Journal of Sound and Vibration (1997) **206**(3), 425–434

LETTERS TO THE EDITOR

EXACT SOLUTIONS FOR THE ANALYSIS OF A NON-CONSERVATIVE BEAM SYSTEM WITH GENERAL NON-HOMOGENEOUS BOUNDARY CONDITIONS

S. M. Lin

Mechanical Engineering Department, Kung Shan Institute of Technology, Tainan, Taiwan 710, *Republic of China*

(*Received* 28 *May* 1996, *and in final form* 26 *March* 1997)

1. Introduction

Many physical problems such as diffusion, heat conduction, vibration, wave propagation etc., can be described by a one-dimensional, nth order, linear partial differential equation. So far, a few closed form solutions of some special partial equations have been obtained. By taking the Laplace transform with respect to time, the partial differential equation becomes a one-dimensional, nth order, linear ordinary differential equation. Yang and Tan [1] obtained the closed form matrix general solution of the nth order ordinary differential equation by transforming the equation into a first order system and taking some appropriate procedures. By taking the inverse Laplace transform, the solution in time domain will be obtained. However, the solution seems to be complicated. Stakgold [2] discussed some properties of the Green's function of boundary value problems for a nth order ordinary differential equation. But no simpler general solution of a one-dimensional, nth order, linear ordinary differential equation with variable coefficients has been obtained.

The vibration of a uniform Bernoulli–Euler beam with classical time dependent boundary conditions can be solved by using the method of Laplace transform [3, 4] and the method of Mindlin–Goodman [5, 6]. In the Mindlin–Goodman method, a procedure of change of dependent variable together with four shifting polynomials of the fifth degree are introduced. By properly selecting these shifting polynomial functions, the non-homogeneous boundary conditions are transformed into homogeneous ones. Consequently, the method of separation of variables was used to solve the problem. However, the method requires exact eigensolutions, which may be difficult to obtain particularly for some non-conservative system. Lee and Lin [7] study the dynamic response of a non-uniform Bernoulli–Euler beam with time dependent elastic boundary conditions by generalizing the method of Mindlin–Goodman [5] and utilizing the general solutions of general elastically restrained non-uniform beams given by Lee and Kuo [8]. The general solutions by Lee and Kuo are restricted to a conservative system only. For the uniform Timoshenko beams, the vibration of beams with classical time dependent boundary conditions was solved by Herrmann by adopting the method of Mindlin–Goodman [9]. In general, the study of non-conservative instability of beams was restricted to determining the critical loads and the instability mechanism by using the frequency equation [10]. There is still no study on the vibrational analysis of non-conservative system of a beam with time dependent elastic boundary conditions.

The purpose of this paper is to find a simple general solution and the Green's function of a boundary value problem with a nth order governing differential equation with variable coefficients. The general solution presented here is simpler than that given by Yang and Tan [1]. The general solution of the boundary value system is expressed in terms of n linearly independent homogeneous solutions. It is the generalization of that given by

0022–460X/97/380425 + 10 $25.00/0/sv971059

Stakgold [2]. The proposed method can be applied to both the boundary value problems and the initial value problems. With the proposed method, the limitation of the method given by Lee and Kuo [8] to a self-adjoint system is eliminated. Without transforming the non-homogeneous boundary conditions into the homogeneous ones by using the shifting functions, the general solution of a boundary value problem with time-dependent boundary conditions can be obtained directly [7]. A systematic theoretical development of the static and dynamic analysis of a non-uniform Bernoulli–Euler beam subjected to a partial tangential follower force, with non-homogeneous elastic boundary conditions, are presented. If the coefficients of the fourth order governing differential equation can be expressed in polynomial form, the closed form homogeneous solution can be obtained [11]. If the closed form homogeneous solutions are not available, then approximate solutions can be obtained through a simple and efficient numerical method given by Lee and Lin [12]. The suitability of the method to determine the flutter buckling loads of a non-conservative system through finding roots of the characteristic equation is discussed.

2. GENERAL SOLUTION

In general, a nth order ordinary differential equation with variable coefficients can be written as

$$q_n(x)\frac{\mathrm{d}^nV}{\mathrm{d}x^n}+q_{n-1}(x)\frac{\mathrm{d}^{n-1}V}{\mathrm{d}x^{n-1}}+\cdots+q_1(x)\frac{\mathrm{d}V}{\mathrm{d}x}+q_0(x)V=p(x), \qquad x\in(0, L) \tag{1}$$

where the coefficients $\{q_i(x)\}$ are variable on the closure domain [0, L], and the leading coefficient $q_n(x)$ does not vanish anywhere on the closure domain. Letting the Green's function be

$$E_\zeta(x)=G_\zeta(x)\mathrm{H}(x-\zeta), \tag{2}$$

where $\mathrm{H}(x-\zeta)$ is the Heaviside function and $G_\zeta(x)$ satisfies the following conditions

$$\mathrm{d}^jG_\zeta/\mathrm{d}x^j|_{x=\zeta}=0, \qquad j=0, 1, 2, \ldots, n-2, \tag{3}$$

$$\mathrm{d}^{n-1}G_\zeta/\mathrm{d}x^{n-1}|_{x=\zeta}=1/q_n(\zeta), \tag{4}$$

then the Green's function satisfies, in the distributional sense,

$$q_n(x)\frac{\mathrm{d}^nE_\zeta}{\mathrm{d}x^n}+q_{n-1}(x)\frac{\mathrm{d}^{n-1}E_\zeta}{\mathrm{d}x^{n-1}}+\cdots+q_1(x)\frac{\mathrm{d}E_\zeta}{\mathrm{d}x}+q_0(x)E_\zeta=\delta(x-\zeta). \tag{5}$$

The general solution $V(x)$ of equation (1) can be expressed in terms of a particular solution $V_p(x)$ and the n linearly independent homogeneous solutions $V_i(x)$ of equation (1)

$$V(x)=V_p(x)+\sum_{i=1}^{n}C_iV_i(x), \tag{6}$$

where $\{C_i\}$ are the constants to be determined. The particular solution can be obtained

$$V_p(x)=\int_0^L E_\zeta(x)p(\zeta)\,\mathrm{d}\zeta. \tag{7}$$

Letting the unknown function $G_\zeta(x)$ be

$$G_\zeta(x)=\sum_{i=1}^{n}\varepsilon_i(\zeta)V_i(x), \tag{8}$$

and substituting it into equations (3–4), the following coefficients $\{\varepsilon_i\}$ can be obtained

$$\begin{bmatrix} \varepsilon_1(\zeta) \\ \varepsilon_2(\zeta) \\ \vdots \\ \varepsilon_{n-1}(\zeta) \\ \varepsilon_n(\zeta) \end{bmatrix} = \begin{bmatrix} V_1(\zeta) & V_2(\zeta) & \cdots & V_{n-1}(\zeta) & V_n(\zeta) \\ V_1'(\zeta) & V_2'(\zeta) & \cdots & V_{n-1}'(\zeta) & V_n'(\zeta) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V_1^{(n-2)}(\zeta) & V_2^{(n-2)}(\zeta) & \cdots & V_{n-1}^{(n-2)}(\zeta) & V_n^{(n-2)}(\zeta) \\ V_1^{(n-1)}(\zeta) & V_2^{(n-1)}(\zeta) & \cdots & V_{n-1}^{(n-1)}(\zeta) & V_n^{(n-1)}(\zeta) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{1}{q_n(\zeta)} \end{bmatrix}.$$

Then the general solution $V(x)$ becomes

$$V(x) = \sum_{i=1}^{n} \left[\int_0^x \varepsilon_i(\zeta) p(\zeta)\, \mathrm{d}\zeta + C_i \right] V_i(x). \tag{10}$$

Substituting the general solution into the specified boundary conditions, the coefficients $\{C_i\}$ can be obtained. It should be noted that the proposed method can be applied to both boundary value problems and initial value problems. If the order of the differential equation (1) is four, the corresponding general solution (10) can be applied to the static and dynamic analysis of beams.

3. NON-CONSERVATIVE BEAM SYSTEM

3.1. *Static Analysis*

Consider the static deflection of a symmetric non-uniform Bernoulli–Euler beam with non-homogeneous elastic boundary conditions, resting on a non-uniform Winkler foundation $K(x)$ and subjected to any transverse force and partial tangential follower force, as shown in Figure 1.

In terms of the following dimensionless quantities,

$$b(\xi) = E(x)I(x)/E(0)I(0), \quad f_1 = F_1, \quad f_2 = F_2/L, \quad f_3 = F_3, \quad f_4 = F_4/L,$$
$$f_1^* = F_1^* L/E(0)I(0), \quad f_2^* = F_2^* L^2/E(0)I(0), \quad f_3^* = F_3^* L/E(0)I(0),$$
$$f_4^* = F_4^* L^2/E(0)I(0), \quad k(\xi) = K(x)L^4/E(0)I(0), \quad m(\xi) = \rho(x)A(x)/\rho(0)A(0),$$

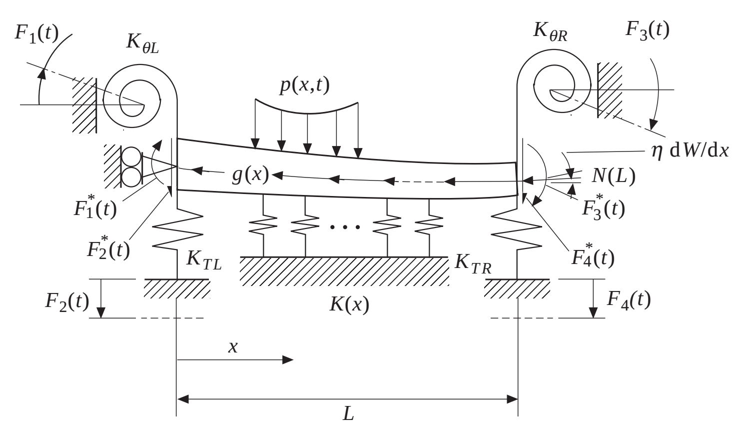

Figure 1. Geometry and co-ordinate system of a general elastically restrained non-uniform beam with general elastic non-homogeneous boundary conditions, subjected to the transverse force and the partial tangential follower force.

$$n(\xi) = N(x)L^2/E(0)I(0), \qquad p(\xi) = P(x)L^3/E(0)I(0), \qquad w(\xi) = W(x)/L, \qquad \xi = x/L,$$
$$\beta_1 = K_{\theta L}L^3/E(0)I(0), \qquad \beta_2 = K_{TL}L/E(0)I(0), \qquad \beta_3 = K_{\theta R}L^3/E(0)I(0),$$
$$\beta_4 = K_{TR}L/E(0)I(0), \tag{11}$$

the governing differential equation is

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[b(\xi)\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2}\right] + n(\xi)\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} + k(\xi)w = p(\xi), \qquad \xi \in (0, 1) \tag{12}$$

which is a non-conservative operator. The associated boundary conditions are at $\xi = 0$:

$$\gamma_{12}\,\mathrm{d}^2w/\mathrm{d}\xi^2 - \gamma_{11}\,\partial w/\partial\xi = \alpha_1, \qquad \gamma_{22}\,\mathrm{d}/\mathrm{d}\xi(b\,\mathrm{d}^2w/\mathrm{d}\xi^2) + \gamma_{21}w = \alpha_2, \tag{13, 14}$$

at $\xi = 1$:

$$\gamma_{32}b\,\mathrm{d}^2w/\mathrm{d}\xi^2 + \gamma_{31}\,\mathrm{d}w/\mathrm{d}\xi = \alpha_3, \qquad \gamma_{42}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(b\frac{\mathrm{d}^2w}{\mathrm{d}\xi^2}\right) + (1-\eta)n\frac{\mathrm{d}w}{\mathrm{d}\xi}\right] - \gamma_{41}w = \alpha_4, \tag{15, 16}$$

where

$$\gamma_{i1} = \beta_i/(1+\beta_i), \qquad \gamma_{i2} = 1/(1+\beta_i), \qquad i = 1, 2, 3, 4, \qquad \alpha_1 = -\gamma_{11}f_1 - \gamma_{12}f_1^*,$$
$$\alpha_2 = \gamma_{21}f_2 + \gamma_{22}f_2^*, \qquad \alpha_3 = \gamma_{31}f_3 + \gamma_{32}f_3^*, \qquad \alpha_4 = -\gamma_{41}f_4 - \gamma_{42}f_4^*, \tag{17}$$

in which $W(x)$ is the flexural displacement. $N(x)$ is a tangential follower force, equal to $N(L) + \int_x^L g(r)\,\mathrm{d}r$ in which $g(x)$ is a distributed tangential follower force. η is the tangential coefficient, $E(x)$, $I(x)$ and $A(x)$ denote Young's modulus, moment of inertia and area per unit length, respectively, $\rho(x)$ is the density per unit volume, $P(x)$ is the transverse force. F_1, F_2, F_1^* and F_2^* and F_3, F_4, F_3^* and F_4^* are the slope of the base, the displacement of the base, the external moment and the shear force excitations at the left end and the right end of the beam, respectively. K_{TL} and K_{TR} are the translational spring constants at $x = 0$ and $x = L$, respectively. $K_{\theta L}$ and $K_{\theta R}$ are the rotational spring constants at $x = 0$ and $x = L$, respectively.

Letting the homogeneous solutions $\{V_1, V_2, V_3, V_4\}$ of equation (12) satisfy the following normalized condition

$$\begin{bmatrix} V_1(0) & V_2(0) & V_3(0) & V_4(0) \\ V_1'(0) & V_2'(0) & V_3'(0) & V_4'(0) \\ V_1''(0) & V_2''(0) & V_3''(0) & V_4''(0) \\ V_1'''(0) & V_2'''(0) & V_3'''(0) & V_4'''(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

and substituting the corresponding general solution (10) into the boundary conditions (13–16), the associated coefficients can be obtained

$$C_i = \frac{1}{\chi}\sum_{j=1}^{4} a_{ij}\alpha_j^*, \qquad i = 1, 2, 3, 4, \tag{19}$$

where

$$\chi = \gamma_{12}[\gamma_{22}(G_1H_2 - G_2H_1) + \gamma_{21}(G_2H_4 - G_4H_2)] + \gamma_{11}\{\gamma_{22}[G_1H_3 - G_3H_1 + b'(0) \times (G_4H_1 - G_1H_4)] + \gamma_{21}(G_3H_4 - G_4H_3)\},$$

$$a_{11} = \gamma_{22}b'(0)(G_2H_1 - G_1H_2) + \gamma_{21}(G_3H_2 - G_2H_3),$$

$$a_{12} = \gamma_{12}(G_1H_2 - G_2H_1) + \gamma_{11}(G_1H_3 - G_3H_1),$$

$$a_{13} = \gamma_{11}[\gamma_{22}b'(0)H_1 - \gamma_{21}H_3] - \gamma_{12}\gamma_{21}H_2, \qquad a_{14} = \gamma_{12}\gamma_{21}G_2 - \gamma_{11}[\gamma_{22}b'(0)G_1 - \gamma_{21}G_3],$$

$$a_{21} = \gamma_{22}(G_1H_2 - G_2H_1) + \gamma_{21}(G_2H_4 - G_4H_2), \qquad a_{22} = \gamma_{11}(G_4H_1 - G_1H_4),$$

$$a_{23} = \gamma_{11}(\gamma_{21}H_4 - \gamma_{22}H_1), \qquad a_{24} = \gamma_{11}(\gamma_{22}G_1 - \gamma_{21}G_4),$$

$$a_{31} = \gamma_{22}(G_3H_1 - G_1H_3) + \gamma_{21}(G_4H_3 - G_3H_4) - \gamma_{22}b'(0)(G_4H_1 - G_1H_4),$$

$$a_{32} = \gamma_{12}(G_4H_1 - G_1H_4), \qquad a_{33} = \gamma_{12}(\gamma_{21}H_4 - \gamma_{22}H_1), \qquad a_{34} = \gamma_{12}(\gamma_{22}G_1 - \gamma_{21}G_4),$$

$$a_{41} = \gamma_{22}[b'(0)(G_4H_2 - G_2H_4) - (G_3H_2 - G_2H_3)],$$

$$a_{42} = \gamma_{12}(G_2H_4 - G_4H_2) + \gamma_{11}(G_3H_4 - G_4H_3),$$

$$a_{43} = \gamma_{22}[\gamma_{11}H_3 + \gamma_{12}H_2 - \gamma_{11}b'(0)H_4], \qquad a_{44} = -\gamma_{22}[\gamma_{11}G_2 + \gamma_{12}G_2 - \gamma_{11}b'(0)G_4],$$

$$\alpha_1^* = \alpha_1, \qquad \alpha_2^* = \alpha_2 - 2\gamma_{22}\varepsilon_1'(0)f'(0), \qquad \alpha_3^* = \alpha_3 - \gamma_{32}b(1)\hat{C} - \gamma_{31}\hat{B},$$

$$\alpha_4^* = \alpha_4 - \gamma_{42}[b(1)\hat{D} + b'(1)\hat{C}(1-\eta)n(1)\hat{B}] + \gamma_{41}\hat{A}, \tag{20}$$

in which

$$G_i = \gamma_{32}b(1)V_i''(1) + \gamma_{31}V_i'(1),$$

$$H_i = \gamma_{42}[b(1)V_i'''(1) + b'(1)V_i''(1) + (1-\eta)n(1)V_i'(1)] - \gamma_{41}V_i(1),$$

$$\hat{A} = \sum_{i=1}^{4}\int_0^1 \varepsilon_i(\zeta)f(\zeta)\,\mathrm{d}\zeta\, V_i(1), \qquad \hat{B} = \sum_{i=1}^{4}\int_0^1 \varepsilon_i(\zeta)f(\zeta)\,\mathrm{d}\zeta\, V_i'(1),$$

$$\hat{C} = \sum_{i=1}^{4}\left[\varepsilon_i(1)f'(1)V_i(1) + \int_0^1 \varepsilon_i(\zeta)f(\zeta)\,\mathrm{d}\zeta\, V_i''(1)\right],$$

$$\hat{D} = \sum_{i=1}^{4}\left\{[2\varepsilon_i'(1)f'(1) + \varepsilon_i(1)f''(1)]V_i(1) + 3\varepsilon_i(1)f'(1)V_i'(1) + \int_0^1 \varepsilon_i(\zeta)f(\zeta)\,\mathrm{d}\zeta\, V_i'''(1)\right\}. \tag{21}$$

It has been shown that if the coefficients of the differential equation are in polynomial forms, then the closed form homogeneous solutions can be obtained. Consequently, the closed form solution for the system is obtained [11]. If the closed form homogeneous solutions are not available, then approximate homogeneous solutions can be obtained through a simple and efficient numerical method [12].

3.2. *Dynamic Analysis*

The dimensionless governing equation for dynamic response $w(\xi, \tau)$ of a non-uniform Bernoulli–Euler beam with time-dependent elastic boundary conditions, as specified in section 3.1, is

$$\frac{\partial^2}{\partial\xi^2}\left[b(\xi)\frac{\partial^2 w}{\partial\xi^2}\right] + n(\xi)\frac{\partial^2 w}{\partial\xi^2} + k(\xi)w + m(\xi)\frac{\partial^2 w(\xi,\tau)}{\partial\tau^2} = p(\xi,\tau), \tag{22}$$

where $m(\xi)$ and $p(\xi, \tau)$ are the dimensionless mass per unit length of the beam and an arbitrary dimensionless transverse excitation force, respectively. τ is the dimensionless time variable. The associated boundary conditions are at $\xi = 0$:

$$\partial^2 w(\xi, t)/\partial\xi^2 - \beta_1\, \partial w(\xi, \tau)/\partial\xi = -\beta_1 f_1(\tau) - f_1^*(\tau), \tag{23}$$

$$\partial/\partial\xi\, (b(\xi)\, \partial^2 w(\xi, \tau)/\partial\xi^2) + \beta_2 w(\xi, \tau) = \beta_2 f_2(\tau) + f_2^*(\tau), \tag{24}$$

at $\xi = 1$:

$$b(\xi)\, \partial^2 w(\xi, \tau)/\partial\xi^2 + \beta_3\, \partial w(\xi, \tau)/\partial\xi = \beta_3 f_3(\tau) + f_3^*(\tau), \tag{25}$$

$$\frac{\partial}{\partial\xi}\left(b(\xi)\,\frac{\partial^2 w(\xi, \tau)}{\partial\xi^2}\right) + (1-\eta)n(1)\,\frac{\partial w(\xi, \tau)}{\partial\xi} - \beta_4 w(\xi, \tau) = -\beta_4 f_4(\tau) - f_4^*(\tau), \tag{26}$$

in terms of the dimensionless quantities (16) and

$$\tau = t/L^2\sqrt{E(0)I(0)/\rho(0)A(0)}, \tag{27}$$

where t is the time variable. The associated initial conditions are

$$w(\xi, 0) = w_0(\xi), \qquad \partial w(\xi, 0)/\partial\tau = \dot{w}_0(\xi), \tag{28}$$

where w_0 and $\dot{w}_0$ are two prescribed initial functions.

After taking the Laplace transform with respect to the dimensionless time variable τ, the following ordinary differential equation (22) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[b(\xi)\,\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\xi^2}\right] + n(\xi)\,\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\xi^2} + [k(\xi) + s^2 m(\xi)]\bar{w} = p^*(\xi, s), \tag{29}$$

and the associated boundary conditions equations (28–31) become at $\xi = 0$:

$$\gamma_{12}\, \mathrm{d}^2\bar{w}/\mathrm{d}\xi^2 - \gamma_{11}\, \mathrm{d}\bar{w}/\mathrm{d}\xi = \bar{\alpha}_1, \tag{30}$$

$$\gamma_{22}\, (\mathrm{d}/\mathrm{d}\xi)(b\, \mathrm{d}^2\bar{w}/\mathrm{d}\xi^2) + \gamma_{21}\bar{w} = \bar{\alpha}_2, \tag{31}$$

at $\xi = 1$:

$$\gamma_{32} b\,\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\xi^2} + \gamma_{31}\,\frac{\mathrm{d}\bar{w}}{\mathrm{d}\xi} = \bar{\alpha}_3, \qquad \gamma_{42}\left[\frac{\mathrm{d}}{\mathrm{d}\xi}\left(b\,\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\xi^2}\right) + (1-\eta)n\,\frac{\mathrm{d}\bar{w}}{\mathrm{d}\xi}\right] - \gamma_{41}\bar{w} = \bar{\alpha}_4, \tag{32, 33}$$

where

$$p^*(\xi, s) = \bar{p}(\xi, s) + m(\xi)[s w_0(\xi) + \dot{w}_0(\xi)], \qquad \bar{w}(\xi, s) = \int_0^\infty w(\xi, \tau)\, \mathrm{e}^{-s\tau}\, \mathrm{d}\tau,$$

$$\bar{\alpha}_1 = -\gamma_{11}\bar{f}_1(s) - \gamma_{12}\bar{f}_1^*(s), \qquad \bar{\alpha}_2 = \gamma_{21}\bar{f}_2(s) + \gamma_{22}\bar{f}_2^*(s), \qquad \bar{\alpha}_3 = \gamma_{31}\bar{f}_3(s) + \gamma_{32}\bar{f}_3^*(s),$$

$$\bar{\alpha}_4 = -\gamma_{41}\bar{f}_4(s) - \gamma_{42}\bar{f}_4^*(s), \qquad \bar{p}(\xi, s) = \int_0^\infty p(\xi, \tau)\, \mathrm{e}^{-s\tau}\, \mathrm{d}\tau,$$

$$\bar{f}_i(s) = \int_0^\infty f_i(\tau)\, \mathrm{e}^{-s\tau}\, \mathrm{d}\tau, \qquad \bar{f}_i^*(s) = \int_0^\infty f_i^*(\tau)\, \mathrm{e}^{-s\tau}\, \mathrm{d}\tau, \tag{34}$$

and s is the Laplace transform parameter. Since equations (29–33) are in the same form as equations (12–16), the transformed general solution $\bar{w}(\xi, s)$ and the Green's function are the same as those given in section 3.1. By taking the inverse Laplace transform, one obtains the dynamic response of the system

$$w(\xi, \tau) = \frac{1}{2\pi \mathrm{j}} \int_{c - \mathrm{j}\infty}^{c + \mathrm{j}\infty} \bar{w}(\xi, s)\, \mathrm{e}^{s\tau}\, \mathrm{d}s, \qquad \xi \in (0, 1) \tag{35}$$

where $\mathrm{j}^2 = -1$.

4. VERIFICATION AND EXAMPLES

The following examples are given to illustrate the validity and the accuracy of the analysis.

Example 1: Consider the vibration of a clamped–hinged uniform beam subjected to a displacement time dependent excitation $f_4 = 0{\cdot}02\tau^2$ at the right end of the beam. For convenience, one takes the initial conditions as $w_0(\xi) = \dot{w}_0(\xi) = 0$ and the transverse force $p(\xi, \tau) = 0$. Therefore, $f_1 = f_2 = f_3 = f_1^* = f_2^* = f_3^* = f_4^* = 0$. After following the solution methods given in section 3 and taking the method for the numerical inversion of Laplace transforms [13], the results are listed in Table 1. The numerical results in the rows with mark "*" are given by the present analysis and those in the rows with mark "**" are given by Grant [6]. It is found that the numerical results are very similar.

Example 2: Consider the vibration of a cantilever uniform beam subjected to a harmonic transverse excitation at the position ξ_0 of the beam and a partial tangential follower force n_0. The applied transverse harmonic force is given as

$$p(\xi, \tau) = \gamma_0 \sin(\omega\tau)\delta(\xi - \xi_0). \tag{36}$$

Since the time dependent excitations on the beam are in harmonic forms, it is not necessary to use the Laplace transform and inverse Laplace transform while solving the steady state solution of the dependent variable $w(\xi, \tau)$ in equation (22) and the boundary conditions equations (23–26). The dependent variable $w(\xi, \tau)$ can be assumed to take the form

$$w(\xi, \tau) = v(\xi) \sin(\omega\tau), \tag{37}$$

TABLE 1

The dynamic response of clamped–hinged uniform beam subjected to a displacement excitation at the right end [$f_1 = f_2 = f_3 = f_1^* = f_2^* = f_3^* = f_4^* = p(\xi, \tau) = 0$, $f_4 = 0{\cdot}02\tau^2$, $w_0(\xi) = \dot{w}_0(\xi) = 0$].

τ	ξ	0·2	0·4	0·6	0·8	1·0
0·5	*	0·0003	0·0011	0·0023	0·0036	0·0050
	**	0·0003	0·0010	0·0021	0·0035	0·0050
1·0	*	0·0011	0·0042	0·0088	0·0142	0·0200
	**	0·0011	0·0040	0·0085	0·0140	0·0200
1·5	*	0·0025	0·0094	0·0196	0·0318	0·0450
	**	0·0025	0·0093	0·0193	0·0316	0·0450
2·0	*	0·0045	0·0167	0·0347	0·0564	0·0800
	**	0·0045	0.0166	0·0346	0·0563	0·0800
2·5	*	0·0070	0·0261	0·0541	0·0881	0·1250
	**	0·0070	0·0260	0·0540	0·0880	0·1250

*, present study; **, by Grant [6].

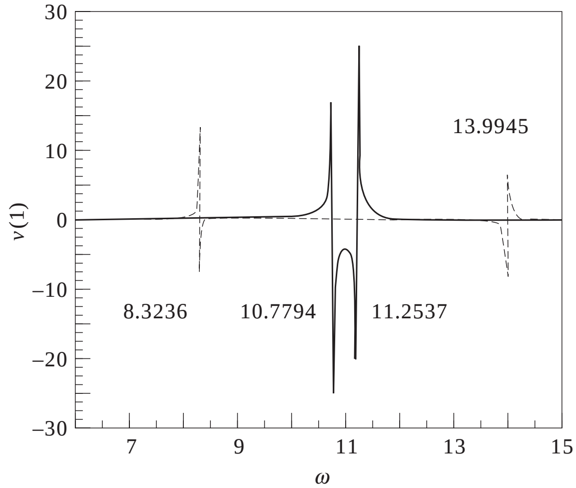

Figure 2. The amplitudes of steady response at the right end of a uniform cantilever beam subjected to different harmonic excitations [$p(\xi, \tau) = 0{\cdot}1 \sin(\omega\tau)\delta(\xi - 0{\cdot}5)$, the critical flutter load $n_{cr} = 20{\cdot}0591$, $\eta = 1{\cdot}0$: ——, $n_0 = 20{\cdot}0391$; ——, $n_0 = 18{\cdot}3333$].

where the coefficient $v(\xi)$ is to be determined. Substituting $w(\xi, \tau)$ into equation (22) and the boundary conditions equations (23–26), and by utilizing the results given in section 3.1, the exact solution is obtained:

$$w(\xi, \tau) = \begin{cases} \sum_{i=1}^{4} [\gamma_0 \varepsilon_i(\xi_0) + C_i] V_i(\xi) \sin \omega\tau, & \text{for } \xi \geqslant \xi_0, \\ \sum_{i=1}^{4} C_i V_i(\xi) \sin \omega\tau, & \text{for } \xi < \xi_0, \end{cases} \tag{38}$$

where the coefficients $\{C_i\}$ are listed in case 2 in the Appendix.

In Figure 2, the vibrational response curves at the tip of the beam are illustrated. It shows that when the excitation frequencies approach the natural frequencies of the beam, the response increases rapidly and becomes infinite as the excitation frequencies coincide with the natural frequencies. If the follower force increases, the first two natural frequencies will approach each other. Based on the method to determine the critical flutter loads determined through finding roots of the characteristic equation, when the follower force increases to the critical value, the first two natural frequencies coincide with each other. The critical flutter load of the beam is 20·0591. When the follower force is 20·0391 and the excitation frequencies are between the first two natural frequencies, the amplitudes of dynamic response at the tip of the beam are greater than 5. Then it is not a system with small deformation. But the system is derived on Bernoulli–Euler beam theory. So, it is concluded that the instability will occur under some load smaller than that determined through finding roots of the characteristic equation.

5. CONCLUSION

In this paper, a simple Green's function for a nth order ordinary differential equation with variable coefficients is proposed. The Green's function and the general solution are expressed in terms of n linearly independent normalized homogeneous solutions of the differential equation. It is the generalization of that given by Stakgold. The proposed method can be applied to both the boundary value problems and the initial value problems. The static and dynamic analysis of a non-uniform Bernoulli–Euler beam subjected to partial tangential follower force, with non-homogeneous elastic boundary conditions, are presented. The general solution is expressed in terms of the four normalized homogeneous solutions. The general solution can be obtained directly without transforming the non-homogeneous boundary conditions into the homogeneous ones. If the coefficients of the fourth order governing differential equation can be expressed in a polynomial form, the exact solution can be obtained. The instability will occur under a load smaller than that determined through finding roots of the characteristic equation.

ACKNOWLEDGMENT

The support of the National Science Council of Taiwan, R.O.C., is gratefully acknowledged (Grant number: Nsc85-2212-E168-001).

REFERENCES

1. B. Yang and C. A. Tan 1992 *Journal of Applied Mechanics* **59**, 1009–1014. Transfer functions of one-dimensional distributed parameter systems.
2. I. Stakgold 1979 *Green's Functions and Boundary Value Problems*. New York: John Wiley and Sons.
3. G. A. Nothmann 1984 *Journal of Applied Mechanics* **15**, 327–334. Vibration of a cantilever beam with prescribed end motion.
4. T. C. Yen and S. Kao 1959 *Journal of Applied Mechanics* **26**, 353–356. Vibration of beam–mass system time-dependent boundary conditions.
5. R. D. Mindlin and L. E. Goodman 1950 *Journal of Applied Mechanics* **17**, 377–380. Beam vibrations with time-dependent boundary conditions.
6. D. A. Grant 1983 *Journal of Sound and Vibration* **89**, 519–522. Beam vibrations with time-dependent boundary conditions.
7. S. Y. Lee and S. M. Lin 1996 *Journal of Applied Mechanics* **63**, 474–478. Dynamic analysis of non-uniform beams with time dependent elastic boundary conditions.
8. S. Y. Lee and Y. H. Kuo 1992 *Journal of Applied Mechanics* **59**, s205–s212. Exact solutions for the analysis of general elastically restrained non-uniform beams.
9. G. Herrmann 1985 *Journal of Applied Mechanics* **22**, 53–56. Forced motions of Timoshenko beams.
10. S. Y. Lee, W. R. Wang and T. Y. Chen 1995 *Journal of Sound and Vibration* **188**, 25–38. Non-conservative instability of a Timoshenko beam subjected to a partially tangential follower force.
11. S. Y. Lee and S. M. Lin 1992 *American Institute of Aeronautics and Astronautics Journal* **30**, 2930–2934. Exact vibration solutions for non-uniform Timoshenko beams with attachments.
12. S. Y. Lee and S. M. Lin 1995 *Journal of Sound and Vibration* **183**, 403–415. Vibrations of elastically restrained non-uniform Timoshenko beams.
13. G. Honig and U. Hirdes 1984 *Journal of Computational and Applied Mathematics* **10**, 113–132. A method for the numerical inversion of Laplace transforms.

APPENDIX

Case 1: Clamped–Hinged

In this case, β_1, β_2 and β_4 are infinite, β_3 is zero, the non-homogeneous terms become

$$\alpha_1 = -f_1, \qquad \alpha_2 = f_2, \qquad \alpha_3 = f_3^*, \qquad \alpha_4 = -f_4.$$

The coefficients of the general solution are

$$C_1 = (1/b(1)[V_4''(1)V_3(1) - V_4(1)V_3''(1)])\{\alpha_1 b(1)[V_2''(1)V_3(1) - V_2(1)V_3''(1)]$$

$$+ \alpha_2 b(1)[V_3''(1)V_1(1) - V_3(1)V_1''(1)] + (\alpha_3 - b(1)\bar{C})V_3(1) + \alpha_4 \bar{A} b(1)V_3''(1)\},$$

$$C_2 = (1/b(1)[V_4''(1)V_3(1) - V_4(1)V_3''(1)])\{\alpha_1 b(1)[V_4''(1)V_2(1) - V_4(1)V_2''(1)]$$

$$+ \alpha_2 b(1)[V_1''(1)V_4(1) - V_1(1)V_4''(1)] - (\alpha_3 - b(1)\bar{C})V_4(1) - \alpha_4 \bar{A} b(1)V_4''(1)\},$$

$$C_3 = -\alpha_1, \qquad C_4 = \alpha_2.$$

Case 2: Clamped–Free

In this case, β_1 and β_2 are infinite, β_3 and β_4 are zero, the non-homogeneous terms become

$$\alpha_1 = -f_1, \qquad \alpha_2 = f_2, \qquad \alpha_3 = f_3^*, \qquad \alpha_4 = -f_4^*.$$

The coefficients of the general solution are

$$C_1 = (1/[G_3 H_4 - G_4 H_3])\{\alpha_1(G_3 H_2 - G_2 H_3) + \alpha_2(G_1 H_3 - G_3 H_1) + [\alpha_3 - b(1)\bar{C}]H_3$$

$$+ (\alpha_4 - b(1)\bar{D} - b'(1)\bar{C})b(1)V_3''(1)\},$$

$$C_2 = (1/[G_3 H_4 - G_4 H_3])\{\alpha_1(G_2 H_4 - G_4 H_2) + \alpha_2(G_4 H_1 - G_1 H_4) + [\alpha_3 - b(1)\bar{C}]H_4$$

$$+ (\alpha_4 - b(1)\bar{D} - b'(1)\bar{C})b(1)V_4''(1)\},$$

$$C_3 = -\alpha_1, \qquad C_4 = \alpha_2,$$

in which

$$G_i = b(1)V_i''(1), \qquad H_i = b(1)V_i'''(1) + b'(1)V_i''(1) + (1 - \eta)n(1)V_i'(1).$$