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EXACT ANALYSIS FOR AXISYMMETRIC VIBRATION AND BUCKLING OF THE THICK LAMINATED CLOSED CYLINDRICAL SHELLS IN A HAMILTON SYSTEM

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1. INTRODUCTION

The current theories of plates and shells are established on certain hypotheses. For example, some assume that the mechanical quantities are the polynomials of a certain co-ordinate variable. It is shown later that the true solution for each mechanical quantity cannot be a polynomial of any co-ordinate variable. If the form of a polynomial is adopted, the incompatibility among the fundamental equations must appear in the deductive process, and only some of the elastic constants can be taken into account. This contradiction results in errors in all of the current theories, especially in thicker plates and shells. In addition, all the existing theory is invalid for plates and of considerable thickness.

With no initial assumptions regarding stress and deformation models [1–4], and using three-dimensional elasticity, the free vibration problems of homogeneous isotropic, orthotropic, and laminated thick cylindrical shells and plate were solved. In these papers, the thick shells were divided into N fictitious subcylinders in order to simplify the variable coefficient differential equations into a set of simpler ones that was solved by using a method of successive approximations. The frequencies with desired accuracy were obtained, by increasing the value of N. A similar approach was suggested by Bhimaraddi to analyze the free vibration of doubly curved shallow shells [5]. But all of the papers had to deal with the many unknowns that appear at the real or fictitious interfaces. Based on the three-dimensional theory of elasticity, the buckling and vibration of isotropic, orthotropic and laminated cylindrical shells have also been studied by Kardomateas [6, 7], and by Ye and Soldators [8–10]. In reference [11–12] the method of state space was developed and exact solutions for single-ply and laminated rectangular plates were given. Ding and Fan [13] also used this method for the axisymmetric statics problem of the closed laminated continuous cylindrical shell. In this paper, by introducing the Hellinger–Reissner variational principle, the Hamilton canonical equation is presented. Exact solutions are expressed for the dynamics and buckling of the axisymmetric problem of thin, moderately thick and thick laminated closed cylindrical shells. Numerical results are given to compare with those of FEM calculated using SAP5.

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2. HAMILTON CANONICAL EQUATION FOR THE AXISYMMETRIC PROBLEM OF A CLOSED CYLINDRICAL SHELL

The modified Hamilton-type Hellinger–Reissner variational principle can be shown to be of the form

$$\Pi^* = \int_V \left\{ \frac{\partial U}{\partial r}\tau_{rx} + \frac{\partial W}{\partial r}\sigma_r - H \right\} 2\pi r \, \mathrm{d}r \, \mathrm{d}x - \int_{S_\sigma} (\bar{p}_r W + \bar{p}_x U) \, \mathrm{d}s, \tag{1}$$

in which the usual index notation is used and Figure 1 shows the co-ordinate system. S_σ denotes the portion of the edge boundary where tractions $\bar{p}_i$ are prescribed. The quadratic form of the Hamilton function H can be written as

$$-H = \sigma_x \frac{\partial U}{\partial x} + \sigma_\theta \frac{W}{r} + \tau_{rx}\left(\frac{\partial w}{\partial x} + \frac{\partial U}{\partial r}\right) - \tfrac{1}{2}\{\boldsymbol{\sigma}\}^{\mathrm{T}}[\mathbf{C}]\{\boldsymbol{\sigma}\} - \tfrac{1}{2}\rho\left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial W}{\partial t}\right)^2\right], \tag{2}$$

where ρ is the mass density and the matrix $[\mathbf{C}]$ is the elastic stiffness matrix. For an orthotropic body, one has

$$[\mathbf{C}] = \begin{bmatrix} C_{11} & C_{12} & c_{13} & 0 \\ C_{12} & C_{22} & C_{23} & 0 \\ C_{13} & C_{23} & C_{33} & 0 \\ 0 & 0 & 0 & C_{55} \end{bmatrix}. \tag{3}$$

Using $\delta\Pi^* = 0$, the following relations can be obtained

$$\partial U/\partial r = \partial H/\partial \tau_{rx}, \qquad \partial W/\partial r = \partial H/\partial \sigma_r, \qquad \partial \tau_{rx}/\partial r = -\partial H/\partial U, \qquad \partial \sigma_r/\partial r = -\partial H/\partial W. \tag{4}$$

One denotes

$$q = (U \quad W), \qquad p = (\tau_{rx} \quad \sigma_r). \tag{5}$$

Equation (4) can also be written in a simplified form

$$\partial q/\partial r = \partial H/\partial p, \qquad \partial p/\partial r = -\partial H/\partial q \tag{6}$$

This is a classical Hamilton canonical equation.

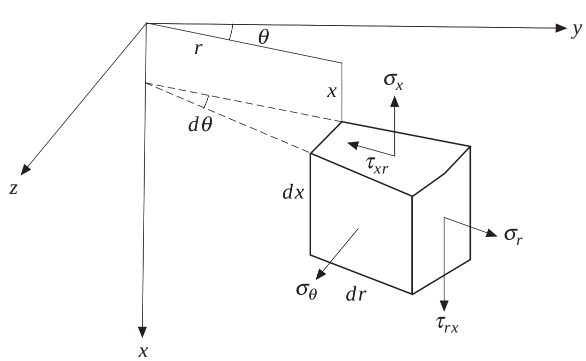

Figure 1. A cylindrical co-ordinate system.

Elimination σ_x and σ_θ from equation (4) by letting

$$\alpha = \partial/\partial x, \qquad \zeta = \rho\,\partial^2/\partial t^2, \qquad \sigma = \sigma_r, \qquad X = \tau_{rx}, \qquad C_1 = -C_{13}/C_{33},$$
$$C_2 = C_{11} - C_{13}^2 C_{33},$$
$$C_3 = C_{12} - C_{13}C_{23}/C_{33}, \qquad C_4 = C_{22} - C_{23}^2/C_{33}, \qquad C_5 = -C_{23}/C_{33},$$
$$C_7 = 1/C_{33}, \qquad C_8 = 1/C_{55}$$

one has

$$(\partial/\partial r)[U \quad \sigma \quad X \quad W]^{\mathrm{T}} = D(r)[U \quad \sigma \quad X \quad W]^{\mathrm{T}} \tag{7}$$

in which

$$D(r) = \begin{bmatrix} 0 & 0 & C_8 & -\alpha \\ (C_3/r)\alpha & -(C_5+1)/r & -\alpha & C_4/r^2 + \zeta^2 \\ \zeta^2 - C_2\alpha^2 & C_1\alpha & -1/r & -(C_3/r)\alpha \\ C_1\alpha & C_7 & 0 & C_5/r \end{bmatrix}. \tag{8}$$

The mechanical expression of equation (6) is that they constitute mixed simpletic space. When the modulus of the medium changes or two different median are in contact, the simpletic variables must be kept continuous, but the variables σ_x and σ_θ may be discontinuous, and can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \end{Bmatrix} = \begin{bmatrix} C_2\alpha & -C_1 & C_3/r \\ C_3\alpha & -C_5 & C_4/r \end{bmatrix} \begin{Bmatrix} U \\ \sigma \\ W \end{Bmatrix}. \tag{9}$$

For a simply supported shell, the quantities in equation (7) are expanded into the following series system

$$U = \sum_m U_m(r) \cos\frac{m\pi x}{l}\,\mathrm{e}^{\mathrm{i}\omega_m t}, \qquad \sigma = \sum_m \sigma_m(r) \sin\frac{m\pi x}{l}\,\mathrm{e}^{\mathrm{i}\omega_m t},$$

$$X = \sum_m X_m(r) \cos\frac{m\pi x}{l}\,\mathrm{e}^{\mathrm{i}\omega_m t}, \qquad W = \sum_m W_m(r) \sin\frac{m\pi x}{l}\,\mathrm{e}^{\mathrm{i}\omega_m t}. \tag{10}$$

Considering equations (10) and (9), it can be seen that the boundary conditions are satisfied,

$$x = 0, l, \qquad W = \sigma_x = 0. \tag{11}$$

Introducing equations (10) into equation (7) yields for each m:

$$(\mathrm{d}/\mathrm{d}r)[U_m(r) \quad \sigma_m(r) \quad X_m(r) \quad W_m(r)]^{\mathrm{T}} = \mathbf{D}(r)[U_m(r) \quad \sigma_m(r) \quad X_m(r) \quad W_m(r)]^{\mathrm{T}}, \tag{12}$$

where

$$\mathbf{D}(r) = \begin{bmatrix} 0 & 0 & C_8 & -\xi \\ -(C_3/r)\xi & -(C_5+1)/r & \xi & C_4/r^2 - \rho\omega^2 \\ C_2\xi^2 - \rho\omega^2 & C_1\xi & -1/r & -(C_3/r)\xi \\ -C_1\xi & C_7 & 0 & C_5/r \end{bmatrix}, \tag{13}$$

with $\xi = m\pi/l$, $\omega = \omega_m$. Equation (12) is called the variable coefficient non-homogeneous state equation. Based on equation (12) one can prove that each mechanical quantity cannot be a polynomial of co-ordinated r. If $U_m(r)$ and $\sigma_m(r)$ were the polynomials of degree l for variable r, from the third and fourth row of equation (12), $X_m(r)$ and $W_m(r)$ would have to be the polynomials of degree $l+1$ for r. If so, observing the two other rows of the same equation, $U_m(r)$ and $\sigma_m(r)$ would be the polynomials of degree $l+2$ of r, which contradicts what has been supposed.

3. THE SOLUTION OF THE AXISYMMETRIC PROBLEM OF A LAMINATED CLOSED CYLINDRICAL SHELL

A p-plied laminated thick cylindrical shell is made up of orthotropic layers. The length and the thickness of the shell are l and h $(=a-b)$, respectively. The state equation for the jth ply is obtained from equation (12), as follows:

$$(\mathrm{d}/\mathrm{d}r)[U_m(r) \quad \sigma_m(r) \quad X_m(r) \quad W_m(r)]_j^{\mathrm{T}} = \mathbf{D}_j[U_m(r) \quad \sigma_m(r) \quad X_m(r) \quad W_m(r)]_j^{\mathrm{T}} \tag{14}$$

The solution of equation (14) is

$$\mathbf{R}_j(r) = \mathbf{G}_j(r-a_j)\mathbf{R}_j(a_j), \tag{15}$$

where

$$\mathbf{R}_j(r) = [U_m(r) \quad \sigma_m(r) \quad X_m(r) \quad W_m(r)]_j^{\mathrm{T}}, \tag{16a}$$

$$\mathbf{R}_j(a_j) = [U_m(r) \quad \sigma_m(r) \quad X_m(r) \quad W_m(r)]_j^{\mathrm{T}}, \tag{16b}$$

$$\mathbf{G}_j(r-a_j) = \exp[\mathbf{D}_j(r-a_j)]. \tag{17}$$

Applying the transfer matrix method [13], the mechanical quantities of the interior surface and outer surfaces for the entire laminated shell can be linked together in the form:

$$\mathbf{R}_p(b) = \mathbf{\Pi} \quad \mathbf{R}_1(a), \tag{18}$$

where

$$\mathbf{\Pi} = \prod_{p}^{1} \mathbf{G}_j.$$

$\mathbf{R}_1(a)$ and $\mathbf{R}_p(b)$ in equation (18) are the mechanical quantities for the outer and interior surfaces of the laminated shell, respectively. $\mathbf{R}_1(a)$ is called the initial value. $\mathbf{\Pi}$ is a (4×4) constant matrix. Usually, the loads acting on the interior and outer surfaces of a shell are given *a priori*. Actually, equation (18) is a matrix equation for four displacements of the outer and interior surfaces of a shell. When normal pressure $q(q=\text{const})$ acts on the interior surface of the shell, the load is expanded in the same series as the σ-series in equation (10) and $X_m(a) = X_m(b) = \sigma_m(a) = 0$ thus equation (18) becomes:

$$\begin{bmatrix} \Pi_{21} & \Pi_{24} \\ \Pi_{31} & \Pi_{34} \end{bmatrix} \begin{Bmatrix} U_m(a) \\ W_m(a) \end{Bmatrix} = \begin{Bmatrix} \sigma_m(b) \\ 0 \end{Bmatrix}. \tag{19}$$

In the calculation of natural frequencies, let the right side of equation (19) be zero. The non-trivial solution of equation (19) gives

$$\begin{vmatrix} \Pi_{21} & \Pi_{24} \\ \Pi_{31} & \Pi_{34} \end{vmatrix} = 0. \tag{20}$$

TABLE 1

Frequency parameters Ω and critical stress parameters λ for a single-ply shell with different ratios h/R_0

ρ_1/ρ_2	$C_{11}^{(1)}/C_{11}^{(2)}$	h/R_0	SAP5 Ω_1	present study I_1	I_2	I_3	Ω_1, λ_1	Ω_2, λ_2	Ω_3, λ_3
1	1	0·1	0·0443	1	4	1	0·0453	0·0749	1·2579
1	1	0·2	0·0886	1	4	1	0·0906	0·1499	1·2626
1	1	0·4		2	5	2	0·1814	0·3017	1·2814
1	1	0·6		2	8	2	0·2727	0·4571	1·3131
1	1	0·8		3	6	3	0·3644	0·6169	1·3574
1	1	1·0		3	8	3	0·4567	0·7807	1·4151

It should be mentioned that instead of being a polynomial in ω^2 as in the ordinary theories, equation (20) is a transcendental one. In fact, equation (20) is the exact frequency equation of the laminated shell in the sense of satisfying a prescribed precision. Equation (20) has an infinite number of roots corresponding to an infinite number of frequencies, which can be determined by using the procedure for finding the zero points of a function. Obviously, the frequency bandwidth in the present study is much larger than that produced by other theories.

In order to solve the buckling problem, equation (2) should be written as

$$-H = \sigma_x \frac{\partial U}{\partial x} + \sigma_\theta \frac{W}{r} + \tau_{rx}\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial r}\right) - \tfrac{1}{2}\{\boldsymbol{\sigma}\}^{\mathrm{T}}[\mathbf{C}]\{\boldsymbol{\sigma}\} - \tfrac{1}{2}p_x\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2\right] \quad (21)$$

where p_x are the uniformly distributed compressive stresses acting on the two edges of a shell, respectively, along the x direction. Comparing equation (2) and equation (21) one sees that if ζ^2 in equation (8) is replaced with $p_x\alpha^2$, an infinite number of critical stresses can be obtained from equation (20). Of course, the minimum critical stress is the most useful.

4. NUMERICAL EXAMPLE

The example given below was done on a SIEMENS/7570c processor in four-fold precision, I_1, I_2 and I_3 in the following tables are the number of thin plies corresponding to the first, second and third layer, respectively.

Example. A three-ply closed cylindrical shell is used. The materials for the first and third layers are identical. Each layer has the same elastic constants: $C_{12}/C_{11} = 0{\cdot}246269$, $C_{13}/C_{11} = 0{\cdot}0831715$, $C_{22}/C_{11} = 0{\cdot}543103$, $C_{23}/C_{11} = 0{\cdot}115017$, $C_{33}/C_{11} = 0{\cdot}530172$, $C_{55}/C_{11} = 0{\cdot}159914$, $C_{11}^{(1)}/C_{11}^{(2)} = 5$, where $C_{11}^{(1)}$ and $C_{11}^{(2)}$ denote C_{11} of the materials corresponding to the first and second layers, respectively. When $C_{11}^{(1)} = C_{11}^{(2)} = C_{11}$, the three-ply shell degenerates into a homogeneous one. The laminated shell has the following geometry parameters: $h_1 = h_3 = 0{\cdot}1h$, $h_2 = 0{\cdot}8$, $l = s = R_0$, where l = length of the shell, s = the arc length of middle surface and R_0 = the radius of middle surface.

The densities for the outer and middle layers are denoted by ρ_1 and ρ_2 respectively. When $m = 1$, the first three natural frequencies and critical stresses for the single-ply shell ($C_{11}^{(1)} = C_{11}^{(2)} = C_{11}$, $\rho_1 = \rho_2 = \rho$) and the three-ply shell are indicated in Table 1 and Table 2, respectively. The results for a three-dimensional finite element method (FEM) using SAP5 are also shown in Tables 1 and 2. Because of the symmetry, 24 three-dimensional

TABLE 2

Frequency parameters Ω and critical stress parameters λ for a three-plied laminated shell with different ratios h/R_0

			SAP5	present study								
ρ_1/ρ_2	$C_{11}^{(1)}/C_{11}^{(2)}$	h/R_0	Ω_1	I_1	I_2	I_3	Ω_1	Ω_2	Ω_3	λ_1	λ_2	λ_3
3	5	0·1		1	4	1	0·0514	0·0849	0·9498	0·0453	0·0749	0·7890
3	5	0·2		1	4	1	0·1028	0·1703	0·9572	0·0908	0·1515	0·7344
3	5	0·4		2	5	2	0·2058	0·3438	0·9867	0·1824	0·3082	0·8111
3	5	0·6		2	8	2	0·3092	0·5222	1·0349	0·2730	0·4598	0·8627
3	5	0·8	0·4013	3	6	3	0·4127	0·7024	1·1002	0·3649	0·6165	0·9207
3	5	1·0	0·4991	3	8	3	0·5158	0·8744	1·1808	0·4568	0·7623	0·9938

isoparametric elements (for 1/4 shell) with 20 nodes are employed in the calculation. It can be seen that the first-order frequency calculated by SAP5 is accurate enough. Where $\Omega = \omega h\sqrt{\rho_2/C_{11}^{(2)}}$, $\lambda = (\pi h/l)\sqrt{\rho_{xcr}/C_{11}}$, $(\rho_{xcr}/C_{11})_i = \text{cons tan } t$, $i = 1, 2, 3$, ρ_{xcr} = critical stress.

5. CONCLUSION

The exact analysis of axisymmetric vibration and buckling using the Hamilton equation is an efficient method. Exact frequencies and critical stresses are given for the axisymmetric problem of thin, moderately thick and thick closed laminated cylindrical shells. The principle and method suggested here have clear physical concepts and can overcome the contradictions and limitations that arise from incompatibilty among the fundamental equation in various theories of plates and shells. Numerical results denote that the method of dividing the layer into several thin plies has the characteristics of fast convergence rate, satisfactory precision, and controlled error. The present study satisfies the continuity conditions of stresses and displacements at the interfaces, and the number of variables is reduced greatly.

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