} nz\right)\rho \, \mathrm{d}\Gamma$$

$$- \int_{\Gamma - \Delta\Gamma} \left[\int_0^{2\pi} \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial z^{\mathrm{P}}} n_z^{\mathrm{P}}\right)\left(\frac{\partial G}{\partial n} - M^2 \frac{\partial G}{\partial z} n_z\right) \mathrm{d}\theta\right] \phi\rho \, \mathrm{d}\Gamma$$

$$- \int_{\Delta\Gamma} \left[\int_0^{2\pi} \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial z^{\mathrm{P}}} n_z^{\mathrm{P}}\right)\left(\frac{\partial (G - G_0)}{\partial n} - M^2 \frac{\partial (G - G_0)}{\partial z} n_z\right) \mathrm{d}\theta\right] \phi\rho \, \mathrm{d}\Gamma$$

$$- \int_{\Delta\Gamma} \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial z^{\mathrm{P}}} n_z^{\mathrm{P}}\right)\left(\frac{\partial \tilde{G}_0}{\partial n} - M^2 \frac{\partial \tilde{G}_0}{\partial z} n_z\right)$$

$$\times \left[\phi - \phi(\mathbf{P}) - \frac{\partial \phi}{\partial z}(\mathbf{P})(z - z^{\mathrm{P}}) + \frac{\partial \phi}{\partial \rho}(\mathbf{P})\rho^{\mathrm{P}}\right] \rho \, \mathrm{d}\Gamma$$

$$+ \frac{\partial \phi}{\partial \rho}(\mathbf{P}) \int_{\Delta\Gamma} \left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial}{\partial z^{\mathrm{P}}} n_z^{\mathrm{P}}\right)\left(\frac{\partial \tilde{G}_1}{\partial n} - M^2 \frac{\partial \tilde{G}_1}{\partial z} n_z\right) \rho^2 \, \mathrm{d}\Gamma$$

$$- (1 - M^2) \frac{\partial \phi}{\partial z}(\mathbf{P}) \int_{\Delta\Gamma} \left(\frac{\partial \tilde{G}_0}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial \tilde{G}_0}{\partial z^{\mathrm{P}}} n_z^{\mathrm{P}}\right) n_z \rho \, \mathrm{d}\Gamma$$

$$- \frac{\partial \phi}{\partial \rho}(\mathbf{P}) \int_{\Delta\Gamma} \left(\frac{\partial \tilde{G}_1}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial \tilde{G}_1}{\partial z^{\mathrm{P}}} n_z^{\mathrm{P}}\right) n_\rho \rho \, \mathrm{d}\Gamma$$

$$- \left[\frac{\partial \phi}{\partial n}(\mathbf{P}) - M^2 \frac{\partial \phi}{\partial z}(\mathbf{P}) n_z^{\mathrm{P}}\right] \int_{\Delta\Gamma} \left(\frac{\partial \tilde{G}_0}{\partial n} - M^2 \frac{\partial \tilde{G}_0}{\partial z} n_z\right) \rho \, \mathrm{d}\Gamma$$

$$+ \phi(\mathbf{P}) \beta_j \beta_i n_j^{\mathrm{P}} \varepsilon_{rij} \oint_C \frac{\partial G_0}{\partial x_i} \, \mathrm{d}x_r + \frac{\partial \phi}{\partial x_k}(\mathbf{P}) \beta_j \beta_i n_j^{\mathrm{P}} \varepsilon_{rij} \oint_C \frac{\partial G_0}{\partial x_i} [x_k - x_k(\mathbf{P})] \, \mathrm{d}x_r$$

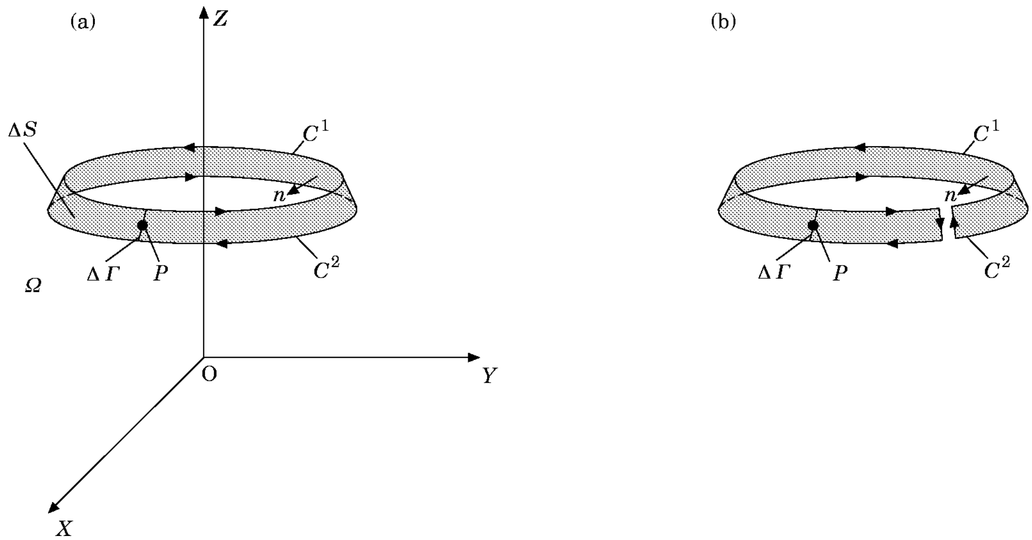

Figure 3. (a) Contour integration for an axisymmetric problem; (b) right-hand rule for the contour integration.

$$+\frac{\partial\phi}{\partial x_i}(\mathbf{P})\beta_j\beta_i n_j^{\mathrm{P}}\varepsilon_{rji}\oint_C G_0\,\mathrm{d}x_r, \tag{18}$$

where

$$C^0(\mathbf{P}) = \int_\Gamma\left(\frac{\partial\tilde{G}_0}{\partial n} - M^2\frac{\partial\tilde{G}_0}{\partial z}n_z\right)\rho\,\mathrm{d}\Gamma. \tag{19}$$

It should be noted that in the axisymmetric formulation, the singular surface element ΔS in equation (12) becomes a belt-like ring surface, as shown in Figure 3(a). Therefore, the contour C along the edge of ΔS actually consists of two circles, C_1 and C_2, at the top and bottom of the ring, respectively. Since the contour and the normal vector have to follow the right-hand rule (Appendix A), the direction of the contour integration is counterclockwise (from $\theta = 0$ to $\theta = 2\pi$) along C_1, and clockwise (from $\theta = 2\pi$ to $\theta = 0$) along C_2. This rule of thumb will be much easier to see if one imagines that the ring is cut to become a belt with two ends (see Figure 3(b)). The integration contributions along the two cut ends cancel out because they are equal and opposite. Since the normal vector of the belt surface is pointing inward (or away from the domain Ω), the direction of integration along C_1 and C_2 should become clear if one uses the right-hand rule.

The three axisymmetric Green's function of equations (15–17) can be evaluated by the complete elliptic integrals of the first and second kinds. This procedure is briefly described in Appendix B. A detailed discussion for the evaluation of $\tilde{G}$ and $\tilde{G}_0$ can be found in reference [6]. Here, one simply summarizes the results:

$$\tilde{G} = \mathrm{e}^{-\mathrm{i}\tilde{k}M(z-z^{\mathrm{P}})}\left[\tilde{G}_0 + \int_0^{2\pi}\frac{\mathrm{e}^{-\mathrm{i}\tilde{k}\tilde{r}} - 1}{\tilde{r}}\,\mathrm{d}\theta\right], \tag{20}$$

$$\tilde{G}_0 = \frac{4}{R}F\left(\frac{\pi}{2}, m\right), \qquad \tilde{G}_1 = \frac{4}{R}\left\{\frac{2}{m^2}\left[F\left(\frac{\pi}{2}, m\right) - E\left(\frac{\pi}{2}, m\right)\right] - F\left(\frac{\pi}{2}, m\right)\right\}, \quad (21, 22)$$

where $F(\pi/2, m)$ is the complete elliptic integral of the first kind, $E(\pi/2, m)$ is the complete elliptic integral of the second kind, and

$$\tilde{k} = k/(1 - M^2), \qquad \tilde{r} = \sqrt{(z - z^P)^2 + (1 - M^2)[\rho^2 + (\rho^P)^2 - 2\rho\rho^P \cos\theta]}, \quad (23, 24)$$

$$R = \sqrt{(z - z^P)^2 + (1 - M^2)(\rho + \rho^P)^2}, \qquad m^2 = 4(1 - M^2)\rho\rho^P/R^2. \quad (25, 26)$$

Equations (18) and (19) also contain first and second order derivatives of the Green's functions. This will in turn require derivatives of the elliptic integrals. The formulas for the first order derivatives are

$$\frac{\mathrm{d}F}{\mathrm{d}m} = \left[E\left(\frac{\pi}{2}, m\right) - (1 - m^2)F\left(\frac{\pi}{2}, m\right)\right]\Big/ m(1 - m^2), \qquad \frac{\mathrm{d}E}{\mathrm{d}m} = \left[E\left(\frac{\pi}{2}, m\right) - F\left(\frac{\pi}{2}, m\right)\right]\Big/ m. \quad (27, 28)$$

The second order derivatives will follow by differentiating the first order derivatives.

To apply the Burton and Miller method, one takes a linear combination of equation (3) (in an axisymmetric form, see reference [6]) and equation (18) with a pure-imaginary coupling constant i/k. Since equation (18) requires the C^1 continuity condition at **P**, the linear combination is taken only when **P** is collocated at the mid node of each three-noded quadratic element. When **P** is at one of the two end nodes, only equation (3) is used. This "reduced" version of the Burton and Miller method has been shown to be effective by Ingber and Hickox [11].

In numerical implementation, the generator Γ is discretized into a number of quadratic line elements. The co-ordinates ρ and z as well as the variables ϕ and $\partial\phi/\partial n$ at any point on the boundary are interpolated by a set of quadratic shape functions as:

$$\rho(\xi) = \sum_{\alpha=1}^{3} N_\alpha(\xi)\rho_\alpha, \qquad z(\xi) = \sum_{\alpha=1}^{3} N_\alpha(\xi) z_\alpha, \qquad \phi(\xi) = \sum_{\alpha=1}^{3} N_\alpha(\xi)\phi_\alpha, \quad (29a–c)$$

$$\frac{\partial\phi}{\partial n}(\xi) = \sum_{\alpha=1}^{3} N_\alpha(\xi)\left(\frac{\partial\phi}{\partial n}\right)_\alpha, \quad (29d)$$

where $N_\alpha(\xi)$ are the shape functions, ξ is the local co-ordinate, and the subscript α on ρ, z, ϕ, and $\partial\phi/\partial n$ denotes the corresponding nodal values.

Notice that the integral equations also contain the derivative terms, $\partial\phi/\partial z$ and $\partial\phi/\partial\rho$. Through a local co-ordinate transformation, the two derivatives can be converted into a linear combination of the normal derivative and the tangential derivative. The result is

$$\partial\phi/\partial z = (n_\rho/J)\,\partial\phi/\partial\xi + n_z\,\partial\phi/\partial n, \qquad \partial\phi/\partial\rho = -(n_z/J)\,\partial\phi/\partial\xi + n_\rho\,\partial\phi/\partial n, \quad (30, 31)$$

where

$$J = \sqrt{(\mathrm{d}\rho/\mathrm{d}\xi)^2 + (\mathrm{d}z/\mathrm{d}\xi)^2}. \quad (32)$$

The tangential derivative $\partial\phi/\partial\xi$ is obtained by differentiating the shape functions in equation (29c). Therefore, only ϕ and $\partial\phi/\partial n$ are retained as the nodal variables.

4. TEST CASES

The formulation given in the paper is also valid under the no-flow condition when M is set to zero. Under such circumstances, equation (3) reduces to the Helmholtz integral

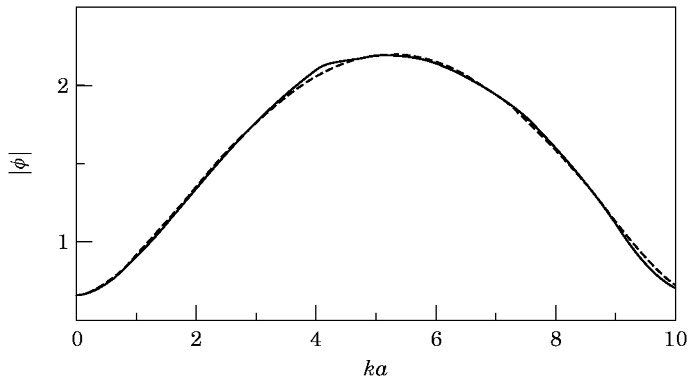

Figure 4. Comparison between the BEM solution (solid line) and the analytical solution (dash line) for the case of $M = 0$.

equation, and the generalized normal derivative integral equation becomes the standard normal derivative integral equation. The first test case is to overcome the non-uniqueness difficulty presented previously in Figure 1, where two out-of-phase point sources were placed inside an imaginary sphere surface. Eight quadratic elements with a total of 17 nodes are used to model this problem for frequencies up to $ka = 10$. It is noted that the Burton and Miller method is applied only at the mid node of each element. Therefore, there are only 8 Burton and Miller equations among the total of 17 equations. The result is shown in Figure 4. It is seen that all the non-uniqueness peaks are successfully removed.

The second test case is to overcome the non-uniqueness difficulty presented in Figure 2, where $M = 0{\cdot}5$. Since the Mach number is higher, one uses 32 quadratic elements with a total of 65 nodes to model the problem for frequencies up to $ka = 10$. As shown in Figure 5, all the non-uniqueness peaks are successfully removed. One also examines the directivity pattern of the solution at $ka = 4{\cdot}579$, which is one of the characteristic frequencies revealed in Figure 2. At this intermediate frequency, only 8 quadratic elements with a total of 17 nodes are used to model the problem. The BEM solution (asterisks) on the sphere surface is compared to the analytical solution (solid line) on a polar plot in Figure 6. It should be pointed out that the mean flow ($M = 0{\cdot}5$) is moving to the right in Figure 6. The mesh on the generator only produces the solution on the lower half of the polar plot. The upper half is obtained by reflecting the lower half.

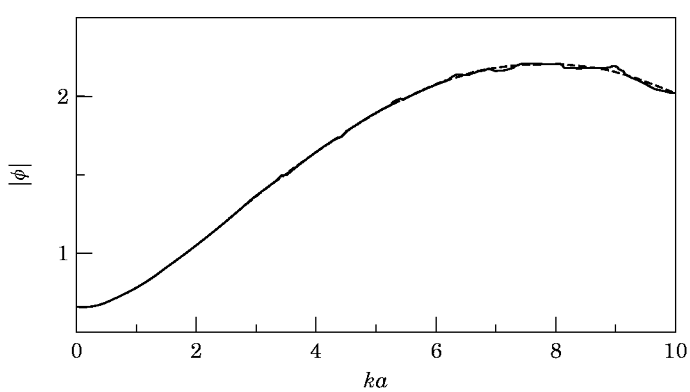

Figure 5. Comparison between the BEM solution (solid line) and the analytical solution (dash line) for the case of $M = 0{\cdot}5$.

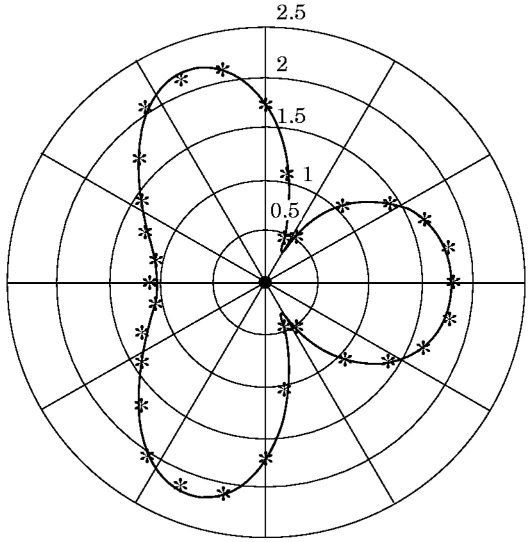

Figure 6. Comparison between the BEM solution (*) and the analyical solution (solid line) at $M = 0{\cdot}5$ and $ka = 4{\cdot}579$.

In the third test case, the Mach number is raised to $M = 0{\cdot}8$ and the same radiation problem is run for frequencies up to $ka = 10$. 72 quadratic elements with a total of 145 nodes are used to model the problem. The comparison between the BEM solution and the analytical solution is shown in Figure 7. The directivity pattern at $ka = 8$ is also shown in Figure 8. This problem is also solved by the conventional CHIEF method [6, 7]. The same accuracy is obtained by using six to twelve CHIEF points.

5. CONCLUSIONS

A generalized normal derivative integral equation is derived for three-dimensional acoustic radiation in a subsonic uniform flow. The hypersingular integral is regularized by Stokes' theorem. The three-dimensional formulation is then reduced to the axisymmetric case for numerical implementation. The generalized normal derivative integral equation is used in a reduced version of the Burton and Miller method to overcome the

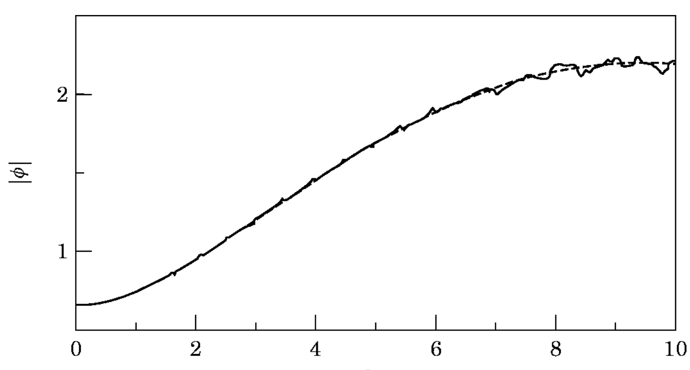

Figure 7. Comparison between the BEM solution (solid line) and the analytical solution (dash line) for the case of $M = 0{\cdot}8$.

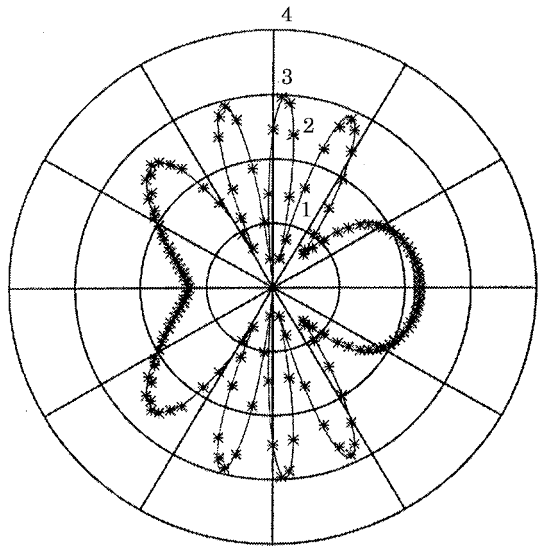

Figure 8. Comparison between the BEM solution (*) and the analytical solution (solid line) at $M = 0{\cdot}8$ and $ka = 8$.

non-uniqueness difficulty. Numerical results at different Mach numbers and frequencies show that this approach is effective. The hypersingular formulation derived in this paper is also valid under the no-flow condition by setting $M = 0$.

For all the axisymmetric test cases shown in this paper, the more conventional CHIEF method can also produce the same accuracy as the Burton and Miller method if enough CHIEF points are used. The CHIEF method is much easier to implement and is also less computational intensive due to its simplicity. However, there is always an uncertainty when deciding how many CHIEF points should be used. This really creates a burden on the user side. Furthermore, for truly three-dimensional problems, the interior modes at higher characteristic frequencies will become much more complicated. That means a three-dimensional interior space may require more CHIEF points than an axisymmetric interior area, even at the same frequency. Under such circumstances, the Burton and Miller method should be a more practical choice because it provides a "black-box" solution and requires no user input with regard to this matter.

ACKNOWLEDGMENT

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APPENDIX A: REGULARIZATION OF THE LAST TWO INTEGRALS OF EQUATION (10)

Let I represent the second integral on the right side of equation (10), that is,

$$I = \fint_{\Delta S}\left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2\frac{\partial}{\partial x_1^{\mathrm{P}}}n_1^{\mathrm{P}}\right)\left(\frac{\partial G_0}{\partial n} - M^2\frac{\partial G_0}{\partial x_1}n_1\right)\mathrm{d}S. \tag{A1}$$

Using the Cartesian tensor notation, equation (A1) is rewritten as

$$I = -\beta_j\,\beta_i n_j^{\mathrm{P}} \fint_{\Delta S}\frac{\partial^2 G_0}{\partial x_j\,\partial x_i}n_i\,\mathrm{d}S, \qquad i, j = 1, 2, 3, \tag{A2}$$

where $\beta_1 = 1 - M^2$, $\beta_2 = \beta_3 = 1$ and the summation convention is used for repeated indices. Since this integral is hypersingular, one tries to eliminate the singularity before **P** approaches the boundary surface. Subtracting and adding back the term $(\partial^2 G_0/\partial x_i^2)n_j$, one has

$$I = -\beta_j\beta_i n_j^{\mathrm{P}}\int_{\Delta S}\left(\frac{\partial^2 G_0}{\partial x_j\,\partial x_i}n_i - \frac{\partial^2 G_0}{\partial x_i^2}n_j\right)\mathrm{d}S - \beta_j\beta_i n_j^{\mathrm{P}}\int_{\Delta S}\frac{\partial^2 G_0}{\partial x_i^2}n_j\,\mathrm{d}S. \tag{A3}$$

Recalling that G_0 is the Green's function (or a fundamental solution) of the adjoint operator of equation (1) with $k = 0$ when **P** is still in the domain, the second integral of equation (A3) is simply zero because $\beta_i\, \partial^2 G_0/\partial x_i^2 = 0$ for **P** not exactly on S. One then applies Stokes' theorem [10],

$$\int_{\Delta S}\left(\frac{\partial F}{\partial x_j}n_i - \frac{\partial F}{\partial x_i}n_j\right)\mathrm{d}S = \varepsilon_{rij}\oint_C F\,\mathrm{d}x_r, \tag{A4}$$

where F is a function, ε_{rij} is the alternating symbol, and C is the contour along the edge of ΔS, to the first integral of equation (A3) with the substitution of $F = \partial G_0/\partial x_i$. The direction of the contour integration in Stokes' theorem should be taken in such a way that the contour C and the normal vector **n** follow the right-hand rule. Equation (A3) finally reduces to a summation of several non-singular contour integrals:

$$I = -\beta_j\beta_i n_j^{\mathrm{P}}\varepsilon_{rij}\oint_C \frac{\partial G_0}{\partial x_i}\,\mathrm{d}x_r. \tag{A5}$$

Since the singularity has been removed, one can now take the limit as **P** approaches the boundary surface S.

Let J represent the last integral of equation (10), that is,

$$J = \int\!\!\!\!\!\!-_{\Delta S}\left(\frac{\partial}{\partial n^{\mathrm{P}}} - M^2\frac{\partial}{\partial x_1^{\mathrm{P}}}n_1^{\mathrm{P}}\right)\left(\frac{\partial G_0}{\partial n} - M^2\frac{\partial G_0}{\partial x_1}n_1\right)\left\{\frac{\partial \phi}{\partial x_k}(\mathbf{P})[x_k - x_k(\mathbf{P})]\right\}\mathrm{d}S. \tag{A6}$$

Rewriting equation (A6) as

$$J = -\beta_j\beta_i n_j^{\mathrm{P}}\frac{\partial\phi}{\partial x_k}(\mathbf{P})\int\!\!\!\!\!\!-_{\Delta S}\frac{\partial^2 G_0}{\partial x_j\,\partial x_i}[x_k - x_k(\mathbf{P})]n_i\,\mathrm{d}S, \tag{A7}$$

or equivalently,

$$J = -\beta_j\beta_i n_j^{\mathrm{P}}\frac{\partial\phi}{\partial x_k}(\mathbf{P})\int\!\!\!\!\!\!-_{\Delta S}\frac{\partial}{\partial x_j}\left[\frac{\partial G_0}{\partial x_i}(x_k - x_k(\mathbf{P}))\right]n_i\,\mathrm{d}S$$

$$+\ \beta_j\beta_i n_j^{\mathrm{P}}\frac{\partial\phi}{\partial x_k}(\mathbf{P})\,\delta_{jk}\int_{\Delta S}\frac{\partial G_0}{\partial x_i}n_i\,\mathrm{d}S, \tag{A8}$$

where δ_{jk} is Kronecker's delta symbol. The second integral of equation (A8) is only weakly singular because its kernel can be written as

$$\beta_i(\partial G_0/\partial x_i)n_i = \partial G_0/\partial n - M^2(\partial G_0/\partial x_1)n_1, \tag{A9}$$

which is the same as the kernel in equation (5). The density distribution of that integral can also be written as

$$\beta_j n_j^{\mathrm{P}}\frac{\partial\phi}{\partial x_k}(\mathbf{P})\,\delta_{jk} = \beta_j n_j^{\mathrm{P}}\frac{\partial\phi}{\partial x_j}(\mathbf{P}) = \left[\frac{\partial\phi}{\partial n}(\mathbf{P}) - M^2\frac{\partial\phi}{\partial x_1}(\mathbf{P})n_1^{\mathrm{P}}\right]. \tag{A10}$$

Therefore, equation (A8) becomes

$$J = -\beta_j\beta_i n_j^{\mathrm{P}}\frac{\partial\phi}{\partial x_k}(\mathbf{P})\int\!\!\!\!\!\!-_{\Delta S}\frac{\partial}{\partial x_j}\left[\frac{\partial G_0}{\partial x_i}(x_k - x_k(\mathbf{P}))\right]n_i\,\mathrm{d}S$$

$$+\left[\frac{\partial\phi}{\partial n}(\mathbf{P})-M^2\frac{\partial\phi}{\partial x_1}(\mathbf{P})n_1^{\mathrm{P}}\right]\int_{\Delta S}\left(\frac{\partial G_0}{\partial n}-M^2\frac{\partial G_0}{\partial x_1}n_1\right)\mathrm{d}S. \tag{A11}$$

To regularize the first integral of equation (A11), one subtracts and adds back the term

$$\frac{\partial}{\partial x_i}\left[\frac{\partial G_0}{\partial x_i}(x_k-x_k(\mathbf{P}))\right]n_j$$

while **P** is still in the domain. Doing so yields

$$J=-\beta_j\beta_i n_j^{\mathrm{P}}\frac{\partial\phi}{\partial x_k}(\mathbf{P})\int_{\Delta S}\left\{\frac{\partial}{\partial x_j}\left[\frac{\partial G_0}{\partial x_i}(x_k-x_k(\mathbf{P}))\right]n_i-\frac{\partial}{\partial x_i}\left[\frac{\partial G_0}{\partial x_i}(x_k-x_k(\mathbf{P}))\right]n_j\right\}\mathrm{d}S$$

$$-\beta_j\beta_i n_j^{\mathrm{P}}\frac{\partial\phi}{\partial x_k}(\mathbf{P})\int_{\Delta S}\frac{\partial^2 G_0}{\partial x_i^2}[x_k-x_k(\mathbf{P})]n_j\,\mathrm{d}S-\beta_j\beta_i n_j^{\mathrm{P}}\frac{\partial\phi}{\partial x_k}(\mathbf{P})\,\delta_{ik}\int_{\Delta S}\frac{\partial G_0}{\partial x_i}n_j\,\mathrm{d}S$$

$$+\left[\frac{\partial\phi}{\partial n}(\mathbf{P})-M^2\frac{\partial\phi}{\partial x_1}(\mathbf{P})n_1^{\mathrm{P}}\right]\int_{\Delta S}\left(\frac{\partial G_0}{\partial n}-M^2\frac{\partial G_0}{\partial x_1}n_1\right)\mathrm{d}S, \tag{A12}$$

where the second integral is simply zero because $\beta_i\,\partial^2 G_0/\partial x_i^2=0$ for **P** not exactly on S. One then applies Stokes' theorem (A4) to the first integral of equation (A12). This leads to

$$J=-\frac{\partial\phi}{\partial x_k}(\mathbf{P})\beta_j\beta_i n_j^{\mathrm{P}}\varepsilon_{rij}\oint_C\frac{\partial G_0}{\partial x_i}[x_k-x_k(\mathbf{P})]\mathrm{d}x_r-\beta_j\beta_i n_j^{\mathrm{P}}\frac{\partial\phi}{\partial x_i}(\mathbf{P})\int_{\Delta S}\frac{\partial G_0}{\partial x_i}n_j\,\mathrm{d}S$$

$$+\left[\frac{\partial\phi}{\partial n}(\mathbf{P})-M^2\frac{\partial\phi}{\partial x_1}(\mathbf{P})n_1^{\mathrm{P}}\right]\int_{\Delta S}\left(\frac{\partial G_0}{\partial n}-M^2\frac{\partial G_0}{\partial x_1}n_1\right)\mathrm{d}S. \tag{A13}$$

The second integral of equation (A13) still needs to be regularized before the limit (as **P** approaches the boundary) is taken. Subtracting and adding back the term $(\partial G_0/\partial x_j)n_i$, one rewrites the second integral of equation (A13) as

$$\int_{\Delta S}\frac{\partial G_0}{\partial x_i}n_j\,\mathrm{d}S=\int_{\Delta S}\left(\frac{\partial G_0}{\partial x_i}n_j-\frac{\partial G_0}{\partial x_j}n_i\right)\mathrm{d}S+\int_{\Delta S}\frac{\partial G_0}{\partial x_j}n_i\,\mathrm{d}S. \tag{A14}$$

Stokes' theorem is now applied to the first integral of equation (A14) to get

$$\int_{\Delta S}\frac{\partial G_0}{\partial x_i}n_j\,\mathrm{d}S=\varepsilon_{rji}\oint_C G_0\,\mathrm{d}x_r+\int_{\Delta S}\frac{\partial G_0}{\partial x_j}n_i\,\mathrm{d}S. \tag{A15}$$

Substituting equation (A15) into equation (A13), and noting that

$$\beta_j n_j^{\mathrm{P}}\frac{\partial G_0}{\partial x_j}=-\beta_j n_j^{\mathrm{P}}\frac{\partial G_0}{\partial x_j^{\mathrm{P}}}=-\left(\frac{\partial G_0}{\partial n^{\mathrm{P}}}-M^2\frac{\partial G_0}{\partial x_1^{\mathrm{P}}}n_1^{\mathrm{P}}\right), \tag{A16}$$

one obtains

$$J = -\frac{\partial \phi}{\partial x_k}(\mathbf{P})\beta_j\beta_i n_j^{\mathrm{P}} \varepsilon_{rij} \oint_C \frac{\partial G_0}{\partial x_i}[x_k - x_k(\mathbf{P})]\,\mathrm{d}x_r - \frac{\partial \phi}{\partial x_i}(\mathbf{P})\beta_j\beta_i n_j^{\mathrm{P}} \varepsilon_{rji} \oint_C G_0\,\mathrm{d}x_r$$

$$+ \frac{\partial \phi}{\partial x_i}(\mathbf{P})\beta_i \int_{\Delta S}\left(\frac{\partial G_0}{\partial n^{\mathrm{P}}} - M^2 \frac{\partial G_0}{\partial x_1^{\mathrm{P}}} n_1^{\mathrm{P}}\right) n_i\,\mathrm{d}S$$

$$+ \left[\frac{\partial \phi}{\partial n}(\mathbf{P}) - M^2 \frac{\partial \phi}{\partial x_1}(\mathbf{P}) n_1^{\mathrm{P}}\right] \int_{\Delta S}\left(\frac{\partial G_0}{\partial n} - M^2 \frac{\partial G_0}{\partial x_1} n_1\right)\mathrm{d}S, \tag{A17}$$

where the last two integrals are only weakly singular. One can now take the limit as **P** approaches the boundary.

APPENDIX B: EVALUATION OF THE AXISYMMETRIC GREEN'S FUNCTIONS

One first rewrites the static Green's function G_0 in equation (6) in terms of the cylindrical co-ordinates (ρ, θ, z). Recalling that $\theta^{\mathrm{P}} = 0$. The result is

$$G_0 = 1/\tilde{r}, \tag{B1}$$

where

$$\tilde{r} = \sqrt{(z - z^{\mathrm{P}})^2 + (1 - M^2)[\rho^2 + (\rho^{\mathrm{P}})^2 - 2\rho\rho^{\mathrm{P}} \cos\theta]}. \tag{B2}$$

Let $\theta = \pi + 2\gamma$, and one has $\cos\theta = -\cos 2\gamma = 2\sin^2\gamma - 1$. Equation (B2) then becomes

$$\tilde{r} = R\sqrt{1 - m^2 \sin^2\gamma}, \tag{B3}$$

where

$$R = \sqrt{(z - z^{\mathrm{P}})^2 + (1 - M^2)(\rho + \rho^{\mathrm{P}})^2}, \tag{B4}$$

and

$$m^2 = 4(1 - M^2)\rho\rho^{\mathrm{P}}/R^2. \tag{B5}$$

The Green's function $\tilde{G}_0$ in equation (16) then becomes

$$\tilde{G}_0 = \int_0^{2\pi} G_0\,\mathrm{d}\theta = \int_0^{2\pi} \frac{1}{\tilde{r}}\,\mathrm{d}\theta = \frac{4}{R}\int_0^{\pi/2} \frac{\mathrm{d}\gamma}{\sqrt{1 - m^2 \sin^2\gamma}} = \frac{4}{R} F\left(\frac{\pi}{2}, m\right), \tag{B6}$$

where

$$F\left(\frac{\pi}{2}, m\right) \equiv \int_0^{\pi/2} \frac{\mathrm{d}\gamma}{\sqrt{1 - m^2 \sin^2\gamma}}, \tag{B7}$$

which is the complete elliptic integral of the first kind.

Similarly, the Green's function $\tilde{G}_1$ in equation (17) can be written as

$$\tilde{G}_1 = \int_0^{2\pi} G_0 \cos\theta\,\mathrm{d}\theta = \int_0^{2\pi} \frac{\cos\theta}{\tilde{r}}\,\mathrm{d}\theta = \frac{4}{R}\int_0^{\pi/2} \frac{2\sin^2\gamma - 1}{\sqrt{1 - m^2 \sin^2\gamma}}\,\mathrm{d}\gamma$$

$$= \frac{4}{R}\int_0^{\pi/2} \frac{2[1 - (1 - m^2 \sin^2\gamma)] - m^2}{m^2\sqrt{1 - m^2 \sin^2\gamma}}\,\mathrm{d}\gamma = \frac{4}{R}\left\{\frac{2}{m^2}\left[F\left(\frac{\pi}{2}, m\right)\right.\right.$$

$$- E\left(\frac{\pi}{2}, m\right)\Bigg] - F\left(\frac{\pi}{2}, m\right)\Bigg\}, \tag{B8}$$

where

$$E\left(\frac{\pi}{2}, m\right) \equiv \int_0^{\pi/2} \sqrt{1 - m^2 \sin^2 \gamma}\, \mathrm{d}\gamma, \tag{B9}$$

which is the complete elliptic integral of the second kind.

The complete elliptic integrals of the first and second kinds can be calculated by Chebyshev polynomials [15] or by standard library calls (such as IMSL).