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A SYMMETRIC AND POSITIVE DEFINITE BEM FOR 2-D FORCED VIBRATIONS

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1. INTRODUCTION

Boundary Element Methods (BEMs) have been extensively applied to various structural dynamics problems [1, 2] and many approaches are available in the literature. Starting from the classical direct formulation for elastodynamics, introduced by Cruse and Rizzo [3], the problem has been addressed by developing many methods in which computational advantages are recovered with respect to the classical direct BEM involving dynamic fundamental solutions [4–6]. Among the methods proposed for the analysis of the dynamic elastic response those employing static fundamental solutions [7] have received particular attention because they lead to a standard linear resolving system. However in this case the inertial term gives rise to a domain integral which invalidates the boundary-only character of the model [8–10]. This drawback has been overcome by employing the dual reciprocity technique proposed by Nardini and Brebbia [11] and then used by other researchers [12–14]. Actually the BEM models proposed for elastodynamics are characterized by the loss of two fundamental properties of the continuum: that is, symmetry and definiteness of the structural operators. More recently, to get over this deficiency, some variational formulations of BEM have been proposed [15–18]. In this paper a variational formulation of BEM for 2-D elastostatics and free vibration analysis, previously presented by the authors [17, 18], is extended to the analysis of forced vibrations. The dynamic model, obtained from the stationarity conditions of a modified hybrid functional, is expressed in terms of boundary displacements only. The structural operators, namely the stiffness matrix and the mass matrix, preserve the symmetry and definiteness properties of the continuum and are calculated performing boundary integrations of regular kernels. The resolving system exhibits the same nature of the more popular finite element models and therefore the standard procedures can be applied for the numerical solution. The formulation is here validated presenting two solutions of forced vibration problems which are in excellent agreement with the results found in the literature.

2. FORMULATION

In this section the fundamentals of the formulation are briefly presented, with reference to references [17] and [18] for a more detailed discussion.

Consider an homogeneous, isotropic, linear elastic body, occupying a region Ω bounded by the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. Let the body be constrained on the boundary Γ_1 and let it be subjected to the tractions $\bar{\mathbf{t}}$ on the free boundary Γ_2 and to the body forces $\mathbf{f}$ in the domain Ω. Denoting by $\mathbf{u}$, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{t}}$ the displacements in the domain Ω and the displacements

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and tractions on the boundary Γ respectively, the dynamic model can be obtained by the stationarity of the hybrid modified functional [18]

$$\Pi = \int_{\Omega} [\tfrac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\boldsymbol{\varepsilon} - \mathbf{u}^{\mathrm{T}}(\mathbf{f} - \rho\ddot{\mathbf{u}})]\,\mathrm{d}\Omega - \int_{\Gamma} (\mathbf{u} - \tilde{\mathbf{u}})^{\mathrm{T}}\,\tilde{\mathbf{t}}\,\mathrm{d}\Gamma - \int_{\Gamma_2} \tilde{\mathbf{u}}^{\mathrm{T}}\,\bar{\mathbf{t}}\,\mathrm{d}\Gamma, \tag{1}$$

where the displacement and traction functions $\mathbf{u}$, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{t}}$ are supposed as independent one of the other. In equation (1) $\boldsymbol{\varepsilon}$ indicates the strain field, $\mathbf{E}$ is the elasticity matrix, ρ is the mass density and the overdots denote time derivatives. The stationarity conditions of Π with respect to $\mathbf{u}$, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{t}}$ provide the relations between the boundary and domain variables and the equilibrium equations [18]. To obtain the discrete model for the dynamic problem consider the body boundary discretized by boundary elements. On the boundary Γ the displacements and tractions are expressed by means of their nodal values $\boldsymbol{\delta}$ and $\mathbf{p}$ through shape functions

$$\tilde{\mathbf{u}} = \mathbf{N}\boldsymbol{\delta} = [\mathbf{N}_1 \quad \mathbf{N}_2]\begin{bmatrix}\boldsymbol{\delta}_1\\ \boldsymbol{\delta}_2\end{bmatrix}, \qquad \tilde{\mathbf{t}} = \boldsymbol{\Psi}\mathbf{p}, \tag{2, 3}$$

where the subscripts 1 and 2 refer to the constrained and free boundary nodal displacements respectively. The domain displacement field is approximated by a linear combination of regular, static fundamental solutions $\mathbf{u}_i^*$:

$$\mathbf{u} = \sum_{i=1}^{m} \mathbf{u}_i^*\, s_i, \tag{4}$$

or in matrix notation

$$\mathbf{u} = \mathbf{U}^*\mathbf{s}, \tag{5}$$

where $\mathbf{s}$ is the vector of the time dependent unknown coefficients s_i and $\mathbf{U}^*$ is the matrix of the fundamental solutions $\mathbf{u}_i^*$ whose source point is located outside the domain. With these assumptions the stationarity of the functional Π yields [18]

$$\int_{\Gamma} \mathbf{U}^{*\mathrm{T}}\mathbf{P}^*\,\mathrm{d}\Gamma\mathbf{s} - \int_{\Gamma} \mathbf{U}^{*\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}\Gamma\mathbf{p} - \int_{\Omega} \mathbf{U}^{*\mathrm{T}}\mathbf{f}\,\mathrm{d}\Omega + \rho\int_{\Omega} \mathbf{U}^{*\mathrm{T}}\mathbf{U}^*\,\mathrm{d}\Omega\ddot{\mathbf{s}} = \mathbf{0}, \tag{6}$$

$$\int_{\Gamma} \mathbf{N}_2^{\mathrm{T}}\boldsymbol{\Psi}\,\mathrm{d}\Gamma\mathbf{p} - \int_{\Gamma_2} \mathbf{N}_2^{\mathrm{T}}\,\bar{\mathbf{t}}\,\mathrm{d}\Gamma = \mathbf{0}, \qquad \int_{\Gamma} \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{U}^*\,\mathrm{d}\Gamma\mathbf{s} - \int_{\Gamma} \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}\Gamma\boldsymbol{\delta} = \mathbf{0}. \tag{7, 8}$$

From equation (8) one has for any choice of $\boldsymbol{\Psi}$

$$\mathbf{U}^*\mathbf{s} = \mathbf{N}\boldsymbol{\delta}, \tag{9}$$

and then, if the number of fundamental solutions m is equal to the number of nodal displacements and these fundamental solutions are linearly independent, by collocating equation (9) at the nodes one obtains

$$\mathbf{s} = \bar{\mathbf{U}}^{*-1}\boldsymbol{\delta} = \boldsymbol{\Phi}\boldsymbol{\delta} = [\boldsymbol{\Phi}_1 \quad \boldsymbol{\Phi}_2]\begin{bmatrix}\boldsymbol{\delta}_1\\ \boldsymbol{\delta}_2\end{bmatrix}. \tag{10}$$

It is worth noting that from equations (9) and (10) one deduces the following expression for **N** in terms of fundamental solutions:

$$\mathbf{N} = \mathbf{U}^*\bar{\mathbf{U}}^{*-1} = \mathbf{U}^*\mathbf{\Phi} = \mathbf{U}^*[\mathbf{\Phi}_1 \quad \mathbf{\Phi}_2] = [\mathbf{N}_1 \quad \mathbf{N}_2]. \tag{11}$$

Pre-multiplying equation (6) by $\mathbf{\Phi}_2^{\mathrm{T}}$, by using equations (7) and (10), the BEM model is obtained and it can be written as

$$\mathbf{M}\ddot{\boldsymbol{\delta}}_2 + \mathbf{K}\boldsymbol{\delta}_2 = \int_{\Gamma_2} \mathbf{N}_2^{\mathrm{T}}\,\bar{\mathbf{t}}\,\mathrm{d}\Gamma - \mathbf{\Phi}_2^{\mathrm{T}}\mathbf{B}\mathbf{\Phi}_1\ddot{\boldsymbol{\delta}}_1 - \mathbf{\Phi}_2^{\mathrm{T}}\mathbf{A}\mathbf{\Phi}_1\boldsymbol{\delta}_1, \tag{12}$$

where the body force term has been dropped. The stiffness matrix **K** and the mass matrix **M** are given by

$$\mathbf{K} = \mathbf{\Phi}_2^{\mathrm{T}} \int_{\Gamma} \mathbf{U}^{*\mathrm{T}}\mathbf{P}^*\,\mathrm{d}\Gamma\,\mathbf{\Phi}_2 = \mathbf{\Phi}_2^{\mathrm{T}}\mathbf{A}\mathbf{\Phi}_2 \tag{13}$$

$$\mathbf{M} = \rho \int_{\Omega} \mathbf{N}_2^{\mathrm{T}}\mathbf{N}_2\,\mathrm{d}\Omega = \rho\mathbf{\Phi}_2^{\mathrm{T}} \int_{\Omega} \mathbf{U}^{*\mathrm{T}}\mathbf{U}^*\,\mathrm{d}\Omega\,\mathbf{\Phi}_2 = \rho\mathbf{\Phi}_2^{\mathrm{T}}\mathbf{B}\mathbf{\Phi}_2, \tag{14}$$

where $\mathbf{P}^*$ is the matrix of the tractions $\mathbf{p}_i^*$ associated to the fundamental solutions $\mathbf{u}_i^*$. The domain integral that appears in the definition of the mass matrix **M** requires a domain discretization of the body. To obtain a dynamic model of the body discretized by boundary elements only, a transformation of this domain integral into an equivalent boundary integral needs to be performed [18]. Consider the body loaded by a fictitious system of body forces $\mathbf{u}_j^*$, where $\mathbf{u}_j^*$ is the jth fundamental solution. Let $\mathbf{v}_j$ be a displacement field due to the fictitious forces $\mathbf{u}_j^*$ and let again $\mathbf{q}_j$ be the associated boundary tractions. By using the reciprocity theorem and recalling that the fundamental solutions are regular in Ω one obtains

$$\int_{\Omega} \mathbf{u}_i^{*\mathrm{T}}\mathbf{u}_j^*\,\mathrm{d}\Omega = \int_{\Gamma} [\mathbf{p}_i^{*\mathrm{T}}\mathbf{v}_j - \mathbf{u}_i^{*\mathrm{T}}\mathbf{q}_j]\,\mathrm{d}\Gamma. \tag{15}$$

Equation (15) allows one to express the mass matrix in terms of boundary integrals. Actually, by applying equation (15) to all the fundamental solutions used, one has

$$\mathbf{B} = \int_{\Omega} \mathbf{U}^{*\mathrm{T}}\mathbf{U}^*\,\mathrm{d}\Omega = \int_{\Gamma} [\mathbf{P}^{*\mathrm{T}}\mathbf{V} - \mathbf{U}^{*\mathrm{T}}\mathbf{Q}]\,\mathrm{d}\Gamma, \tag{16}$$

where **V** satisfies the equilibrium equation

$$\mathbf{D}^{\mathrm{T}}\mathbf{E}\mathbf{D}\mathbf{V} + \mathbf{U}^* = \mathbf{0} \text{ in } \Omega \qquad \text{and} \qquad \mathbf{Q} = \mathbf{D}_n\mathbf{E}\mathbf{D}\mathbf{V} \qquad \text{on } \Gamma. \tag{17, 18}$$

In the previous expressions **D** and $\mathbf{D}_n$ denote the strain and boundary traction operators defined as

$$\mathbf{D}^{\mathrm{T}} = \begin{bmatrix} \partial/\partial x_1 & 0 & 0 & \partial/\partial x_2 & \partial/\partial x_3 & 0 \\ 0 & \partial/\partial x_2 & 0 & \partial/\partial x_1 & 0 & \partial/\partial x_3 \\ 0 & 0 & \partial/\partial x_3 & 0 & \partial/\partial x_1 & \partial/\partial \mathrm{x}_2 \end{bmatrix}, \tag{19}$$

$$\mathbf{D}_n = \begin{bmatrix} \alpha_1 & 0 & 0 & \alpha_2 & \alpha_3 & 0 \\ 0 & \alpha_2 & 0 & \alpha_1 & 0 & \alpha_3 \\ 0 & 0 & \alpha_3 & 0 & \alpha_1 & \alpha_2 \end{bmatrix}, \quad (20)$$

where α_1, α_2 and α_3 are the direction cosines of the boundary outer normal. The transformation of the domain integral into a boundary integral overcomes the limitation of the domain discretization and allows one to obtain a pure BEM model for structural dynamics. The structural operators **K** and **M** can thus be calculated by performing boundary integrations of regular kernels. Notice that the matrices **K** and **M** are frequency independent, symmetric and positive definite. Therefore, in the approach proposed, these two fundamental properties of the continuum, i.e. symmetry and definiteness of the structural operators, are preserved. Moreover the dynamic model is constituted by a set of linear differential equations that exhibits the same nature of the more common finite element resolving systems.

3. APPLICATIONS AND NUMERICAL RESULTS

Solutions of some problems are presented here in order to show the accuracy and the efficacy of the proposed method. In the applications the structural operators are computed through Gaussian quadrature, performing boundary integrations only in accordance with the expressions given in equations (13), (14) and (16). The regular kernels employed have been obtained by locating the fundamental solution source point outside the domain along the outward directed normal at the nodal point. The solutions presented are relative to a ratio between the distance of the source point from the nodal point and the element length equal to 1. However studies have been carried out proving the solution is stable with respect to the variation of this ratio in the range 0·5–2. Due to the structure of the resolving system the Houbolt recurrent scheme has been used to integrate the equations of motion and then obtain the transient response. The ith fundamental solution employed is the classical plain strain fundamental solution [11],

$$u^*_{k,2i+j-2} = \frac{1}{4G(1-\nu)}\left[(4\nu-3)\delta_{kj}\ln r + \frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_j}\right], \qquad k, j = 1, 2, \quad (21)$$

$$p^*_{k,2i+j-2} = \frac{1}{2(1-\nu)r}\left\{\frac{\partial r}{\partial n}\left[(1-2\nu)\delta_{kj} + 2\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial x_j}\right] + (1-2\nu)\left[\frac{\partial r}{\partial x_k}\alpha_j - \frac{\partial r}{\partial x_j}\alpha_k\right]\right\}, \quad (22)$$

where r is the distance between the ith source point and the observed point, G and ν are the shear modulus and the Poisson's ratio, and δ_{kj} is the Kronecker symbol. The auxiliary solutions $\mathbf{v}$, employed for the transformation of the domain integrals associated with the inertial term into boundary integrals, were obtained by integrating analytically the equilibrium equations (17). They are given by

$$v_{k,2i+j-2} = \frac{1}{G(j+1)}\frac{\partial}{\partial x_j}\left\{\frac{1}{2G(1-2\nu)(1-\nu)}\frac{\partial}{\partial x_k}\left[\frac{r^4}{64}\left(\ln r - \frac{3}{2}\right)\right] + \right.$$

$$\left. - \frac{r^3}{4(1-2\nu)}(\ln r - 1)\frac{\partial r}{\partial x_k}\right\} - \frac{r^2}{4G^2}(\ln r - 1)\delta_{kj}, \quad (23)$$

$$q_{k,2i+j-2} = \frac{r}{4G(1-\nu)}\left[\frac{\partial r}{\partial x_j}\alpha_k \ln r - 2(1-\nu)(\ln r - \tfrac{1}{2})\left(\frac{\partial r}{\partial n}\delta_{kj} - \frac{\partial r}{\partial x_k}\alpha_j + \frac{\partial r}{\partial x_j}\alpha_k\right)\right]. \quad (24)$$

The first solution presented refers to the forced transient response of a rectangular plate subjected to a flexural load with a triangular time variation. The geometry of the plate together with the time variation of the load and the BEM discretization used in the analysis are shown in Figures 1(a) and 1(b), respectively. The material properties are taken as $E = 10^5$, $\rho = 1$ and $\nu = 0{\cdot}25$. The horizontal displacement at the centre of the top end of the plate is plotted in Figure 1(c) where the present results are compared with those obtained by Gallego and Dominguez [5] and with those obtained by performing a finite element analysis with 440 degrees of freedom. Figure 2(a) shows the geometry of a dam-like structure. The dynamic behaviour of this structure is investigated when a sinusoidal excitation is applied to the base. The forcing frequency is chosen to be 16 Hz and the material properties are the same as in the previous example. Figure 2(b) shows the discretization used and the horizontal displacement at the top end versus time is plotted in Figure 2(c). A good agreement between the results obtained using the present method and the existing ones is registered in all the considered examples.

4. CONCLUSIONS

A BEM formulation for 2-D elastodynamics in the time domain has been presented. The formulation gives a resolving system that involves boundary displacements only. The stiffness and mass matrices of the boundary discretized body are frequency independent, symmetric and positive definite. They are evaluated by performing boundary integrations only and further, due to the employment of regular, static fundamental solutions, integrations of singular kernels are not required. For the forced vibration analysis a linear system of ordinary differential equations with constant coefficients is obtained allowing the

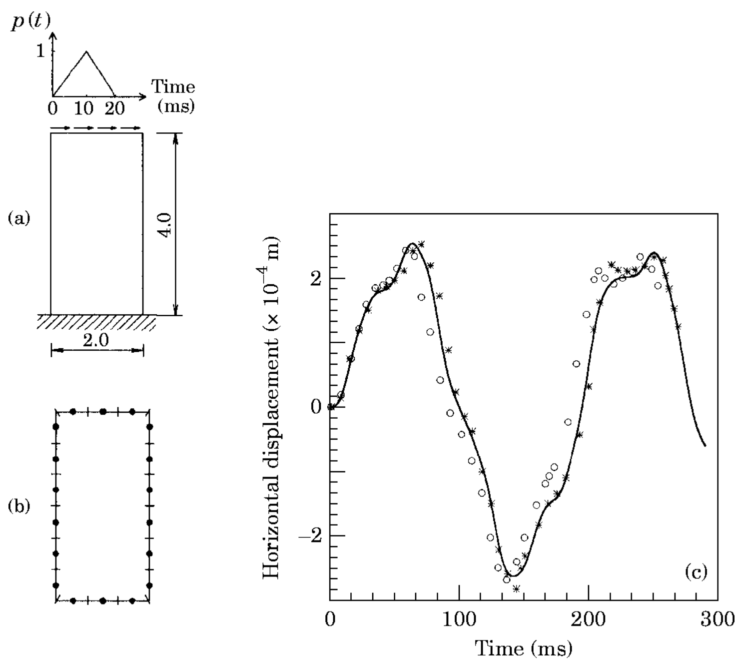

Figure 1. Beam under time-varying flexural load. (a) Geometry and load characteristics; (b) BEM discretization; (c) horizontal displacement at the top end; ○, reference [5]; ∗, FEM (440 d.o.f.); —, present BEM.

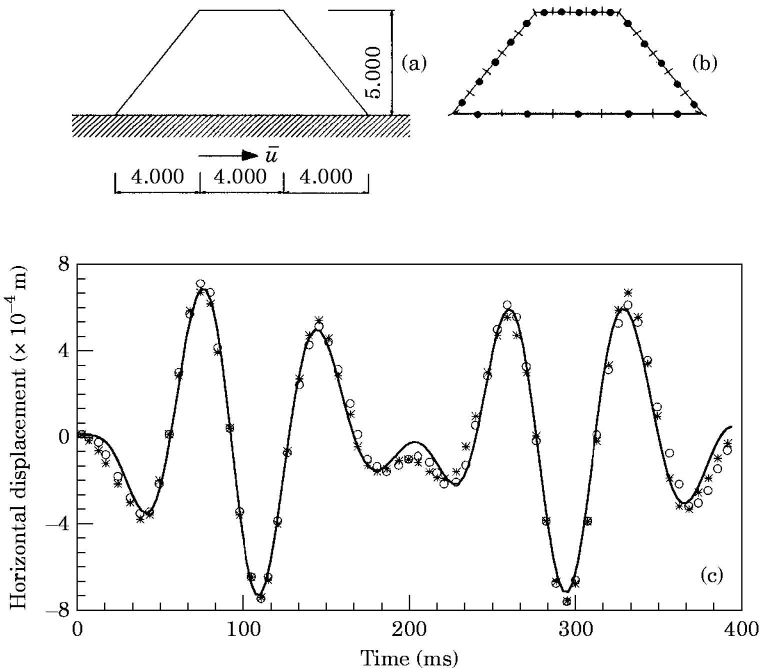

Figure 2. Dam under sinusoidal support excitation. (a) Geometry; (b) BEM discretization; (c) horizontal displacement at the centre of the top end; ○, reference [12]; *, FEM (420 d.o.f.); —, present BEM.

application of the standard procedures of recurrent schemes to solve time dependent problems. The good results obtained suggest the possible application of the method to the solution of many dynamic problems with meaningful computational advantages with respect to the more common field methods.

REFERENCES

1. D. E. Beskos 1987 *Applied Mechanics Reviews* **40**, 1–23. Boundary element methods in dynamic analysis.
2. D. E. Beskos 1997 *Applied Mechanics Reviews* **50**, 149–197. Boundary element methods in dynamic analysis: part II (1986–1996).
3. T. A. Cruse and F. J. Rizzo 1968 *Journal of Mathematical Analysis and Application* **22**, 244–259. A direct formulation and numerical solution of the general transient elastodynamic problem.
4. J. Dominguez and E. Alarcon 1981 *Progress in Boundary Element Methods* **1**, London: Pentech Press. See chapter 7, Elastodynamics.
5. R. Gallego and J. Dominguez 1994 *Boundary Element Communications* **5**, 171–180. A two dimensional boundary element code for time domain formulations using quadratic elements, II: transient elastodynamic problems.
6. G. K. K. Wong and J. R. Hutchinson 1981 *Boundary Element Methods*, Berlin: Springer-Verlag. See pp. 272–289, An improved boundary element method for plate vibration.
7. G. Bezine 1980 *Mechanics Research Communications* **7**, 141–150. A mixed boundary integral finite element approach to plate vibration problems.
8. M. Tanaka, K. Yamagiwa, K. Miyazaky and T. Ueda 1988 *Engineering Analysis with Boundary Elements* **5**, 182–188. Free vibration analysis of elastic plate structures by boundary element method.
9. J. T. Katsikadelis 1990 *Journal of Sound and Vibration* **141**, 313–322. A boundary element solution to the vibration problem of plates.
10. C. P. Providakis and D. E. Beskos 1989 *International Journal for Numerical Methods in Engineering* **28**, 1977–1994. Free and forced vibrations of plates by boundary and interior elements.

11. D. Nardini and C. A. Brebbia 1982 *Proceedings of the 4th International Seminar on BEM, Southampton*, 312–326. A new approach to free vibration analysis using boundary elements.
12. D. Nardini and C. A. Brebbia 1985 *Topics in Boundary Element Research*, Berlin and New York: Springer-Verlag. See chapter 7, Boundary integral formulation of mass matrices for dynamics analysis.
13. P. W. Partridge and C. A. Brebbia 1991 *The Dual Reciprocity Boundary Element Method*. Southampton: Computational Mechanics Publications.
14. J. Agnantiaris, D. Polyzos and D. E. Beskos 1995 *Proceedings of the IABEM 95 Conference*, Hawaii, 2890–2896. Some studies on dual reciprocity BEM for elastodynamic analysis.
15. T. G. B. De Figueredo and C. A. Brebbia 1989 *Proceedings of the 11th International Conference on BEM, Cambridge, USA*, 47–57. A new hybrid displacement variational formulation of BEM for elastostatics.
16. N. A. Dumont and R. de Oliveira 1995 *Proceedings of the IABEM 95 Conference, Hawaii*, 3068–3073. On the determination of approximate frequency-dependent mass matrices in the hybrid boundary element method.
17. G. Davì 1992 *Engineering Analysis with Boundary Elements* **10**, 219–224. A hybrid displacement variational formulation of BEM for elastostatics.
18. G. Davì and A. Milazzo 1994 *Engineering Analysis with Boundary Elements* **14**, 343–348. A symmetric and positive definite variational BEM for 2-D free vibration analysis.