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COUPLED BENDING–BENDING–TORSION VIBRATION ANALYSIS OF ROTATING PRETWISTED BLADES: AN INTEGRAL FORMULATION AND NUMERICAL EXAMPLES

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A new approximate method is presented for the analysis of the modal characteristics of straight, pretwisted non-uniform blades corresponding to the coupled flapwise bending, chordwise bending and torsion of both rotating and non-rotating blades. An integral approach is described based on the use of Green functions (structural influence functions), which are used to develop the equations of motion. A clamped–free blade is analyzed and comparisons are made with numerical results from the literature. Several examples regarding specific aspects of the flapwise bending, coupled bending–bending, coupled bending–torsion and coupled bending–bending–torsion vibration analysis are presented. The method presented gives good results and can be used for modelling of turbomachine blades, aircraft propellers or helicopter rotor blades which may be considered as straight non-uniform beams with built-in pretwist.

1. INTRODUCTION

The determination of the dynamic characteristics of rotating beams is of great importance in the design of several engineering components, such as blades in turbines, compressors, propellers or helicopter rotors. Indeed, in order to avoid possible resonances, for transient response problems and in flutter analysis, it is necessary to determine accurate estimates of the natural frequencies of the structure under test.

The general differential equations of motion for combined bending–bending–torsion of a pretwisted non-uniform blade were derived in reference [1] and are, in the general form, too complex to be solved exactly. As a consequence, several methods have been developed to obtain approximate solutions for the general case or specific subcases as has been well summarized in reference [2]. For example, the Integrating Matrix Method (I.M.M.) presented in references [3–5] has also been used for studying buckling of rotating beams [6] and more recently in rotary wing aeroelastic analysis [7]. Murthy [8] used the Transmission Matrix Method (T.M.M.) and in the Lang and Nemat-Nasser paper [9], the

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method of the new quotient, based on a variational statement proposed by Nemat-Nasser, was presented. Many works in this field are based on finite element models.

With reference to the bending vibration of rotating beams or blades, reference [10] contains a review of several approximate methods such as the Myklestad method, the Galerkin method, the Rayleigh–Ritz method, the finite element method, etc.

There are also several known exact solutions, and in references [11] and [12] a semi-analytic technique was presented which is based on the Frobenius power series method and which represents centrifugal forces exactly, including shear deformation and rotatory inertia effects. In reference [13] the method of Frobenius was used for the dynamic analysis of rotating beams having uniform or linear distributions of mass and flexural stiffness. Results were presented for hinged or fixed root beams with root offset and tip mass.

In the present paper an integral approach is introduced based on Green functions and in which the analysis of the modal characteristics corresponding to the coupled flapwise bending–chordwise bending and torsion of rotating/non-rotating straight, non-uniform, pretwisted blades is addressed. Initially, the use of such a formulation for the study of the transverse vibrations of rotating clamped–free beams is outlined and, subsequently, an extension to the bending–bending–torsion case is described. Furthermore, the method is limited to linear problems and, therefore, to free vibration analysis.

The main feature of this alternative method is the use of Green functions and, in the case of a pretwisted blade, the introduction of a coupling structural influence coefficient between flap and lag bending. In this approach weighting matrices we used for integration and differentiation in a similar way to I.M.M. [5]. The boundary conditions are incorporated in the formulation using appropriate Green functions (for the clamped–free blade in the applications presented in this paper). The numerical results determined are compared with available experimental data and other numerical solutions, demonstrating that this method produces good results in terms of the prediction of the modal characteristics.

2. THEORETICAL BACKGROUND

2.1. TRANSVERSE BENDING VIBRATION

Upon assuming simple harmonic motion, the differential equation governing the transverse (flapwise) vibration of a rotating beam, shown in Figure 1, is

$$[EI(x)w''(x)]'' - [T(x)w'(x)]' - m(x)\omega^2 w(x) = 0, \tag{1}$$

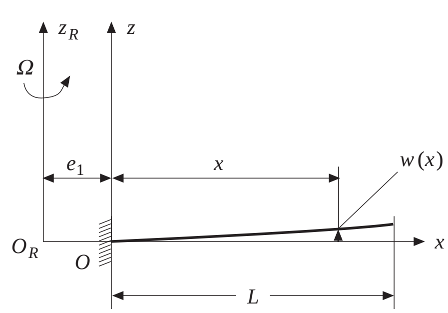

Figure 1. The rotating beam of configuration.

where $EI(x)$ is the flapwise bending stiffness of the beam and $T(x)$ is the centrifugal force in a section given by (a list of notation is given in the Appendix)

$$T(x) = \int_x^L m(x)\Omega^2(x + e_1)\,\mathrm{d}x. \tag{2}$$

According to references [14] and [15], a differential equation of the form

$$[EI(x)w''(x)]'' = p(x) \tag{3}$$

may be written in the integral form

$$w(x) = \int_0^L G(x, \xi)p(\xi)\,\mathrm{d}\xi, \tag{4}$$

where $G(x, \xi)$ is the Green function which represents the bending deflection $w(x)$ at x due to a unit force applied at ξ. The Green function for this case is given in reference [15] as

$$G(x, \xi) = \int_0^{\min(x,\xi)} \frac{(x - \xi_1)(\xi - \xi_1)}{EI(\xi_1)}\,\mathrm{d}\xi_1. \tag{5}$$

The fundamental concept is to consider equation (1) of the form (3) with

$$p(x) = m\omega^2 w - \Omega^2\left[m(x + e_1)w' - w''\int_x^L m(\xi)(\xi + e_1)\,\mathrm{d}\xi\right], \tag{6}$$

so that equation (3) becomes

$$w(x) = \omega^2\int_0^L G(x, \xi)mw\,\mathrm{d}\xi - \Omega^2\int_0^L G(x, \xi)m(e_1 + \xi)w'\,\mathrm{d}\xi$$
$$+ \Omega^2\int_0^L G(x, \xi)\left[\int_\xi^L m(\xi_1)(e_1 + \xi_1)\,\mathrm{d}\xi_1\right]w''\,\mathrm{d}\xi. \tag{7}$$

After choosing n collocation points along the beam axis, each integral can be approximated by the summation

$$\int_0^L f(\xi)\,\mathrm{d}\xi = \sum_{i=1}^{n} f_i W_i, \tag{8}$$

where the W_i are weighting numbers which depend on the method employed for numerical integration [14]. As a consequence, relation (7) becomes

$$\{w\} = \omega^2[G][W][M_1]\{w\} + \Omega^2[G][W][[M_{in}][D_2] - [M_x][D_1]]\{w\}, \tag{9}$$

where $[G]$ contains the values $G(x_i, \xi_j)$ and $[M_1]$, $[M_{in}]$ and $[M_x]$ are diagonal matrices with the values $m(x)$, $\int_x^L m(x + e_1)\,\mathrm{d}x$ and $m(x + e_1)$ respectively along the diagonal. The $[W]$ matrix is an (n, n) diagonal weighting matrix with values depending on the method of integration employed (Simpson's method in the present work) and $[D_1]$ and $[D_2]$ are differentiating matrices used to obtain the vectors $\{w'\}$ and $\{w''\}$. In this paper, a central-difference operator is used to obtain the differentiating matrices. It should be noted that in the formulation presented, for a non-rotating beam, the differentiating matrices are not required, while in the case of I.M.M., these matrices can also be avoided for the rotating beam through properly chosen dependent variables [6]. Equation (9) has the form

$$\{w\} = \omega^2[G_1]\{w\} + \Omega^2[[G_2] - [G_3]]\{w\}, \tag{10}$$

and can be written as a standard eigenproblem,

$$[[A] - \omega^2[I]]\{w\} = \{0\}, \tag{11}$$

with $[A] = [G_1]^{-1}[[I] - \Omega^2[G_2] + \Omega^2[G_3]]$, which can provide the natural frequencies of bending vibration of the beam. The method presented yields non-symmetric fully populated matrices. However, this approach makes it possible to use values $G(x_i, \xi_j)$ obtained from experimental measurements.

2.2. COUPLED VIBRATION ANALYSIS

In this section, coupled bending–torsion vibration is considered for the case of a pretwisted blade. The differential equations of the free bending–bending–torsion vibration of a pretwisted rotating blade, shown in Figure 2, can be written as

$$(EI_y w'' + EI_{zy} v'')'' = (Tw')' + p_{z1} = p_z, \tag{12}$$

$$(EI_z v'' + EI_{zy} w'')'' = (Tv')' + p_{y1} = p_y, \qquad (GJ\phi')' + m_x = 0, \tag{13, 14}$$

where p_{y1}, p_{z1} and m_x include the linear inertial terms of the equations of Houbolt and Brooks:

$$p_{z1} = [\Omega^2 em(x + e_1)\phi \cos\theta]' + \omega^2 m(w + e\phi\cos\theta), \tag{15}$$

$$p_{y1} = -[\Omega^2 em(x + e_1)\phi \sin\theta]' - \Omega^2 em\phi \sin\theta + \omega^2 m(v - e\phi\sin\theta) + \Omega^2 mv, \tag{16}$$

$$\begin{aligned} m_x = {} & \Omega^2 em(x + e_1)(v' \sin\theta - w'\cos\theta) - \Omega^2 emv \sin\theta \\ & - \Omega^2 m(k_{m2}^2 - k_{m1}^2)\phi\cos 2\theta + \omega^2 mk_m^2\phi - \omega^2 em(v\sin\theta - w\cos\theta). \end{aligned} \tag{17}$$

The expressions (15), (16) and (17) are specialized to the case of simple harmonic motion with $k_A = e_A = EB_1 = EB_2 = e_o = 0$ and a singly symmetric cross-section, with the introduction of a horizontal offset e_1. The profile of the blade is symmetric relative to axis $O\eta$ and therefore

$$I_y = I_\eta \cos^2\theta + I_\zeta \sin^2\theta, \qquad I_z = I_\zeta \cos^2\theta + I_\eta \sin^2\theta, \qquad I_{zy} = (I_\zeta - I_\eta)\sin\theta\cos\theta. \tag{18}$$

The linear equations (12), (13) and (14) are coupled due to the angle θ, which includes the pretwist and the setting angle (in the case of the controlled blade), and to the offset e between the elastic center and the center of gravity of the section.

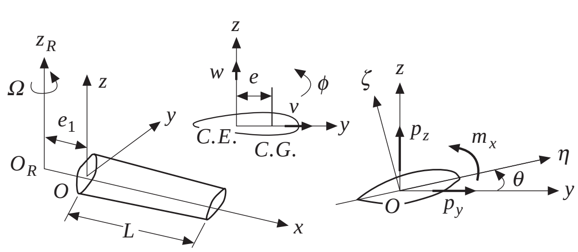

Figure 2. The blade configuration.

For the analysis of the coupled bending equations (12) and (13), a key element is the introduction of different structural influence functions. These functions are called G_{ww}, G_{wv}, G_{vv} and G_{vw}, and their physical significance, illustrated in Figures 3(a) and (b), is as follows: $G_{ww}(x, \xi)$ is the displacement w at x due to a unit force at ξ acting on the direction z; $G_{vv}(x, \xi)$ is the displacement v at x due to a unit force at ξ acting on the direction y; $G_{wv}(x, \xi)$ is the displacement w at x due to a unit force at ξ acting on the direction y; $G_{vw}(x, \xi)$ is the displacement v at x due to a unit force at ξ acting on the direction z. For equation (14), the function $G_t(x, \xi)$, which represents the torsion angle ϕ at x due to a unit torque at ξ, is used.

The main idea is to regard the solution of equations (12) and (13) for the displacements $v(x)$ and $w(x)$ as a superposition of two components:

$$w(x) = \int_0^L G_{ww}(x, \xi) p_z(\xi)\, \mathrm{d}\xi + \int_0^L G_{wv}(x, \xi) p_y(\xi)\, \mathrm{d}\xi, \tag{19}$$

$$v(x) = \int_0^L G_{vv}(x, \xi) p_y(\xi)\, \mathrm{d}\xi + \int_0^L G_{vw}(x, \xi) p_z(\xi)\, \mathrm{d}\xi. \tag{20}$$

Furthermore, according to references [14–16], equation (14) can be written in the integral form

$$\phi(x) = \int_0^L G_t(x, \xi) m_x(\xi)\, \mathrm{d}\xi. \tag{21}$$

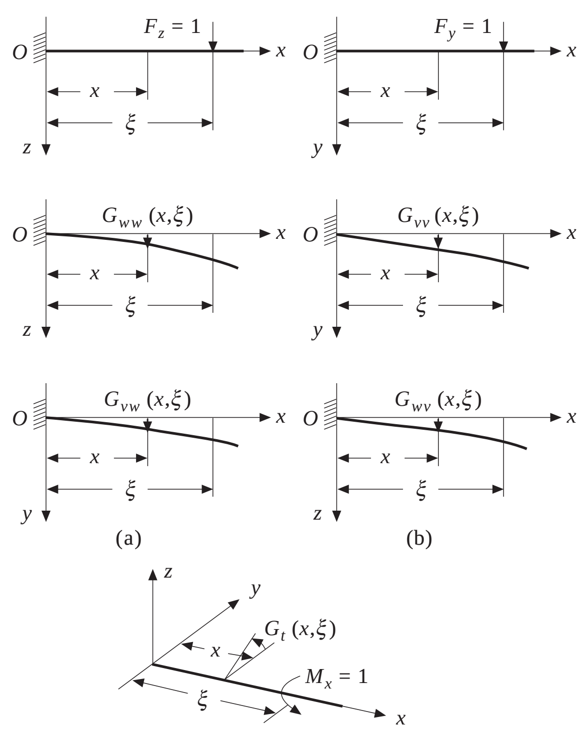

Figure 3. The physical significance of the Green functions. (a) G_{ww} and G_{vw}; (b) G_{vv} and G_{wv}; (c) G_t.

In the case of a clamped–free beam, Green functions are calculated according to some simple concepts of strength of materials reported in reference [17]:

$$G_{ww}(x,\xi)=\int_0^{\min(x,\xi)}\frac{(x-\xi_1)(\xi-\xi_1)I_z(\xi_1)}{EI_\zeta(\xi_1)I_\eta(\xi_1)}\,\mathrm{d}\xi_1, \tag{22}$$

$$G_{vv}(x,\xi)=\int_0^{\min(x,\xi)}\frac{(x-\xi_1)(\xi-\xi_1)I_y(\xi_1)}{EI_\zeta(\xi_1)I_\eta(\xi_1)}\,\mathrm{d}\xi_1, \tag{23}$$

$$G_{wv}(x,\xi)=G_{vw}(x,\xi)=\int_0^{\min(x,\xi)}\frac{(x-\xi_1)(\xi-\xi_1)I_{zy}(\xi_1)}{EI_\zeta(\xi_1)I_\eta(\xi_1)}\,\mathrm{d}\xi_1, \tag{24}$$

$$G_t(x,\xi)=\int_0^{\min(x,\xi)}\frac{\mathrm{d}\xi_1}{GJ(\xi_1)}. \tag{25}$$

For $\theta(x)=0$, $I_{zy}=0$, $I_z=I_\zeta$ and $I_y=I_\eta$, $G_{wv}=G_{vw}=0$ and the Green functions established in reference [15] are obtained.

In the manner presented in section 2.1, by choosing n collocation points along the blade and transforming in summations all the integrals in equations (19), (20) and (21), the following three relations in matrix form are obtained:

$$\begin{aligned}\{w\}=\;&\omega^2[G_{ww}][W][M]\{w\}+\Omega^2[G_{ww}][W][[M_{in}][D_2]-[M_x][D_1]]\{w\}\\&+e\omega^2[G_{ww}][W][M_c]\{\phi\}+\Omega^2e[G_{ww}][W][[M_{xc1}]+[M_c]-[M_{xstp}]\\&+[M_{xc}][D_1]]\{\phi\}+(\omega^2+\Omega^2)[G_{wv}][W][M]\{v\}\\&+\Omega^2[G_{wv}][W][[M_{in}][D_2]-[M_x][D_1]]\{v\}-e\omega^2[G_{wv}][W][M_s]\{\phi\}\\&-\Omega^2e[G_{wv}][W][[M_s]+[M_{xs1}]+[M_{xs}][D_1]+[M_{xctp}]]\{\phi\},\end{aligned} \tag{26}$$

$$\begin{aligned}\{v\}=\;&(\omega^2+\Omega^2)[G_{vv}][W][M]\{v\}+\Omega^2[G_{vv}][W][[M_{in}][D_2]-[M_x][D_1]]\{v\}\\&-e\omega^2[G_{vv}][W][M_s]\{\phi\}-\Omega^2e[G_{vv}][W][[M_s]\\&+[M_{xs1}]+[M_{xs}][D_1]+[M_{xctp}]]\{\phi\}+\omega^2[G_{vw}][W][M]\{w\}\\&+\Omega^2[G_{vw}][W][[M_{in}][D_2]-[M_x][D_1]]\{w\}+e\omega^2[G_{vw}][W][M_c]\{\phi\}\\&+e\Omega^2[G_{vw}][W][[M_{xc1}]+[M_c]-[M_{xstp}]+[M_{xc}][D_1]]\{\phi\},\end{aligned} \tag{27}$$

$$\begin{aligned}\{\phi\}=\;&\omega^2[G_t][W][k_m^2[M]\{\phi\}+e[M_c]\{w\}-e[M_s]\{v\}]\\&-\Omega^2[G_t][W][(k_{m2}^2-k_{m1}^2)[M_{2c}]\{\phi\}+e[M_{xc}][D_1]\{w\}\\&-e[M_{xs}][D_1]\{v\}+e[M_s]\{v\}].\end{aligned} \tag{28}$$

Here $[M]$, $[M_{in}]$, $[M_x]$, $[M_c]$, $[M_{xc1}]$, $[M_{xstp}]$, $[M_s]$, $[M_{xs1}]$, $[M_{xs}]$, $[M_{xctp}]$, $[M_{2c}]$ and $[M_{xc}]$ are (n,n) diagonal matrices having on the main diagonal the following values: $m(x)$, $\int_x^L m(x+e_1)\,\mathrm{d}x$, $m(x+e_1)$, $m\cos\theta$, $m'(x+e_1)\cos\theta$, $m(x+e_1)\theta'\sin\theta$, $m\sin\theta$, $m'(x+e_1)\sin\theta$, $m(x+e_1)\sin\theta$, $m(x+e_1)\theta'\cos\theta$, $m\cos 2\theta$ and $m(x+e_1)\cos\theta$, respectively.

TABLE 1

Comparison of approximate frequency ratios with exact values for a uniform cantilever beam

ω/Ω	Rotation speed ratio, η	Frequency ratio, ω/Ω Exact [13]	This paper (I.F.) $n = 10$	$n = 20$	$n = 40$	$n = 80$
ω_1/Ω	0	3·516	3·516	3·516	3·516	3·516
	3	4·7973	4·7863	4·7945	4·7966	4·7971
	6	7·3604	7·3226	7·3509	7·3580	7·3598
	12	13·1702	13·024	13·1349	13·1614	13·168
ω_2/Ω	0	22·0345	22·0344	22·0344	22·0345	22·0345
	3	23·3203	23·2292	23·2965	23·3143	23·3188
	6	26·8091	26·4964	26·7283	26·7887	26·8040
	12	37·6031	36·7341	37·3828	37·5478	37·5893
ω_3/Ω	0	61·6972	61·7148	61·6954	61·6971	61·6972
	3	62·9850	62·7402	62·9120	62·9666	62·9804
	6	66·6840	65·7144	66·4160	66·6159	66·6668
	12	79·6145	76·3843	78·7535	79·3964	79·5598
ω_4/Ω	0	120·902	121·212	120·889	120·900	120·901
	3	122·236	122·067	122·083	122·198	122·226
	6	126·140	124·592	125·589	125·999	126·105
	12	140·534	134·175	138·628	140·045	140·411

The $[W]$ matrix is the weighting matrix, while $[D_1]$ and $[D_2]$ are differentiating matrices used to obtain the vectors $\{w'\}$, $\{w''\}$, $\{v'\}$, $\{v''\}$ and $\{\phi'\}$. The equations may be written simply as

$$\{z\} = [\omega^2[A_1] + [B_1(\Omega^2)]]\{z\}, \tag{29}$$

where $\{z\} = [[w], [v], [\phi]]^T$ is a vector of dimension $3n$ and $[A_1]$, $[B_1(\Omega^2)]$ are $(3n, 3n)$ matrices. If $\theta = 0$ and $e = 0$, the motions become uncoupled. Equation (29) can be written in the form

$$[[A] - \omega^2[I]]\{z\} = \{0\}, \tag{30}$$

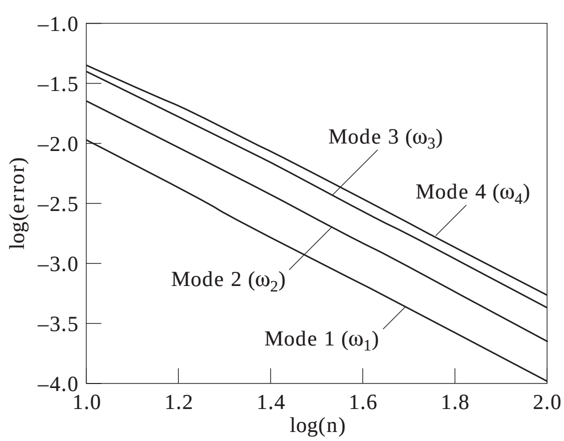

Figure 4. The relative error versus n in the case of a uniform rotating clamped–free beam. $e_1 = 0$, $\eta = 12$.

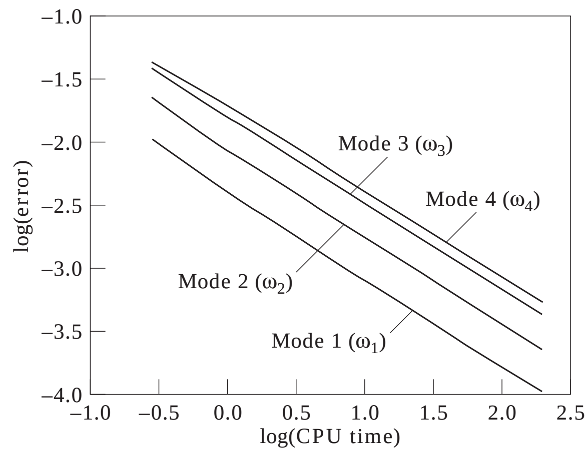

Figure 5. The relative error versus *CPU* time in the case of a uniform rotating clamped–free beam. $e_1 = 0$, $\eta = 12$.

where

$$[A] = [A_1]^{-1}[[I] - [B_1]]. \tag{31}$$

Equation (30) represents an eigenvalue problem, the solution of which yields eigenfrequencies and corresponding mode shapes. When just two motions are considered, the dimensions of the matrix $[A]$ in equation (30) is $(2n, 2n)$ and the vector $\{z\}$ becomes $[[w]\,[v]]^{\mathrm{T}}$ for coupled bending–bending or $[[w]\,[\phi]]^{\mathrm{T}}$ for coupled flap bending–torsion.

3. ILLUSTRATIVE EXAMPLES

3.1. INTRODUCTION

In this section, several examples of flap–bending, lag–bending and coupled vibration analysis are presented, together with an evaluation of the convergence of the method (all of the calculations presented have been performed by using MATLAB on a Pentium PC (90 MHz)).

3.2. UNCOUPLED FLAP–BENDING AND LAG–BENDING VIBRATION

If it is assumed that $v = \phi = e = \theta = 0$, only flap deflection is considered, such that $\{z\} = \{w\}$ and it is possible to use either equation (9) or equation (26). In order to evaluate the merits of the proposed method, the numerical results are compared with the exact

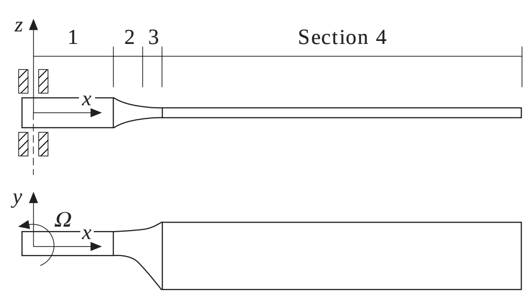

Figure 6. The non-uniform cross-section clamped–free aluminum blade.

TABLE 2

Model data of an aluminum rotor blade

Section number	Length (m)	Area (cm^2)	m (kg/m)	Bending stiffness, EI_y (Nm^2)
1	0·6	42	11·7054	92 106
2	0·12	50	13·9350	76 170
3	0·08	51	14·2137	35 819
4	4	50	13·9350	19 006

solution presented in reference [13]. To facilitate this comparison, the following non-dimensional coefficients are introduced:

$$\lambda = mL^4\omega^2/EI, \qquad \alpha = \eta^2 = mL^4\Omega^2/EI. \tag{32}$$

For the simple case of a uniform cantilever beam with no root offset, in Table 1 are shown both the exact solution and the numerical results, indicating that errors tend to increase with the order of the modes. In Figure 4 is shown the behaviour of relative errors for the first four natural frequencies in the case of rotation speed ratio $\eta = 12$. It can be seen that relative errors decrease increasing number of collocation points n which, in this graph, are varied from 10 to 100. Furthermore, the authors have observed that for $\eta < 12$ errors are smaller. For the same case study, relative errors are shown versus CPU time in Figure 5.

The natural frequencies of the flap–bending modes of a non-uniform untwisted clamped–free aluminum blade (Figure 6) were calculated by using the data given in reference [18] and are reproduced in Table 2. The properties of the blade are piecewise constant along the axial co-ordinate. In this reference, a finite element analysis of this blade is performed and the first five eigenfrequencies are given for $\Omega = 0$ and for $\Omega = 45$ rad/s.

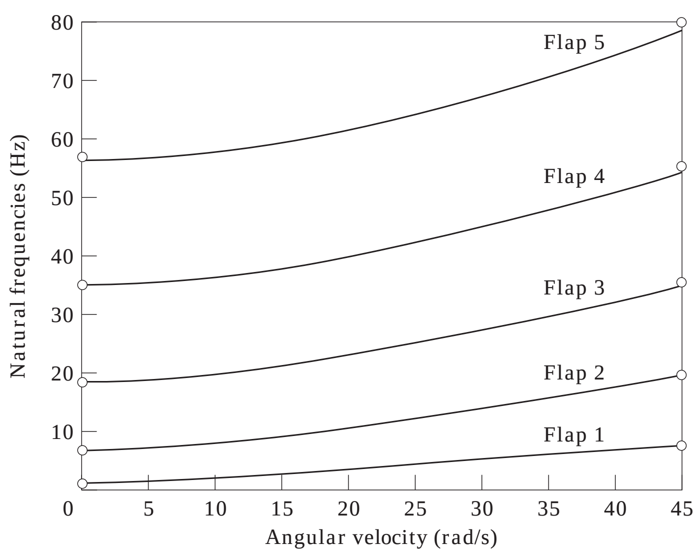

Figure 7. Natural frequencies for flap–bending modes of the aluminum blade. —, I.F. results, $n=50$; ○, ANSYS results.

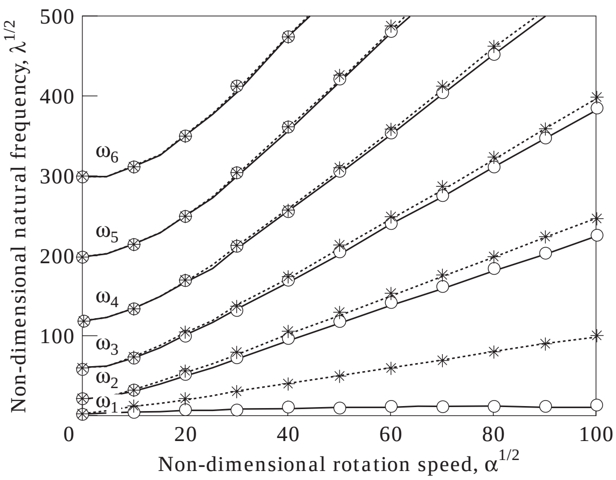

Figure 8. A comparison of results for a cantilevered rotating beam with a symmetric section. $*$, ○, ETB [11]; —, lead–lag; ·······, flap.

The results of the present integral formulation (I.F.) are plotted in Figure 7, and good agreement can be observed.

To evaluate the method also in the case of lag bending vibrations by using available results of reference [11], a study was made of a cantilevered beam having a symmetric section with $mL^4/EI = 1$, $\eta = \alpha^{1/2} = \Omega$ and $\lambda^{1/2} = \omega$. The uncoupled equations (26) and (27) were used for $\phi = 0$, $\theta = 0$, $e = e_1 = 0$ and $EI_z = EI_y = EI$. In reference [11], the dimensionless frequency $\lambda^{1/2}$ for the first six flap and lag modes with and without Timoshenko corrections were given. In Figure 8 the results obtained by using the proposed

TABLE 3

Comparison of results for a non-uniform rotating twisted blade; bending–bending natural frequencies (Hz)

Ω (rpm)	Mode number	Results				
		Exp., [3]	[8]	[9]	I.F., $n = 30$	I.F., $n = 50$
1567	1	40·08	39·89	40·96	40·73	40·87
	2	—	107·40	109·22	108·16	108·83
	3	—	276·32	279·79	276·18	278·18
1589	1	—	40·26	41·35	41·12	41·26
	2	107·53	107·93	109·77	108·72	109·38
	3	—	276·97	280·47	276·86	278·86
2609	1	58·73	58·05	60·07	59·92	60·04
	2	—	135·99	139·52	138·52	139·18
	3	—	313·98	319·40	315·48	317·71
2614	1	—	58·14	60·16	60·01	60·13
	2	137·02	136·14	139·68	138·68	139·34
	3	—	314·19	319·62	315·70	317·93
3583	1	76·52	75·30	78·34	78·19	78·31
	2	—	166·25	—	170·94	171·62
	3	—	357·70	—	362·08	364·54

TABLE 4

Comparison of results for a uniform non-rotating untwisted blade; bending–torsion natural frequencies (Hz)

Mode	I.M.M. [5], $n = 5$	I.M.M. [5], $n = 15$	T.M.M. [8]	This paper (I.F.), $n = 30$
1	31·05	31·05	31·05	31·06
2	189·37	193·74	193·74	193·79
3	390·80	390·87	390·87	390·91
4	578·93	539·54	539·54	539·64
5	1168·22	1043·94	1041·72	1043·87

formulation with $n = 50$ collocation points and the solution by the Engineer's Theory of Bending (E.T.B.) given in Figure 4 of reference [11] are shown; again, good agreement can be observed.

3.3. COUPLED FLAP–LAG AND FLAP–TORSION VIBRATION

In this section an example is given of a non-uniform pretwisted rotating blade. The same example was used in reference [9], where the method of the new quotient was used, and also in reference [8] in which the analysis was conducted by using T.M.M. In this case $\phi = 0$ and coupled equations (26) and (27) are used.

The length of the blade is $L = 18$ in and it is cantilevered at $e_1 = 6$ in from the axis of rotation. The properties $m(x)$, $EI_\eta(x)$ and $EI_\zeta(x)$ of the non-uniform blade are presented in references [8] and [9]. In this example, for each collocation point, these properties are obtained by linear interpolation. In Table 3 are presented the first three natural frequencies of this clamped–free blade for several rotational speeds. The dashes in the table indicate the lack of data in the corresponding references. In this case study, the results of the present integral formulation agree better with those of reference [9].

Natural frequencies have been evaluated also for an untwisted clamped–free non-rotating blade having flap–bending and torsional degrees of freedom. According to reference [8], the data of this blade are the following: $L = 40$ in, $EI_\zeta = 25\,000$ lb in^2, $GJ = 9000$ lb in^2, $e = 0{\cdot}4$ in, $e_1 = 0$, $k_{m1}^2 = 0{\cdot}18$ in^2, $k_{m2}^2 = 0{\cdot}71$ in^2, $m = 0{\cdot}0015$ slugs/in. The results are shown in Table 4 and agree well with those obtained in reference [8] with T.M.M. and in reference [5] with I.M.M.

TABLE 5

Comparison of results for a uniform non-rotating twisted blade; bending–bending–torsion natural frequencies (Hz)

Mode	T.M.M. [9]	This paper (I.F.) $n = 10$	This paper (I.F.) $n = 20$	This paper (I.F.) $n = 30$
1	30·8295	30·8374	30·8379	30·838
2	53·8277	53·8403	53·8404	53·8404
3	184·6175	184·5628	184·661	184·6821
4	337·3333	337·4104	337·4105	337·4116
5	484·3373	482·3053	483·9182	484·2932

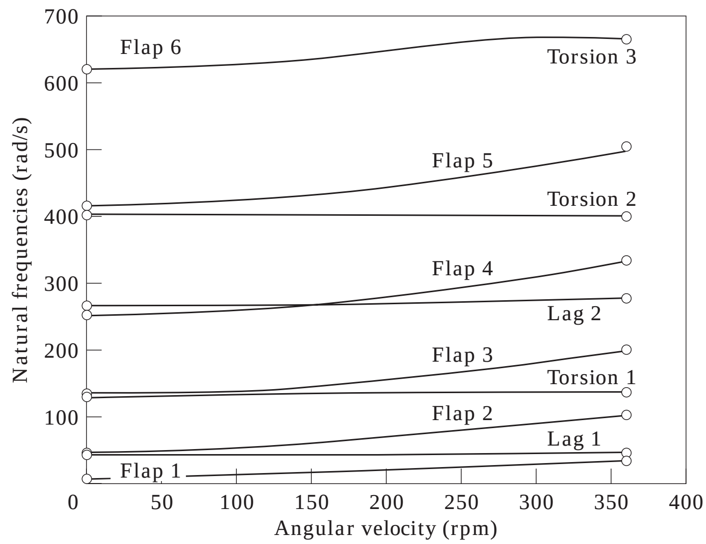

Figure 9. A comparison of results for a uniform rotating twisted blade. —, I.F. results, $n = 30$; ○, T.M.M. results.

3.4. COUPLED FLAP–LAG–TORSION VIBRATION

An example of a pretwisted non-rotating blade having flap, lag and torsional degrees of freedom is taken also from reference [8]. The characteristics of the uniform blade studied are $L = 40$ in, $\theta = 45^\circ$, $EI_\eta = 25\,000$ lb in^2, $EI_\zeta = 75\,000$ lb in^2, $GJ = 9000$ lb in^2, $m = 0{\cdot}0015$ slugs/in, $k_{m1}^2 = 1$ in^2, $k_{m2}^2 = 1$ in^2, $e = \sqrt{2}$ in. The results for this case are compared with those obtained with T.M.M. [8] in Table 5.

Finally, a complete example is the evaluation of the dynamic characteristics of a fixed–free uniform rotating blade having the following properties [19, 20]: $L = 260$ in, $\theta_c = 15{\cdot}026^\circ$, $\theta_B = 0^\circ$, $EI_\eta = 0{\cdot}2977 \times 10^8$ lb in^2, $EI_\zeta = 10^9$ lb in^2, $GJ = 9000$ lb in^2, $EA = 10^{11}$ lb, $m = 0{\cdot}0015$ lb s^2/in^2, $mk_{m1}^2 = 0{\cdot}89545 \times 10^{-3}$ lb s^2, $mk_{m2}^2 = 0{\cdot}04$ lb s^2, $e = -0{\cdot}6$ in. The calculated natural frequencies are shown in Figure 9 in comparison with the available results at $\Omega = 0$ and $\Omega = 360$ rpm of references [19, 20] obtained by using the Transfer Matrix Method. Good agreement is obtained for these values of Ω.

4. CONCLUSIONS

The approach described in this paper addresses the vibration analysis of non-uniform pretwisted rotating/non-rotating straight blades. The complete flap–lag–torsion vibration analysis may be performed by using this method with good results.

A system of appropriate Green functions for the clamped–free beam was utilized to solve the Houbolt and Brooks equations for the free vibration motion of a rotating/non-uniform, pretwisted blade in a simple matrix manner. The dynamic characteristics obtained with this approach are in good agreement with the results of other methods.

The proposed approach uses weighting matrices for integration and also differentiating matrices like the more general I.M.M. [5, 21]. For a non-rotating blade the differentiating matrices are unnecessary. Boundary conditions are included through properly chosen structural influence (Green) functions.

This technique yields unsymmetric, non-banded matrices. It may be concluded that the proposed approach can provide an interesting alternative in this kind of analysis, as the influence coefficients at specific points can be computed or, in specific cases, estimated by static testing to find stiffness distributions [14]. The matrix form of this method also makes possible a simple MATLAB implementation.

In this paper, the formulation is limited to the clamped–free beam case, which is of particular interest for several engineering devices, but the structural influence functions may be obtained also for other boundary conditions. For example, in reference [22], including a tip mass in the formulation, the method is used for analyzing the bending vibration of rotating beams with a flexible root.

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APPENDIX: NOTATION

B_1, B_2	cross-section constants
E	Young's modulus of elasticity
e	distance between center of mass and elastic center
e_1	root blade offset
e_A	distance between tension center and elastic center
e_o	distance at root between elastic axis and pitch change axis of the blade, positive when elastic axis lies ahead
G	shear modulus of elasticity
I_η, I_ζ	bending moments of inertia about major and minor neutral axes
J	torsional stiffness constant
k_A	polar radius of gyration of cross-sectional area effective in supporting tensile stresses about the elastic axis
k_m	polar radius of gyration of cross-sectional mass about elastic axis, $k_m^2 = k_{m1}^2 + k_{m2}^2$
k_{m1}, k_{m2}	mass radii of gyration of cross-sectional mass about major and minor neutral axes
L	blade length
m	mass of unit length of the blade
$\min(x, \xi)$	smallest value from x or ξ
n	number of collocation points
T	tension in the blade
v	bending lag displacement
w	bending flap displacement
x, ξ, ξ_1	co-ordinates along the Ox axis
ϕ	torsion deformation, positive leading edge upwards
θ_B	blade section pretwist
θ_c	collective setting angle
θ	total section setting angle $\theta = \theta_B + \theta_c$
Ω	angular velocity of rotation
ω	frequency of vibration
$[\cdot]'$	differentiation w.r.t. x
$[\cdot]''$	double differentiation w.r.t. x
$[\cdot]^{\mathrm{T}}$	transpose of a matrix

Other symbols are defined in the text.