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AN EFFICIENT COMPLEX MODAL TESTING THEORY FOR ASYMMETRIC ROTOR SYSTEMS: USE OF UNIDIRECTIONAL EXCITATION METHOD

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A complex modal testing theory is newly developed for asymmetric rotor systems. The theoretical development is made strictly in the stationary co-ordinate system, and this enables a unidirectional excitation technique efficiently to estimate the directional frequency response functions, which greatly lessens the testing efforts and enhances the practicality of the theory.

1. INTRODUCTION

In general, a rotor system consists of a rotor and a stator, which may have some degree of non-axisymmetric properties. According to the non-axisymmetric properties, a rotor system may be classified as follows [1–3]: *isotropic* (*symmetric*) *rotor system*, both the rotor and the stator are axisymmetric; *anisotropic rotor system*, the rotor is axisymmetric but the stator is not; *asymmetric rotor system*, the stator is axisymmetric but the rotor is not; *general rotor system*, neither the rotor nor the stator is axisymmetric.

The presence of asymmetric properties can significantly affect the dynamic characteristics of a rotor, such as the unbalance response, critical speeds and stability. Thus, the accurate identification of such asymmetric properties plays an important role in the physical understanding of the dynamic characteristics of asymmetric rotors. Typical examples of a rotor with an asymmetric moment of inertia include a two-bladed propeller and a two-pole generator. A rotor shaft with a rectangular cross-section or keyway is of asymmetric stiffness.

The complex modal testing method, which has been recently developed for rotor systems, utilizes the so-called directional frequency response functions (dFRFs) between complex inputs and outputs for effective modal parameter identification [4–6]. For the unbiased estimation of dFRFs associated with anisotropic rotor systems, Lee *et al*. [7–10] proposed bidirectional random excitation techniques, which require the simultaneous (bidirectional) excitations in two directions at right angle and perpendicular to the rotating axis. As the system anisotropy becomes null, i.e., for isotropic rotors, only a

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unidirectional excitation suffices, with the response measurements along the two perpendicular directions. For asymmetric rotors, Lee and Joh [2] and Joh and Lee [3] proposed a similar bidirectional excitation technique for anisotropic rotors, which converts the measured input and output signals in the stationary co-ordinate system to those in the rotating co-ordinate system. This approach, which is in essence a modulation technique, has been experimentally proved to be valid. However, the testing procedure is quite involved.

The main objective of this work is to develop an efficient modal testing theory for asymmetric rotor systems. This new approach is essentially based upon the direct use of the stationary co-ordinate system to formulate the equation of motion and to derive the dFRFs of asymmetric rotors, which has been prohibited previously owing to the appearance of time-varying properties in the formulations. The key idea is that the use of the stationary co-ordinate system is more sensible than the use of the rotating co-ordinate system in the application of the developed theory to practical asymmetric rotors, since the excitations and responses are usually measured with respect to the stationary co-ordinate system. Then, we confirm the previous finding that the estimation of dFRFs by the bidirectional excitation technique is possible when the complex input processes are jointly stationary. Finally, we propose a unidirectional random excitation technique to estimate dFRFs, which is based upon jointly non-stationary complex input processes.

2. DIRECTIONAL FREQUENCY RESPONSE FUNCTIONS

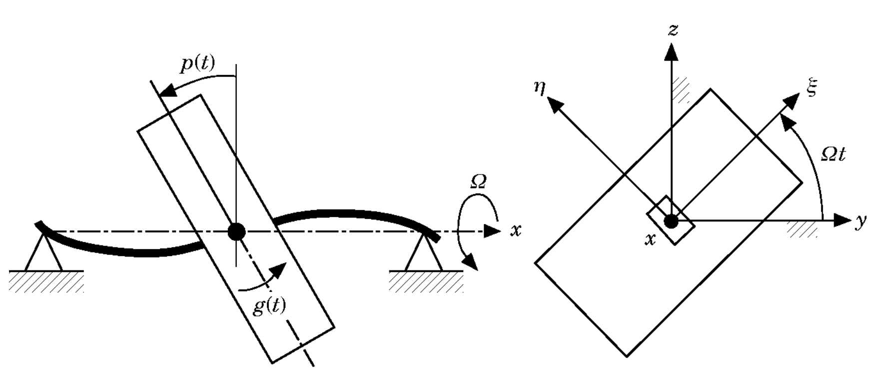
Using the stationary co-ordinate system, xyz, shown in Figure 1(b), the equation of motion of an asymmetric rotor system can be written in the complex domain as [2, 11]

$$\mathbf{M}_f\ddot{\mathbf{p}}(t) + \mathbf{M}_r\,\mathrm{e}^{\mathrm{j}2\Omega t}\ddot{\bar{\mathbf{p}}}(t) + \mathbf{C}_f\dot{\mathbf{p}}(t) + \mathbf{C}_r\,\mathrm{e}^{\mathrm{j}2\Omega t}\dot{\bar{\mathbf{p}}}(t) + \mathbf{K}_f\mathbf{p}(t) + \mathbf{K}_r\,\mathrm{e}^{\mathrm{j}2\Omega t}\bar{\mathbf{p}}(t) = \mathbf{g}(t), \qquad (1)$$

where the $N \times 1$ complex response and input vectors, defined by the real response vectors, $\mathbf{y}(t)$ and $\mathbf{z}(t)$, and the real input vectors, $\mathbf{f}_y(t)$ and $\mathbf{f}_z(t)$, respectively, are

$$\mathbf{p}(t) = \mathbf{y}(t) + \mathrm{j}\mathbf{z}(t), \qquad \bar{\mathbf{p}}(t) = \mathbf{y}(t) - \mathrm{j}\mathbf{z}(t),$$
$$\mathbf{g}(t) = \mathbf{f}_y(t) + \mathrm{j}\mathbf{f}_z(t), \qquad \bar{\mathbf{g}}(t) = \mathbf{f}_y(t) - \mathrm{j}\mathbf{f}_z(t). \qquad (2)$$

Here Ω is the rotational speed of the rotor; the bar denotes the complex conjugate; j is the imaginary number; $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the complex valued $N \times N$ generalized mass, damping and stiffness matrices, respectively; and the subscripts f and r refer to the



(a) (b)

Figure 1. A simple rotor with asymmetric inertia and stiffness. (a) Front view; (b) Side view.

symmetric and asymmetric properties, respectively. Note that the excitations and the response measurements are usually realized with respect to the stationary co-ordinate as in equation (1).

Equation (1), along with its complex conjugate form multiplied by $e^{j2\Omega t}$, can be written as

$$\begin{bmatrix} \mathbf{M}_f & \mathbf{M}_r \\ \bar{\mathbf{M}}_r & \bar{\mathbf{M}}_f \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}}(t) \\ \ddot{\bar{\mathbf{p}}}(t)\,e^{j2\Omega t} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_f & \mathbf{C}_r \\ \bar{\mathbf{C}}_r & \bar{\mathbf{C}}_f \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}(t) \\ \dot{\bar{\mathbf{p}}}(t)\,e^{j2\Omega t} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_f & \mathbf{K}_r \\ \bar{\mathbf{K}}_r & \bar{\mathbf{K}}_f \end{bmatrix} \begin{bmatrix} \mathbf{p}(t) \\ \bar{\mathbf{p}}(t)\,e^{j2\Omega t} \end{bmatrix} = \begin{bmatrix} \mathbf{g}(t) \\ \bar{\mathbf{g}}(t)\,e^{j2\Omega t} \end{bmatrix}. \tag{3}$$

Taking the Fourier transform of equation (3), we obtain

$$\begin{bmatrix} \mathbf{D}_f(j\omega) & \mathbf{D}_r\{j(\omega-2\Omega)\} \\ \hat{\mathbf{D}}_r(j\omega) & \hat{\mathbf{D}}_f\{j(\omega-2\Omega)\} \end{bmatrix} \begin{bmatrix} \mathbf{P}(j\omega) \\ \hat{\mathbf{P}}\{j(\omega-2\Omega)\} \end{bmatrix} = \begin{bmatrix} \mathbf{G}(j\omega) \\ \hat{\mathbf{G}}\{j(\omega-2\Omega)\} \end{bmatrix}, \tag{4}$$

where $\mathbf{P}(j\omega)$, $\hat{\mathbf{P}}(j\omega)$, $\mathbf{G}(j\omega)$ and $\hat{\mathbf{G}}(j\omega)$ are the Fourier transforms of $\mathbf{p}(t)$, $\bar{\mathbf{p}}(t)$, $\mathbf{g}(t)$ and $\bar{\mathbf{g}}(t)$, respectively, and the partitioned dynamic stiffness matrices are

$$\mathbf{D}_f(j\omega) = \mathbf{K}_f - \omega^2\mathbf{M}_f + j\omega\mathbf{C}_f,$$
$$\mathbf{D}_r\{j(\omega-2\Omega)\} = \mathbf{K}_r - (\omega-2\Omega)^2\mathbf{M}_r + j(\omega-2\Omega)\mathbf{C}_r,$$
$$\hat{\mathbf{D}}_r(j\omega) = \bar{\mathbf{K}}_r - \omega^2\bar{\mathbf{M}}_r + j\omega\bar{\mathbf{C}}_r,$$
$$\hat{\mathbf{D}}_f\{j(\omega-2\Omega)\} = \bar{\mathbf{K}}_f - (\omega-2\Omega)^2\bar{\mathbf{M}}_f + j(\omega-2\Omega)\bar{\mathbf{C}}_f.$$

From equation (4), the two-sided directional frequency response matrices (dFRMs) can be defined in the stationary co-ordinate system as

$$\begin{bmatrix} \mathbf{P}(j\omega) \\ \hat{\mathbf{P}}\{j(\omega-2\Omega)\} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{gp}(j\omega) & \mathbf{H}_{g\hat{p}}(j\omega) \\ \mathbf{H}_{\hat{g}p}(j\omega) & \mathbf{H}_{\hat{g}\hat{p}}(j\omega) \end{bmatrix} \begin{bmatrix} \mathbf{G}(j\omega) \\ \hat{\mathbf{G}}\{j(\omega-2\Omega)\} \end{bmatrix}, \tag{5}$$

where

$$\mathbf{H}_{gp}(j\omega) = [\mathbf{D}_f(j\omega) - \mathbf{D}_r\{j(\omega-2\Omega)\}\hat{\mathbf{D}}_f\{j(\omega-2\Omega)\}^{-1}\hat{\mathbf{D}}_r(j\omega)]^{-1},$$
$$\mathbf{H}_{\hat{g}p}(j\omega) = -[\mathbf{D}_f(j\omega) - \mathbf{D}_r\{j(\omega-2\Omega)\}\hat{\mathbf{D}}_f\{j(\omega-2\Omega)\}^{-1}\hat{\mathbf{D}}_r(j\omega)]^{-1}\mathbf{D}_r\{j(\omega-2\Omega)\} \times \hat{\mathbf{D}}_f\{j(\omega-2\Omega)\}^{-1},$$
$$\mathbf{H}_{g\hat{p}}(j\omega) = -[\hat{\mathbf{D}}_f(j(\omega-2\Omega)\} - \hat{\mathbf{D}}_r(j\omega)\mathbf{D}_f(j\omega)^{-1}\mathbf{D}_r\{j(\omega-2\Omega)\}]^{-1}\hat{\mathbf{D}}_r(j\omega)\mathbf{D}_f(j\omega)^{-1},$$
$$\mathbf{H}_{\hat{g}\hat{p}}(j\omega) = [\hat{\mathbf{D}}_f\{j(\omega-2\Omega)\} - \hat{\mathbf{D}}_r(j\omega)\mathbf{D}_f(j\omega)^{-1}\mathbf{D}_r\{j(\omega-2\Omega)\}]^{-1}. \tag{6}$$

Here $\mathbf{H}_{gp}(j\omega)$ and $\mathbf{H}_{\hat{g}\hat{p}}(j\omega)$ are referred to as the *normal* dFRMs, whereas $\mathbf{H}_{\hat{g}p}(j\omega)$ and $\mathbf{H}_{g\hat{p}}(j\omega)$ are referred to as the *reverse* dFRMs, and it is assumed that all inverse matrices exist. From equations (4) and (5), it can easily be proved that

$$\mathbf{H}_{g\hat{p}}(j\omega) = \bar{\mathbf{H}}_{\hat{g}p}\{-j(\omega-2\Omega)\}, \qquad \mathbf{H}_{\hat{g}\hat{p}}(j\omega) = \bar{\mathbf{H}}_{gp}\{-j(\omega-2\Omega)\}. \tag{7}$$

Therefore, in order to completely define the dFRMs, it is sufficient to consider two dFRMs, i.e.,

$$\mathbf{P}(j\omega) = [\mathbf{H}_{gp}(j\omega) \quad \mathbf{H}_{\hat{g}p}(j\omega)] \begin{bmatrix} \mathbf{G}(j\omega) \\ \tilde{\mathbf{G}}(j\omega) \end{bmatrix}, \tag{8}$$

where $\tilde{\mathbf{G}}(j\omega) = \hat{\mathbf{G}}\{j(\omega-2\Omega)\}$ is the Fourier transform of $\tilde{\mathbf{g}}(t) = \bar{\mathbf{g}}(t)\,e^{j2\Omega t}$.

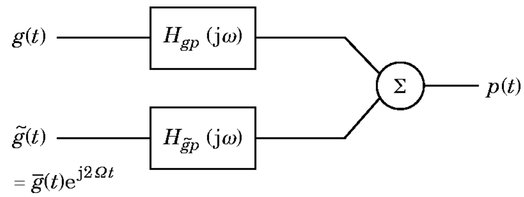


Figure 2. The two-complex input/single-complex output model.

3. ESTIMATION OF DIRECTIONAL FREQUENCY RESPONSE FUNCTIONS

The estimation of dFRFs between complex inputs and outputs is the key feature of the complex modal testing for rotor systems. In this section, we investigate the estimation methods of the dFRFs defined in the previous section, and propose appropriate excitation methods.

3.1. ESTIMATION OF dFRFs USING JOINTLY STATIONARY INPUT PROCESSES

The simple two-complex input/single-complex output model which describes any pair of input/output elements of an asymmetry rotor system expressed by equation (8) is shown in Figure 2. It can easily be proved that the complex random processes, $\{g(t)\}$ and $\{\bar{g}(t)\}$, are jointly wide-sense stationary (WSS) with zero mean if the real random input processes, $\{f_y(t)\}$ and $\{f_z(t)\}$, are jointly WSS with zero mean. Note that the original signals can always be transformed to have zero mean values by subtracting their mean values, and the commonly used random input processes for the complex modal testing, $\{f_y(t)\}$ and $\{f_z(t)\}$, or equivalently $\{g(t)\}$ and $\{\bar{g}(t)\}$, are jointly WSS. This will be assumed henceforth. However, similar to a modulated real signal [12, 13], the complex input processes, $\{g(t)\}$ and $\{\tilde{g}(t)\}$, in Figure 2, are jointly WSS if and only if the complex input processes, $\{g(t)\}$ and $\{\bar{g}(t)\}$, are jointly WSS with zero mean and their correlation functions are such that

$$R_{g\bar{g}}(\tau) = R_{\bar{g}g}(\tau) = 0, \tag{9}$$

or, equivalently, since $R_{g\bar{g}}(\tau) = R_{f_yf_y}(\tau) - R_{f_zf_z}(\tau) - \mathrm{j}\{R_{f_yf_z}(\tau) + R_{f_zf_y}(\tau)\}$,

$$R_{f_yf_y}(\tau) = R_{f_zf_z}(\tau), \qquad R_{f_yf_z}(\tau) = -R_{f_zf_y}(\tau). \tag{10}$$

The proof of this is as follows.

Clearly,

$$\mathrm{E}[g(t)] = \mathrm{E}[f_y(t)] + \mathrm{j}\mathrm{E}[f_z(t)] = 0,$$

$$\mathrm{E}[\tilde{g}(t)] = \mathrm{E}[g(t)]\,\mathrm{e}^{\mathrm{j}2\Omega t} = 0,$$

$$R_{gg}(\tau) = \mathrm{E}\left[\bar{g}\left(t - \frac{\tau}{2}\right)g\left(t + \frac{\tau}{2}\right)\right] = R_{f_yf_y}(\tau) + R_{f_zf_z}(\tau) + \mathrm{j}\{R_{f_yf_z}(\tau) - R_{f_zf_y}(\tau)\},$$

$$R_{\tilde{g}\tilde{g}}(\tau) = R_{gg}(-\tau)\,\mathrm{e}^{\mathrm{j}2\Omega\tau}. \tag{11}$$

Here E[·] indicates the expected value. Furthermore,

$$\mathrm{E}\left[\bar{g}\left(t - \frac{\tau}{2}\right)\tilde{g}\left(t + \frac{\tau}{2}\right)\right] = R_{g\bar{g}}(\tau)\,\mathrm{e}^{\mathrm{j}\Omega\tau}\,\mathrm{e}^{\mathrm{j}2\Omega t}. \tag{12}$$

Note that, if $\{g(t)\}$ is a zero mean WSS complex random process, $\{\tilde{g}(t)\}$ also becomes a zero mean WSS complex random process. However, in order that the complex input processes $\{g(t)\}$ and $\{\tilde{g}(t)\}$ become jointly WSS, the expected value of equation (12) should also be independent of time t. This is possible if and only if equation (9) holds unless the rotational speed, Ω, is zero.

Equations (9) and (12) immediately suggest that the complex random process $\{g(t)\}$ also becomes uncorrelated with $\{\tilde{g}(t)\}$, if it is uncorrelated with $\{\bar{g}(t)\}$. Let us assume that we can properly measure the complex time signals, $g(t)$, $\tilde{g}(t)$ and $p(t)$, from the jointly WSS and uncorrelated complex input processes, $\{g(t)\}$ and $\{\tilde{g}(t)\}$, and the output process, $\{p(t)\}$, respectively. Then the dFRFs can be estimated from [14]

$$H_{gp}(\mathrm{j}\omega) = \frac{S_{gp}(\mathrm{j}\omega)}{S_{gg}(\mathrm{j}\omega)}, \qquad H_{\tilde{g}p}(\mathrm{j}\omega) = \frac{S_{\tilde{g}p}(\mathrm{j}\omega)}{S_{\tilde{g}\tilde{g}}(\mathrm{j}\omega)}, \tag{13}$$

where the quantities $S_{ik}(\mathrm{j}\omega)$, $i = g, \tilde{g}$ and $k = p, g, \tilde{g}$, are the two-sided directional auto- (for $i = k$) and cross- (for $i \neq k$) spectral density functions (dPSDs and dCSDs) between the signals $g(t)$, $\tilde{g}(t)$ and $p(t)$, respectively.

From the condition (10), the spectral density functions of excitations should satisfy

$$S_{f_y f_y}(\mathrm{j}\omega) = S_{f_z f_z}(\mathrm{j}\omega) \qquad \text{and} \qquad \mathrm{Re}\{S_{f_y f_z}(\mathrm{j}\omega)\} = 0. \tag{14}$$

Two conventional excitation methods can be used to estimate the dFRFs for the jointly WSS input processes: one method, with $\mathrm{Im}\{S_{f_y f_z}(\mathrm{j}\omega)\} \neq 0$, is called the directional (or bidirectional rotating) random excitation; and the other, with $\mathrm{Im}\{S_{f_y f_z}(\mathrm{j}\omega)\} = 0$, is called the uncorrelated isotropic (or bidirectional stationary) random excitation [2, 8, 9].

3.2. ESTIMATION OF dFRFs USING JOINTLY NON-STATIONARY INPUT PROCESSES

In practice, it may be difficult to generate excitations that ideally satisfy the condition (9), due to the difficulties associated with the precise alignment and tuning of actuators. This implies that the processes, $\{g(t)\}$ and $\{\bar{g}(t)\}$, may be jointly stationary but correlated. Thus, we will deal, in this section, with the case when the two random input processes, $\{g(t)\}$ and $\{\tilde{g}(t)\}$, are individually stationary but jointly non-stationary, to estimate dFRFs from the signals $g(t)$, $\tilde{g}(t)$ and $p(t)$.

For non-stationary random processes, the double frequency spectral density functions at any pair of fixed frequencies, $\mathrm{j}\omega_1$ and $\mathrm{j}\omega_2$, are defined by the expected values [14], and their relationships are derived, from the model in Figure 2, as

$$\begin{aligned}
S_{gp}(\mathrm{j}\omega_1, \mathrm{j}\omega_2) &= \mathrm{E}[\overline{G}(\mathrm{j}\omega_1)P(\mathrm{j}\omega_2)] \\
&= \mathrm{E}[\overline{G}(\mathrm{j}\omega_1)\{H_{gp}(\mathrm{j}\omega_2)G(\mathrm{j}\omega_2) + H_{\tilde{g}p}(\mathrm{j}\omega_2)\widetilde{G}(\mathrm{j}\omega_2)\}] \\
&= H_{gp}(\mathrm{j}\omega_2)S_{gg}(\mathrm{j}\omega_1, \mathrm{j}\omega_2) + H_{\tilde{g}p}(\mathrm{j}\omega_2)S_{g\tilde{g}}(\mathrm{j}\omega_1, \mathrm{j}\omega_2), \\
S_{\tilde{g}p}(\mathrm{j}\omega_1, \mathrm{j}\omega_2) &= \mathrm{E}[\overline{\widetilde{G}}(\mathrm{j}\omega_1)P(\mathrm{j}\omega_2)] \\
&= \mathrm{E}\overline{\widetilde{G}}(\mathrm{j}\omega_1)\{H_{gp}(\mathrm{j}\omega_2)G(\mathrm{j}\omega_2) + H_{\tilde{g}p}(\mathrm{j}\omega_2)\widetilde{G}(\mathrm{j}\omega_2)\}] \\
&= H_{gp}(\mathrm{j}\omega_2)S_{\tilde{g}g}(\mathrm{j}\omega_1, \mathrm{j}\omega_2) + H_{\tilde{g}p}(\mathrm{j}\omega_2)S_{\tilde{g}\tilde{g}}(\mathrm{j}\omega_1, \mathrm{j}\omega_2),
\end{aligned} \tag{15}$$

where the quantities $S_{ik}(\mathrm{j}\omega_1, \mathrm{j}\omega_2)$, $i = g, \tilde{g}$ and $k = g, \tilde{g}, p$, are the two-sided double frequency directional auto- and cross-spectral density functions (dPSDs and dCSDs) between the signals $g(t)$, $\tilde{g}(t)$ and $p(t)$, respectively. Unless the double frequency directional coherence function (dCOH) between $g(t)$ and $\tilde{g}(t)$ for the line $\omega_1 = \omega_2 = \omega$ in the ($\mathrm{j}\omega_1$, $\mathrm{j}\omega_2$)

plane, $\gamma^2_{g\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega)$, is unity, the estimates of dFRFs, $H_{gp}(\mathrm{j}\omega)$ and $H_{\tilde{g}p}(\mathrm{j}\omega)$, become, from equation (15),

$$H_{gp}(\mathrm{j}\omega) = \frac{S_{gp}(\mathrm{j}\omega, \mathrm{j}\omega)}{S_{gg}(\mathrm{j}\omega, \mathrm{j}\omega)}\left[1 - \frac{S_{\tilde{g}p}(\mathrm{j}\omega, \mathrm{j}\omega)S_{g\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega)}{S_{gp}(\mathrm{j}\omega, \mathrm{j}\omega)S_{\tilde{g}\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega)}\right]\Big/[1 - \gamma^2_{g\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega)],$$

$$H_{\tilde{g}p}(\mathrm{j}\omega) = \frac{S_{\tilde{g}p}(\mathrm{j}\omega, \mathrm{j}\omega)}{S_{\tilde{g}\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega)}\left[1 - \frac{S_{gp}(\mathrm{j}\omega, \mathrm{j}\omega)S_{\tilde{g}g}(\mathrm{j}\omega, \mathrm{j}\omega)}{S_{\tilde{g}p}(\mathrm{j}\omega, \mathrm{j}\omega)S_{gg}(\mathrm{j}\omega, \mathrm{j}\omega)}\right]\Big/[1 - \gamma^2_{g\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega)], \qquad (16)$$

where

$$\gamma^2_{g\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega) = \frac{|S_{g\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega)|^2}{S_{gg}(\mathrm{j}\omega, \mathrm{j}\omega)S_{\tilde{g}\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega)}. \qquad (17)$$

Equations (A12), (A13) and (A14) in the Appendix indicate, for a sufficiently long record length $T \geqslant \pi/2\Omega$, that

$$S_{\tilde{g}g}(\mathrm{j}\omega, \mathrm{j}\omega) = S_{g\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega) = 0. \qquad (18)$$

Equations (17) and (18) imply that the estimate of $\gamma^2_{g\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega)$ will eventually become null as the number of ensemble averages increases. Recalling that the individual random processes $\{g(t)\}$ and $\{\tilde{g}(t)\}$ are stationary, we can write, from equation (A17),

$$S_{gg}(\mathrm{j}\omega, \mathrm{j}\omega) = TS_{gg}(\mathrm{j}\omega), \qquad S_{\tilde{g}\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\omega) = TS_{\tilde{g}\tilde{g}}(\mathrm{j}\omega). \qquad (19)$$

Substituting equations (18) and (19) into equation (16), we obtain the expressions for the estimates of the *normal* and *reverse* dFRFs for the jointly non-stationary input processes, $\{g(t)\}$ and $\{\tilde{g}(t)\}$, as

$$H_{gp}(\mathrm{j}\omega) = \frac{S_{gp}(\mathrm{j}\omega, \mathrm{j}\omega)}{TS_{gg}(\mathrm{j}\omega)}, \qquad H_{\tilde{g}p}(\mathrm{j}\omega) = \frac{S_{\tilde{g}p}(\mathrm{j}\omega, \mathrm{j}\omega)}{TS_{\tilde{g}\tilde{g}}(\mathrm{j}\omega)}, \qquad (20)$$

where

$$S_{gp}(\mathrm{j}\omega, \mathrm{j}\omega) = \mathrm{E}[\bar{G}(\mathrm{j}\omega)P(\mathrm{j}\omega)], \qquad S_{\tilde{g}p}(\mathrm{j}\omega, \mathrm{j}\omega) = \mathrm{E}[\bar{\tilde{G}}(\mathrm{j}\omega)P(\mathrm{j}\omega)],$$

$$S_{gg}(\mathrm{j}\omega) = \frac{1}{T}\,\mathrm{E}[\bar{G}(\mathrm{j}\omega)G(\mathrm{j}\omega)], \qquad S_{\tilde{g}\tilde{g}}(\mathrm{j}\omega) = \frac{1}{T}\,\mathrm{E}[\bar{\tilde{G}}(\mathrm{j}\omega)\tilde{G}(\mathrm{j}\omega)],$$

which are, in essence, not different from the method given in equation (13) developed for the jointly WSS and uncorrelated processes, $\{g(t)\}$ and $\{\tilde{g}(t)\}$. In other words, equations (18) and (20) hold irrespective of the correlation between the jointly WSS processes with zero mean, $\{g(t)\}$ and $\{\tilde{g}(t)\}$. This implies that any unidirectional random stationary excitation in the y–z plane is sufficient to estimate the dFRFs of asymmetric rotor systems, if we can properly measure the complex input and output time signals, $g(t)$, $\tilde{g}(t)$ and $p(t)$. Thus the proposed technique greatly lessens the modal testing effort for asymmetric rotor systems compared with the conventional method [8] developed for anisotropic rotor systems, which requires simultaneous (bidirectional) excitations in the y and z directions.

4. NUMERICAL EXAMPLE

In this section, a numerical simulation is performed to demonstrate and examine the effectiveness of the proposed estimation of dFRFs in asymmetric rotor systems.

Consider a simple asymmetric rotor shown in Figure 1, where an asymmetric rigid disk is located at the mid-span of a massless shaft and the straightening tendency of the bent shaft is modelled as an asymmetric torsional stiffness. Let us assume for simplicity that the orientations of the principal elastic and inertia axes coincide with the rotating co-ordinates (ξ, η) as indicated in Figure 1(b). For the small complex angular displacement of the disk, $p(t)$, and the complex input torque to the system, $g(t)$, the equation of motion in the stationary co-ordinate system can be written as [6, 15]

$$\ddot{p}(t) + \varepsilon\, e^{j2\Omega t}\ddot{\bar{p}}(t) + (2\zeta\omega_0 - j\alpha\Omega)\dot{p}(t) + j2\varepsilon\Omega\, e^{j2\Omega t}\dot{\bar{p}}(t) + \omega_0^2 p(t) + \Delta\omega_0^2\, e^{j2\Omega t}\bar{p}(t) = \frac{g(t)}{J}, \tag{21}$$

where

$$\varepsilon = \frac{J_\xi - J_\eta}{2J}, \qquad \zeta = \frac{c}{2\sqrt{Jk}}, \qquad \omega_0 = \sqrt{\frac{k}{J}}, \qquad \alpha = \frac{J_p}{J} \quad (0 < \alpha < 2),$$

$$\Delta = \frac{k_\xi - k_\eta}{2k}, \qquad J = \frac{J_\xi + J_\eta}{2}, \qquad k = \frac{k_\xi + k_\eta}{2},$$

$$p(t) = \theta_y(t) + j\theta_z(t), \qquad g(t) = M_y(t) + jM_z(t).$$

Here, J_ξ and J_η are the diametrical mass moments of inertia of the disk with respect to the ξ- and η-axes; J_p is the polar mass moment of inertia of the disk; c is the torsional damping coefficient of the bent shaft; k_ξ and k_η are the torsional stiffnesses of the bent shaft with respect to the ξ- and η-axes; $\theta_y(t)$ and $\theta_z(t)$ are the small angular displacements of the disk about the y- and z-axes; and $M_y(t)$ and $M_z(t)$ are the input torques acting on the rotor about the y- and z-axes, respectively. Thus, ε and Δ indicate the degree of the inertia and stiffness asymmetry, respectively, and ζ is the damping ratio. The *normal* and *reverse* dFRFs associated with equation (21) can be expressed theoretically, from equation (6), as

$$H_{gp}(j\omega) = \frac{\hat{D}_f\{j(\omega - 2\Omega)\}}{\hat{D}_f\{j(\omega - 2\Omega)\}D_f(j\omega) - D_r\{j(\omega - 2\Omega)\}\hat{D}_r(j\omega)},$$
$$H_{\bar{g}p}(j\omega) = \frac{-D_r\{j(\omega - 2\Omega)\}}{\hat{D}_f\{j(\omega - 2\Omega)\}D_f(j\omega) - D_r\{j(\omega - 2\Omega)\}\hat{D}_r(j\omega)}, \tag{22}$$

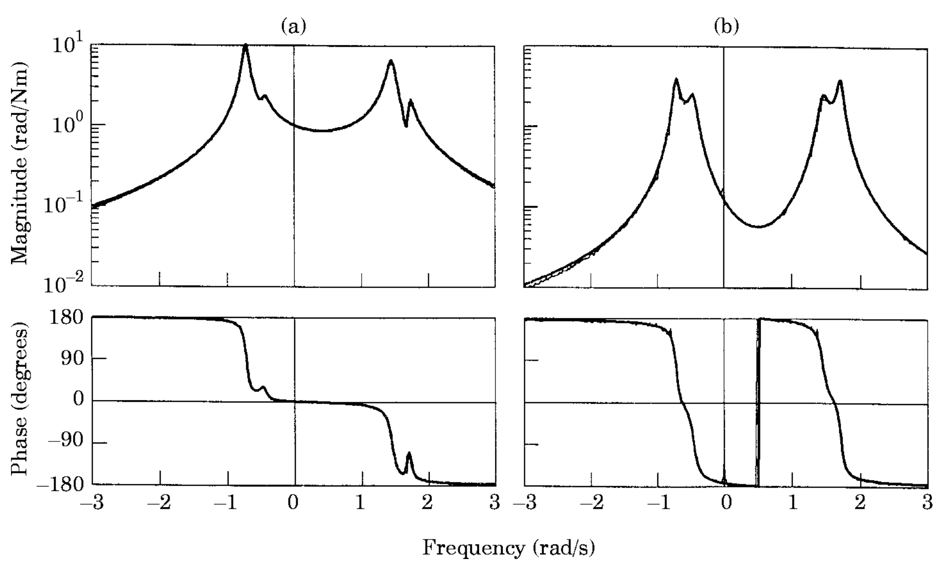


Figure 3. Magnitude–phase plots of the simple asymmetric rotor ($\varepsilon = -0{\cdot}1$ and $\triangle = 0{\cdot}1$). (a) *normal* dFRF, $H_{gp}(j\omega)$; (b) *reverse* dFRF, $H_{\bar{g}p}(j\omega)$. ———, Theoretical; ——, estimated.

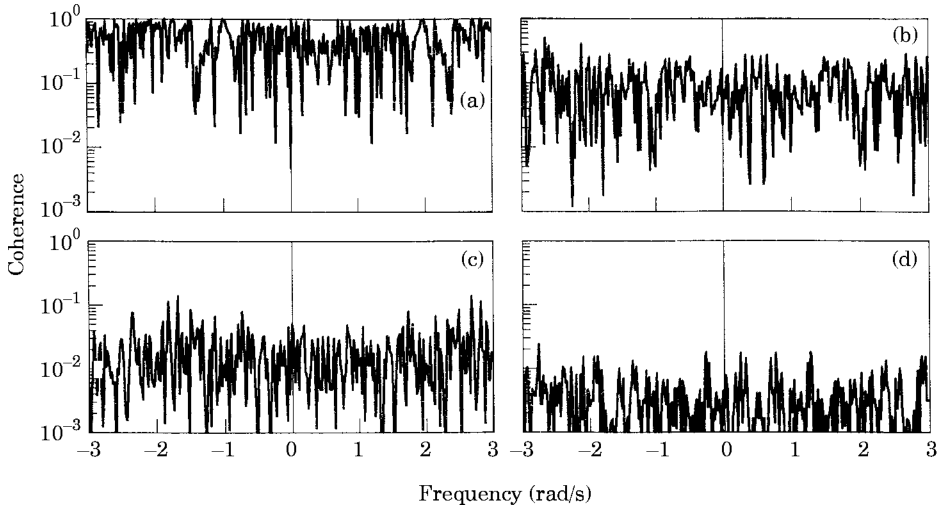


Figure 4. The double frequency dCOH between complex inputs $g(t)$ and $\bar{g}(t)$. (a) $n_d = 2$; (b) $n_d = 10$; (c) $n_d = 50$; (d) $n_d = 250$.

where

$$D_f(\mathrm{j}\omega) = J\{-\omega^2 + (\alpha\Omega + \mathrm{j}2\zeta\omega_0)\omega + \omega_0^2\},$$

$$D_r\{\mathrm{j}(\omega - 2\Omega)\} = J\{-\varepsilon\omega^2 + 2\varepsilon\Omega\omega + \Delta\omega_0^2\},$$

$$\hat{D}_r(\mathrm{j}\omega) = J\{-\varepsilon\omega^2 + 2\varepsilon\Omega\omega + \Delta\omega_0^2\},$$

$$\hat{D}_f\{\mathrm{j}(\omega - 2\Omega)\} = J\{-(\omega - 2\Omega)^2 - (\alpha\Omega - \mathrm{j}2\zeta\omega_0)(\omega - 2\Omega) + \omega_0^2\}.$$

In the simulation, we considered the simple unidirectional excitation method, in which the input signals were of the pseudo-random Gaussian processes with zero mean, and solved the equation of motion using the Runge–Kutta integration method to generate the complex response, $p(t)$. The data used in the simulation were as follows: $\varepsilon = -0{\cdot}1$, $\Delta = 0{\cdot}1$, $\zeta = 0{\cdot}05$, $\alpha = 1{\cdot}6$, $\omega_0 = 1$ rad/s, $\Omega = 0{\cdot}5$ rad/s and $J = 1$ kg m^2. Using equation (16), the dFRFs were estimated with ten averages of 1024 point-FFT using the Hanning window, so that the frequency resolution was 0·02 rad/s when the unidirectional excitation torque was applied about the y direction in Figure 1(b). As pointed out in the previous section, the direction of excitation torque makes no difference in the estimation of dFRFs, so far as it remains perpendicular to the x-axis. The magnitude–phase plots of the theoretical and estimated dFRFs are shown in Figure 3. Note that the theoretical and estimated dFRFs are in good agreement. In Figure 4 it is indicated that the double frequency dCOH, $\gamma^2_{g\bar{g}}(\mathrm{j}\omega, \mathrm{j}\omega)$, approaches null as the number of ensemble averaging, n_d, increases, for $T = 100\pi \gg \pi/2\Omega = \pi$, so that equation (18) holds.

The simulation results clearly show that the theoretical development made in this paper is valid, and that the proposed unidirectional random excitation method in the stationary co-ordinate system is very effective in estimating the *normal* and *reverse* dFRFs of asymmetric rotor systems.

5. CONCLUSIONS

A complex modal testing theory has been newly developed for asymmetric rotor systems, based upon their equations of motion written in the stationary co-ordinate system. The theory essentially assumes that the complex random excitation process and its conjugate defined in the stationary co-ordinate system are jointly wide sense stationary but may be correlated, which often occurs in practice. In particular, the simple unidirectional excitation technique, which requires greatly less testing effort than the previous bidirectional excitation method, in the stationary co-ordinate is proposed for the complex modal testing of asymmetric rotor systems.

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APPENDIX: SPECTRAL STRUCTURE OF NON-STATIONARY COMPLEX RANDOM PROCESSES $\{g(t)\}$ AND $\{\tilde{g}(t)\}$

Consider a pair of complex random processes, $\{g(t)\}$ and $\{\tilde{g}(t)\}$, with the particular relation $\{\tilde{g}(t)\} = \{\bar{g}(t)\}\,\mathrm{e}^{j2\Omega t}$; that is, $\{\tilde{g}(t)\}$ is a modulated complex random process. When the process $\{g(t)\}$ is non-stationary, the process $\{\tilde{g}(t)\}$ also becomes non-stationary. To account for such non-stationary complex random processes, the spectral structure of non-stationary real random processes in reference [14] can be easily extended. Now assume that any complex time signals, $g(t)$ and $\tilde{g}(t)$, from the non-stationary complex random processes with zero means, $\{g(t)\}$ and $\{\tilde{g}(t)\}$, have finite Fourier transforms, for a very long but finite record length T, given by

$$G_T(\mathrm{j}\omega) = G(\mathrm{j}\omega, T) = \int_0^T g(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t, \qquad \tilde{G}_T(\mathrm{j}\omega) = \tilde{G}(\mathrm{j}\omega, T) = \int_0^T \tilde{g}(t)\,\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t. \quad \text{(A1)}$$

From now on, the dependence on T will be omitted for notational simplicity. The inverse Fourier transform pairs to equation (A1) are

$$g(t) = \frac{1}{2\pi}\int G(\mathrm{j}\omega)\,\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega, \qquad \tilde{g}(t) = \frac{1}{2\pi}\int \tilde{G}(\mathrm{j}\omega)\,\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega, \quad \text{(A2)}$$

where limits of integration may be from $-\infty$ to ∞. For any pair of fixed times t_1 and t_2 and frequencies ω_1 and ω_2, it follows from equation (A2) that

$$\bar{g}(t_1)\tilde{g}(t_2) = \left[\frac{1}{2\pi}\int \bar{G}(\mathrm{j}\omega_1)\,\mathrm{e}^{-\mathrm{j}\omega_1 t_1}\,\mathrm{d}\omega_1\right]\left[\frac{1}{2\pi}\int \tilde{G}(\mathrm{j}\omega_2)\,\mathrm{e}^{\mathrm{j}\omega_2 t_2}\,\mathrm{d}\omega_2\right]. \quad \text{(A3)}$$

Taking the expected values of both sides, we obtain the double time cross-correlation function as

$$R_{g\tilde{g}}(t_1, t_2) = \frac{1}{4\pi^2}\int\int S_{g\tilde{g}}(\mathrm{j}\omega_1, \mathrm{j}\omega_2)\,\mathrm{e}^{-\mathrm{j}(\omega_1 t_1 - \omega_2 t_2)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2, \quad \text{(A4)}$$

where the double time cross-correlation and double frequency cross-spectral density function are

$$R_{g\tilde{g}}(t_1, t_2) = \mathrm{E}[\bar{g}(t_1)\tilde{g}(t_2)], \qquad S_{g\tilde{g}}(\mathrm{j}\omega_1, \mathrm{j}\omega_2) = \mathrm{E}[\bar{G}(\mathrm{j}\omega_1)\tilde{G}(\mathrm{j}\omega_2)].$$

Similarly, we obtain the double time autocorrelation functions as

$$R_{gg}(t_1, t_2) = \frac{1}{4\pi^2}\int\int S_{gg}(\mathrm{j}\omega_1, \mathrm{j}\omega_2)\,\mathrm{e}^{-\mathrm{j}(\omega_1 t_1 - \omega_2 t_2)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2,$$

$$R_{\tilde{g}\tilde{g}}(t_1, t_2) = \frac{1}{4\pi^2}\int\int S_{\tilde{g}\tilde{g}}(\mathrm{j}\omega_1, \mathrm{j}\omega_2)\,\mathrm{e}^{-\mathrm{j}(\omega_1 t_1 - \omega_2 t_2)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2, \quad \text{(A5)}$$

where the double time autocorrelation and double frequency autospectral density functions are

$$R_{gg}(t_1, t_2) = \mathrm{E}[\bar{g}(t_1)g(t_2)], \qquad S_{gg}(\mathrm{j}\omega_1, \mathrm{j}\omega_2) = \mathrm{E}[\bar{G}(\mathrm{j}\omega_1)G(\mathrm{j}\omega_2)],$$

$$R_{\tilde{g}\tilde{g}}(t_1, t_2) = \mathrm{E}[\bar{\tilde{g}}(t_1)\tilde{g}(t_2)], \qquad S_{\tilde{g}\tilde{g}}(\mathrm{j}\omega_1, \mathrm{j}\omega_2) = \mathrm{E}[\bar{\tilde{G}}(\mathrm{j}\omega_1)\tilde{G}(\mathrm{j}\omega_2)].$$

A different correlation and spectral structure can be defined by the transformations given by

$$t_1 = t - \tau/2, \qquad t_2 = t + \tau/2, \qquad \omega_1 = \omega - \chi/2, \qquad \omega_2 = \omega + \chi/2. \quad \text{(A6)}$$

That is,

$$R_{ik}(t_1, t_2) = R_{ik}\left(t - \frac{\tau}{2}, t + \frac{\tau}{2}\right) = \mathscr{R}_{ik}(\tau, t), \qquad i, k = g, \tilde{g}, \quad \text{(A7)}$$

$$S_{ik}(\mathrm{j}\omega_1, \mathrm{j}\omega_2) = S_{ik}\left\{\mathrm{j}\left(\omega - \frac{\chi}{2}\right), \mathrm{j}\left(\omega + \frac{\chi}{2}\right)\right\} = \mathscr{S}_{ik}(\mathrm{j}\omega, \mathrm{j}\chi), \qquad i, k = g, \tilde{g}, \tag{A8}$$

with the relationship

$$\mathscr{S}_{ik}(\mathrm{j}\omega, \mathrm{j}\chi) = \int\int \mathscr{R}_{ik}(\tau, t)\, \mathrm{e}^{-\mathrm{j}(\omega\tau + \chi t)}\, \mathrm{d}\tau\, \mathrm{d}t, \qquad i, k = g, \tilde{g}, \tag{A9}$$

where the script letters $\mathscr{R}$ and $\mathscr{S}$ are used in place of R and S to distinguish planes (τ, t) and $(\mathrm{j}\omega, \mathrm{j}\chi)$ from planes (t_1, t_2) and $(\mathrm{j}\omega_1, \mathrm{j}\omega_2)$, respectively.

Now consider the case in which the process $\{g(t)\}$ is stationary with zero mean. Then the complex random processes, $\{g(t)\}$ and $\{\bar{g}(t)\}$, become jointly WSS with zero mean. However, the processes, $\{g(t)\}$ and $\{\tilde{g}(t)\}$, will be WSS but not jointly WSS unless the processes, $\{g(t)\}$ and $\{\bar{g}(t)\}$, are completely uncorrelated or the rotational speed of rotor, Ω, is zero. The double time cross-correlation and double frequency cross-spectral density function between the signals $g(t)$ and $\tilde{g}(t)$ are given, from the relation $\tilde{g}(t) = \bar{g}(t)\,\mathrm{e}^{\mathrm{j}2\Omega t}$ and equations (A7), (A8) and (A9), by

$$\mathscr{R}_{g\tilde{g}}(\tau, t) = R_{g\tilde{g}}(t_1, t_2) = R_{g\bar{g}}(t_2 - t_1)\,\mathrm{e}^{\mathrm{j}2\Omega t_2} = R_{g\bar{g}}(\tau)\,\mathrm{e}^{\mathrm{j}\Omega\tau}\,\mathrm{e}^{\mathrm{j}2\Omega\tau}, \tag{A10}$$

$$\mathscr{S}_{g\tilde{g}}(\mathrm{j}\omega, \mathrm{j}\chi) = 2\pi S_{g\bar{g}}\{\mathrm{j}(\omega - \Omega)\}\delta_1(\chi - 2\Omega). \tag{A11}$$

Here $S_{g\bar{g}}(\mathrm{j}\omega)$ is the dCSD between $g(t)$ and $\bar{g}(t)$, and $\delta_1(\omega)$ is a finite delta function, defined by

$$\delta_1(\omega) = \begin{cases} \dfrac{T}{2\pi}, & \left(-\dfrac{\pi}{T}\right) < \omega < \left(\dfrac{\pi}{T}\right), \\ 0, & \text{otherwise.} \end{cases} \tag{A12}$$

Equation (A11) can be written, from the transformation (A6) and the relation (A8), as

$$S_{g\tilde{g}}(\mathrm{j}\omega_1, \mathrm{j}\omega_2) = 2\pi S_{g\bar{g}}\left\{\mathrm{j}\left(\frac{\omega_1 + \omega_2}{2} - \Omega\right)\right\}\delta_1(\omega_2 - \omega_1 - 2\Omega), \tag{A13}$$

and, similarly,

$$S_{\tilde{g}g}(\mathrm{j}\omega_1, \mathrm{j}\omega_2) = 2\pi S_{\bar{g}g}\left\{\mathrm{j}\left(\frac{\omega_1 + \omega_2}{2} - \Omega\right)\right\}\delta_1(\omega_2 - \omega_1 + 2\Omega). \tag{A14}$$

Here the quantity $S_{\bar{g}g}(\mathrm{j}\omega)$ is the dCSD between $\bar{g}(t)$ and $g(t)$. For the stationary complex random processes, $\{g(t)\}$ and $\{\tilde{g}(t)\}$, we can obtain, from equation (A5), that

$$R_{ii}(t, t) = R_{ii}(0) = \frac{1}{4\pi^2}\int\int S_{ii}(\mathrm{j}\omega_1, \mathrm{j}\omega_2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2 = \frac{1}{2\pi}\int S_{ii}(\mathrm{j}\omega_2)\,\mathrm{d}\omega_2, \qquad i = g, \tilde{g}, \tag{A15}$$

where

$$S_{ii}(\mathrm{j}\omega_2) = \frac{1}{2\pi}\int S_{ii}(\mathrm{j}\omega_1, \mathrm{j}\omega_2)\,\mathrm{d}\omega_1,$$

or, equivalently,

$$S_{ii}(j\omega_1, j\omega_2) = 2\pi S_{ii}(j\omega_2)\delta_1(\omega_2 - \omega_1). \tag{A16}$$

It follows that the double frequency PSDs, $S_{ii}(j\omega_1, j\omega_2)$, $i = g, \tilde{g}$, exist only on the line $\omega_1 = \omega_2$ in the $(j\omega_1, j\omega_2)$ plane, assuming the frequencies ω_1 and ω_2 are spaced $2\pi/T$ apart. Finally, equations (A12) and (A16) lead to the relation given by

$$S_{ii}(j\omega, j\omega) = TS_{ii}(j\omega), \quad i = g, \tilde{g}, \qquad \text{for } \omega_1 = \omega_2 = \omega. \tag{A17}$$