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A NOTE ON THE STRAIN ENERGY STORED IN ROTATIONAL SPRINGS AT THE PLATE EDGES OF NON-UNIFORM THICKNESS

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1. Introduction

The determination of natural frequencies in the transverse vibration of rectangular plates is a problem that has been extensively studied by several researchers. Not only the classical boundary conditions (i.e., clamped, simply-supported or free) with the 21 distinct possible combinations have been considered, but also the cases of elastically restrained edges have been taken into account. On the other hand, among the methods used to study these problems, the Ritz method has been extensively applied. When dealing with this method it is necessary to select a sequence of functions called co-ordinate functions. An important property arises concerning the boundary conditions. It is not necessary to subject the co-ordinate functions to the natural boundary conditions which govern the problem under study. It is sufficient that they satisfy only the geometric ones [1–3]. This property increases the attractiveness of the method specially when dealing with problems for which such satisfaction is difficult to achieve. For instance, the problem of free vibration of a orthotropic rectangular plate with thickness varying in two directions and with edges elastically restrained against rotation is considered. The boundary conditions which correspond to edge $x = a$ are given by

$$W(a, y, t) = 0, \qquad r_2 \frac{\partial W}{\partial x} = -D_x(x, y)\left(\frac{\partial^2 W}{\partial x^2} + \mu_y \frac{\partial^2 W}{\partial y^2}\right), \qquad x = a, \tag{1, 2}$$

where r_2 is the rotational spring constant along the edge, W is the plate deflection; and $D_x(x, y)$ is the flexural rigidity and is given by

$$D_x(x, y) = D_x^{(1)} f^3(x) g^3(y), \tag{3}$$

where $f(x)$ and $g(x)$ describe the thickness variation.

It is very difficult to obtain coordinate functions which satisfy identically equation (2), and it is convenient to replace it, for instance, by

$$r_2\, \partial W/\partial x = -D_x^{(2)}(\partial^2 W/\partial x^2 + \mu_y\, \partial^2 W/\partial y^2), \qquad x = a, \tag{4}$$

where $D_x^{(2)} = D_x^{(1)} f^3(a) g^3(b)$. This procedure leads in most cases to highly satisfactory results. But as stated by Dickinson [4], users of the Ritz method should not deliberately seek

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co-ordinate functions which satisfy more easily applied boundary conditions, specially when using a several term approximating function. When dealing with one or two term approximation function the procedure has been successfully used in various works [5–9]. It is the purpose of this note to demonstrate that when using this procedure in rectangular plates of non-uniform thickness, with edges elastically restrained, the two commonly used expressions that give the stored energy in the elastic supports are not equivalent and that one of them leads to better results.

2. STATEMENT OF THE PROBLEM

A rectangular orthotropic plate with thickness varying in two directions and with edges elastically restrained against rotation is considered. The Rayleigh–Ritz method requires the minimization of the Rayleigh quotient which for the fundamental frequency is given by

$$\omega^2 = U_{max}/T_{max}, \tag{5}$$

where $U_{max} = U_{p,max} + U_{r,max}$, $U_{p,max}$ is the maximum strain energy of the plate, $U_{r,max}$ is the maximum strain energy associated to the rotational restraints in the edges, T_{max} is the maximum kinetic energy of the plate. The maximum strain energy associated to the rotational restraints in the edges is given by

$$U_{r,max} = \frac{1}{2}\left\{r_1 \int_0^b [W_x(0, y)]^2\, \mathrm{d}y + r_2 \int_0^b [W_x(a, y)]^2\, \mathrm{d}y + r_3 \int_0^a [W_x(x, 0)]^2\, \mathrm{d}x + r_4 \int_0^a [W_x(x, b)]^2\, \mathrm{d}x\right\}, \tag{6}$$

where r_i $(i = 1, 4)$ are the rotational spring constants along the corresponding edges. The subscripts denote differentiation of W in respect of the subscripted variable; a and b are side lengths of the plate in the x and y directions; respectively.

For instance, consider the edge which corresponds to boundary conditions (1) and (2). The energy stored in the rotational springs of constant r_2 can also be calculated by means of the alternative expression [8, 9]

$$U_{r_2 max} = -\frac{1}{2}\int_0^b D_x(a, y)[W_{xx}(a, y) + \mu_y W_{yy}(a, y)]W_x(a, y)\, \mathrm{d}y. \tag{7}$$

When dealing with co-ordinate functions which satisfy all the boundary conditions the two expressions are equivalent, but when dealing with the procedure of simplification of the natural boundary condition stated above, the equivalence in general disappears. If one replaces the term W_{xx} in equation (7) by the expression obtained from equation (4) one gets (if the edge is supported, the term $\mu_y W_{xx}$ is equal zero):

$$U_{r_2 max} = -\frac{1}{2}\int_0^b D_x(a, y)[W_{xx}(a, y) + \mu_y W_{yy}(a, y)]W_x(a, y)\, \mathrm{d}y$$

$$= -\frac{1}{2}\int_0^b D_x(a, y)\left[-W_x(a, y)\,\frac{r_2}{D_x^{(2)}}\right]W_x(a, y)\, \mathrm{d}y$$

$$= \frac{r_2}{2D_x^{(1)} f^3(a) g^3(b)} D_x^{(1)} f^3(a) \int_0^b g^3(y) W_x^2(a, y)\, \mathrm{d}y$$

$$= \frac{r_2}{2g^3(b)} \int_0^b g^3(y) W_x^2(a, y)\, \mathrm{d}y.$$

Thus one obtains the expression

$$U_{r_2,max} = \frac{r_2}{2g^3(b)} \int_0^b g^3(y)[W_x(a, y)]^2\, \mathrm{d}y. \tag{8}$$

On the other hand, if one uses the original boundary condition (2) one gets

$$U_{r_2,max} = \frac{r_2}{2} \int_0^b [W_x(a, y)]^2\, \mathrm{d}y. \tag{9}$$

It has been proved that, when dealing with approximating functions which do not satisfy all the boundary conditions identically, expression (7) and (9) are not equivalent. So the repercussion on the numerical values of the natural frequencies, when using expression (8) instead of (9) is considered.

The following approximating function is adopted;

$$W(x, y) = A_1 X_1(x) Y_1(y), \tag{10}$$

TABLE 1

Fundamental frequency coefficient $\Omega_{00} = \sqrt{\rho h^{(1)}/D^{(1)}}\,\omega_{00} b^3$ *of a rectangular isotropic plate of variable thickness with edges* 1, 3 *and* 4 *rigidly clamped*, *and edge* 2 *elastically restrained against rotation*. ($\mu = 0{\cdot}3$, R_1, R_3 y $R_4 = \infty$), ($f(x) = 1 + c_1(x/a)$, $g(y) = 1 + c_2(y/b)$), $D_x^{(1)}/H_{xy}^{(1)} = 1$, $D_y^{(1)}/H_{xy}^{(1)} = 1$

			b/a			
			1		2	
c_1	c_2	R_2	I	II	I	II
$c_1 = -0, 2$	$c_2 = 0, 2$	∞	35·9599	35·9599	98·6029	98·6029
		10	33·8466	34·0666	88·0972	89·4433
		5	33·0945	33·3043	83·9730	85·2896
		1	31·9810	32·0674	77·3580	77·9283
		0	31·5633	31·5633	74·6803	74·6803
0, 2	0, 2	∞	43·8737	43·8737	120·1757	120·1757
		10	41·6823	42·1685	107·6322	110·6207
		5	40·8605	41·3209	102·1277	105·0500
		1	39·5936	39·7816	92·5667	93·8478
		0	39·0998	39·0998	88·4122	88·4122
$-0, 4$	0, 2	∞	32·6474	32·6474	90·6215	90·6215
		10	30·5433	30·6600	80·9276	81·6311
		5	29·7979	29·9098	77·2900	77·9790
		1	28·6993	28·7457	71·6679	71·9654
		0	28·2893	28·2893	69·4722	69·4722

(I) Values obtained with expression (7); (II) Values obtained with expression (9).

TABLE 2

Fundamental frequency $\Omega_{00} = \sqrt{\rho h^{(1)}/H_{xy}^{(1)}}\,\omega_{00} b^2$ of a rectangular orthotropic plate with edges 1, 3 and 4 rigidly clamped, and edge 2 elastically restrained against rotation. (R_1, R_3 y $R_4 = \infty$), ($f(x) = 1 + c_1(x/a)$, $g(y) = 1 + c_2(y/b)$), $D_x^{(1)}/H_{xy}^{(1)} = 1$, $D_y^{(1)}/H_{xy}^{(1)} = 0{\cdot}5$, $\mu_y = 0{\cdot}3$

			b/a			
			1		2	
c_1	c_2	R_2	I	II	I	II
$c_1 = -0, 2$	$c_2 = 0, 2$	∞	32·2813	32·2813	97·3216	97·3216
		10	29·9612	30·2095	86·6787	88·0465
		5	29·1301	29·3682	82·4911	83·8310
		1	27·8939	27·9930	75·7598	76·3420
		0	27·4285	27·4285	73·0289	73·0289
0, 2	0, 2	∞	39·3741	39·3741	118·6069	118·6069
		10	36·8619	37·4108	105·8587	108·8960
		5	35·9080	36·4310	100·2490	103·2245
		1	34·4228	34·6389	90·4758	91·7860
		0	33·8388	33·8388	86·2148	86·2148
−0, 4	0, 2	∞	29·4075	29·4075	89·5054	89·5054
		10	27·1508	27·2821	79·7092	80·4233
		5	26·3495	26·4761	76·0272	76·7275
		1	25·1677	25·2207	70·3282	70·6313
		0	24·7268	24·7268	68·0993	68·0993

(I) Values obtained with expression (7); (II) Values obtained with expression (9).

where

$$X_1(x) = \sum_{i=1}^{4} a_i x^{n_i}, \qquad Y_1(y) = \sum_{i=1}^{4} b_i y^{m_i}, \tag{11}$$

and where $a_4 = 1$, $b_4 = 1$, $n_i = i$, $m_i = i$, $i = 1, 4$. The coefficients a_i and b_i are obtained from the boundary conditions. The co-ordinate functions in equation (10) satisfy the original geometric boundary conditions and the approximate natural ones. Minimization of the Rayleigh quotient (5) with respect to parameter A_i leads to the frequency equation

$$A\Omega^2 + B\Omega = 0. \tag{12}$$

3. NUMERICAL RESULTS AND DISCUSSION

Values of the fundamental frequency coefficient for distinct situations of vibrating plates are shown in Tables 1 and 2. Table 1 contains results of the frequency coefficient $\Omega_{00} = \sqrt{(\rho h^{(1)}/D^{(1)})}\omega_{00} b^2$, for a rectangular isotropic plates of variable thickness with edges 1, 3 and 4 rigidly clamped and edge 2 elastically restrained against rotation. Table 2 contains results of the frequency coefficient $\Omega_{00} = \sqrt{(\rho h^{(1)}/H_{xy}^{(1)})}\omega_{00} b^2$, for a rectangular orthotropic plate with edges 1, 3 and 4 rigidly clamped, and edge 2 elastically restrained against rotation.

4. CONCLUSIONS

Since it is not necessary to subject the co-ordinate functions to the natural boundary conditions which govern the problem under study when dealing with the Rayleigh–Ritz method it is possible to select co-ordinate functions which satisfy more easily the applied boundary conditions. Nevertheless in this procedure when dealing with rectangular plates of non-uniform thickness with edges elastically restrained against rotation, the two commonly used expressions that give the energy stored in the elastic supports are not equivalent. The Rayleigh–Ritz method was applied to the problem with a polynomial expression as approximating function. A frequency equation was thus derived in a very simple form. From the analysis of Tables 1 and 2 we can conclude that the use of expression (7) yields better results.

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