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THE FLEXIBLE STRING'S NEGLECTED TERM

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1. INTRODUCTION

The D'Alembert equation of motion for the flexible string is valid only for a large pre-stressed tension, T, and small values of the string element gradient, $\partial y/\partial x$. In the practical case of the finite string the solution takes on a standing wave form set by the boundary and initial conditions. The theory of the impact of a particle with a D'Alembertian flexible string cannot rely on the elastic properties of the string since it is assumed not to have any: i.e., the tension is assumed not to increase as the string is stretched. The basic problem is how to devise a strategy which will describe the changing string shape and its velocity profile during the impact. It seems possible that the basic assumptions of the D'Alembert equation could be concealing a subtle explanation for the early stages of the collision. In this article the finite string is considered and the solutions of the D'Alembert flexible string equation and another equation which takes some account of finite stretching and significant string gradient are compared.

2. THEORY

D'Alembert's equation of motion for an element of a finite flexible string of cross-sectional area A and density ρ held at tension T is

$$\rho A\, \partial^2 y/\partial t^2 = T\, \partial^2 y/\partial x^2.$$

D'Alembert (1746) also offered a general solution of this equation:

$$y = F(x + c_1 t) + G(x - c_1 t).$$

In this F and G are functions of $(x + ct)$ and $(x - ct)$ and represent two unchanging wave forms which move to the left and right along the string with constant speed c_1 given by $(T/\rho A)^{1/2}$.

In D'Alembert's derivation the string tension was assumed to be much greater than any increase caused by stretching and in so doing the string element length δs was taken to be the same as δx. A modified equation (using an expression for the increase in tension due to stretching [1]) which does take account of finite stretching could be

$$\rho A\, \partial^2 y/\partial t^2 = [T + YA(\mathrm{d}s - \mathrm{d}x)/\mathrm{d}x]\, \partial^2 y/\partial x^2.$$

Y is Young's modulus and the extension is assumed to comply with Hooke's law.

Knowing that $\mathrm{d}s = [1 + (\mathrm{d}y/\mathrm{d}x)^2]^{1/2}\, \mathrm{d}x$ and neglecting fourth and higher powers, one has

$$\partial^2 y/\partial t^2 = [c_1^2 + \tfrac{1}{2}c_2^2(\partial y/\partial x)^2]\, \partial^2 y/\partial x^2,$$

where $c_1 = (T/\rho A)^{1/2}$ is the speed of transverse waves along the D'Alembert string and $c_2 = (y/\rho)^{1/2}$ is the speed of compressional longitudinal waves through the string.

Somewhat surprisingly this equation does not appear to have been considered in the modern literature (or elsewhere?). The closest equivalent could be that of Zabusky [2]. The

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exact solutions of such NPDEs are of extreme complexity and do not in general lend themselves to simple applications. An approximate solution is, however, possible if the expression is linearized. Such a solution is as follows.

The solution of the D'Alembert equation for the particular case of the finite string will be considered first. A flexible string is stretched to a tension T between two fixed points at 0, 0 and L, 0. It can be shown that

$$y(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c_1 t}{L} + \psi\right),$$

where A_n and ψ are arbitrary constants. Furthermore if the string shape $f(x)$ between 0, 0 and L, 0 at time $t = 0$ is known, then

$$A_n = \frac{2}{L}\int_0^L f(x) \sin(n\pi x/L)\, \mathrm{d}x$$

and ψ depends on the speed of the string at time $t = 0$. It is sufficient to say that each term of this series solution will satisfy the D'Alembert equation as could be tested by direct substitution. It also satisfies the boundary conditions and the arbitrary constants can be determined by the initial conditions when $t = 0$. A_n are the Fourier coefficients of the string shape and ψ is related to the phase or velocity profile at this moment. It is convenient though not necessary to interpret this as a set of standing waves produced by continuous reflections between the ends of the string. The solution is, however, no more than a dynamic picture of the changing overall string shape through time. The reality of the standing waves as individual entities has been the subject of some debate [3, 4]. But it is known, for instance, that the sound energy produced by a plucked string can be analyzed in terms of the amplitudes of the vibrational modes predicted by the above expression.

For a plucked string (see Figure 1), the Fourier coefficient A_n at $t = 0$ can be shown to be

$$A_n = 2gL^2 \sin(n\pi a/L)/n^2\pi^2 a(L - a).$$

In this case the velocity profile at $t = 0$ will be zero and $\psi = 0$. If this value is used in the series solution and the string shape is run through one cycle of the fundamental one obtains the characteristic three straight lines construction described by Helmholtz [5]. The longitudinal crest velocity of the shapes is $\pm c_1$. The initial conditions will be duplicated at the start of every cycle.

But suppose the plucked string has a significant gradient or value of g? Quite clearly the D'Alembert conditions will be violated and one will need to consider a solution of the non-linear equation which takes note of a finite increase in tension due to the stretching of the string element. Classical methods are available to attack this problem, harmonic balance etc. [1], but in an attempt to simplify the mathematics a slightly different and less rigorous approach is adopted here to linearize the finite amplitude equation.

Assume that a standing wave solution exists such that

$$y(x, t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L) \cos(2\pi v_n t).$$

The cosine term is concerned with the allowable frequencies of the standing wave components. The increase in tension throughout the string caused by any given finite displacement will cause the mode frequencies to increase. This will also be reflected in the relative magnitude of the finite stretching term in the nonlinear equation of motion: $\frac{1}{2}c_2^2(\partial y/\partial x)^2$. If one considers the nth component in the series solution and assumes that the standing waves are independent entities one has an expression for this term:

$$\tfrac{1}{2}c_2^2 A_n^2 n^2 \pi^2 \cos^2 (n\pi x/L) \cos^2 (2\pi \nu_n t)/L^2.$$

Quite clearly this can take any value between 0 and $\frac{1}{2}c_2^2 A_n^2 n^2 \pi^2/L^2$ but it does show how much the tension will vary over the vibrational cycle. One can now postulate that since the maximum value is proportional to g^2 it is possible to estimate that the "average" tension increase will be one third of this value. This will linearize the equation and the Fourier coefficient A_n will be the same as for the D'Alembert flexible string. It also allows one to make a direct though approximate estimate of the allowable frequencies for the finitely stretched flexible string.

One can compare this with the frequencies of the D'Alembert harmonics: $\nu_n = n\nu_1 = nc_1/2L$. One therefore has, for the dispersive non-linear "harmonics", $\nu_n = [c_1^2 + \frac{1}{6}c_2^2 A_n^2 n^2 \pi^2/L^2]^{1/2} n/2L$. Since the allowable frequencies do not in general form a harmonic sequence the initial conditions will not be recreated periodically. The string shape will change irreversibly with time. The manner in which this occurs will depend on the magnitude of g, the position at which the string is plucked and the elastic properties of the string itself.

Figure 1(a) shows the changing shape of a plucked string expressed as a function of the string flexible fundamental. The parameters used are typical of a piano wire. After

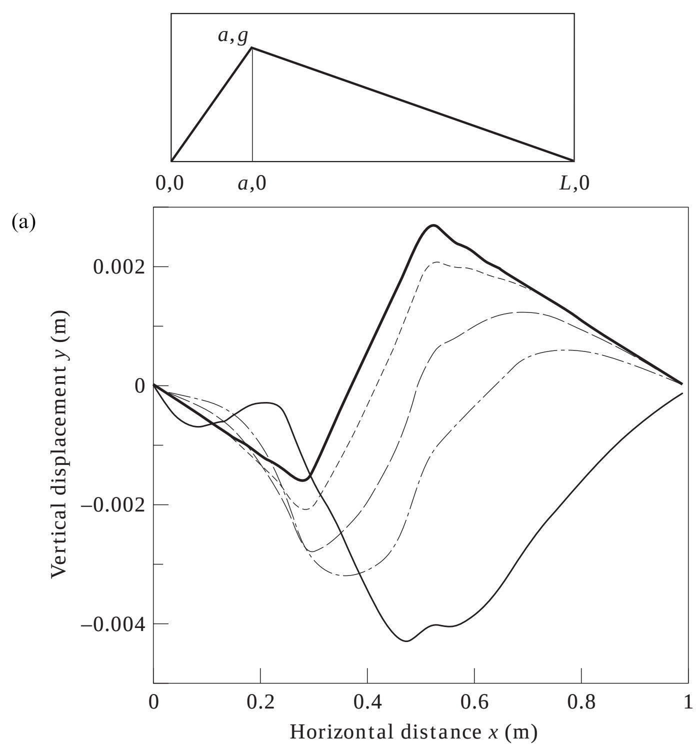

Figure 1. Initial condition for plucked string. (a) Profile of plucked string as a function of the flexible string's fundamental: $L = 1$ m, $N = 9$, $a = L/N$ m, $d = 1$ mm, $T = 7 \times 10^2$ N, $Y = 20 \times 10^{10}$ N/m^2, $\rho = 7{\cdot}8 \times 10^3$ kg/m^3, $g = 5$ mm, cycle = 5·917 ms. Number of cycles: ——, 0·2; - - -, 10·2; — — —, 30·2; –·–, 50·2; ——, 100·2.

100 cycles the string shape retains little of the characteristics of a D'Alembert flexible string.

As g increases in value the allowable frequencies will show a marked departure from the harmonic sequence with the lower frequencies being affected most. This could be described as "negative" inharmonicity which contrasts with the normal (positive) inharmonicity found in a stiff wire. A possible summation of the effects in a practical case will be considered in a future article.

A further curiosity arises from this treatment. The solution can also be applied to a string in which the pre-stressed tension is zero. In this case

$$v_n = (\pi/2L^2\sqrt{6})c_2 A_n n^2.$$

Substituting the value of A_n for the plucked string in this expression yields

$$v_n = (1/\pi\sqrt{6})c_2[g/a(L-a)]\sin(n\pi a/L).$$

The allowable frequencies are markedly different from those of a D'Alembert flexible string and besides having a minimum value when $n = 1$ will also have a maximum of $c_2 g/a(L-a)\pi\sqrt{6}$. In general this does not place a restriction on the number of available frequencies but does set the band in which they can occur. However, if the string is plucked at a node, such that a is L/N and N a whole number, these will be a repeating set of N frequencies for the standing waves. The particular case of $N = 2$, i.e., plucked at the string centre, allows only a single frequency for all the standing waves. A configuration of an isosceles triangle of changing altitude would thus be conserved throughout the vibrational cycle. The D'Alembert string would take the three line dynamic configuration with a vertically moving horizontal line joined to two fixed gradient sides.

3. THE IMPACT PROBLEM

It is not proposed to develop an impact model in this article. Instead the solutions of the equations of motion following an impact which leaves a momentary distortion in a short section of the string will be compared.

If the string is struck by a particle at $a, 0$ there will be a reaction which could result in a V notch shape being formed as the particle leaves. If one assumes such a shape (see Figure 2) one will also have a zero velocity profile, $\psi = 0$, and hence before transverse wave motion commences

$$A_n = (4gL/n^2\pi^2 b)[1 - \cos(n\pi b/L)]\sin(n\pi a/L).$$

Figure 2(a) shows a sequence of snapshots of the string profile, demonstrating how rapidly the form degenerates over successive cycles. The D'Alembert string is shown for reference purposes at the end.

One of the more attractive features of D'Alembert's general solution is the conservation of the string shape once created. This should be qualified slightly. Thus regardless of the boundary conditions, if a momentarily static shape is released onto the string two similar shapes i.e., mirror images and half the crest height of the original, will be produced and will travel in opposite directions along the string with speed c_1. A V notch pair would thus be sustained indefinitely without further distortion in a finite string. The shapes would appear to be bounced backwards and forwards between the supports. This notion may be suspect.

(a)

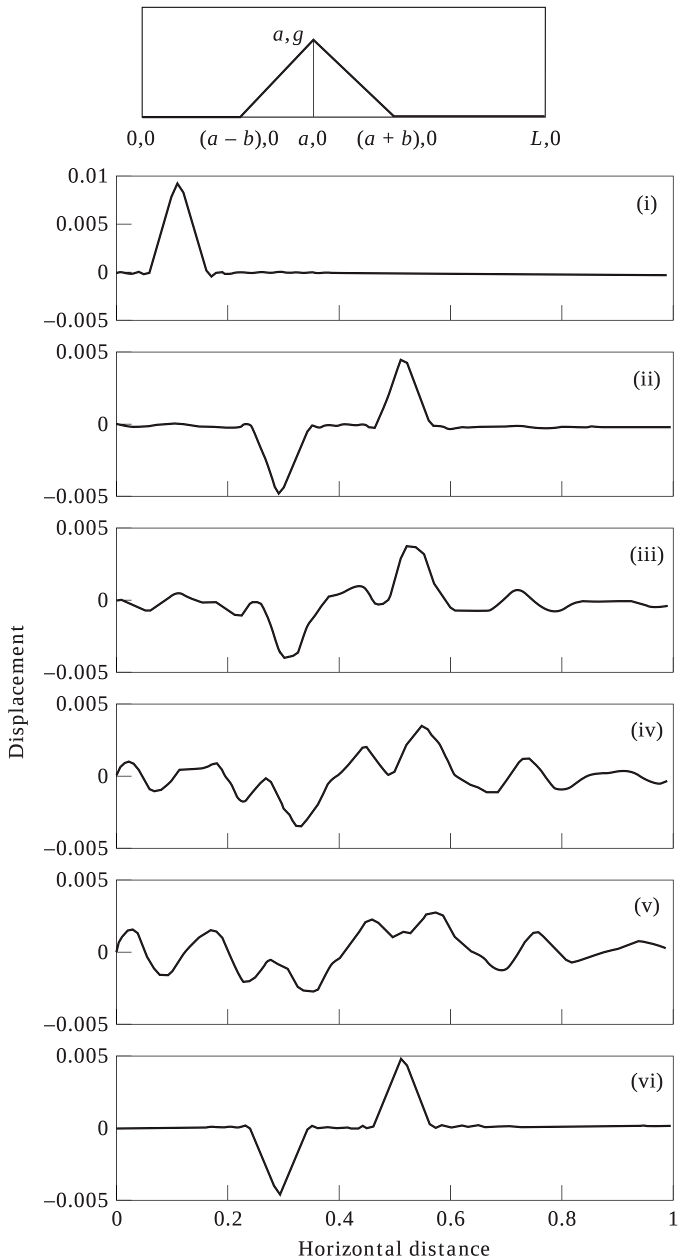

Figure 2. Initial condition for impulse type distortion. (a) String profile following impulse type deformation. $L = 1$ m, $N = 9$, $a = L/N$ m, $b = 50$ mm, $d = 1$ mm, $g = 10$ mm, $y = 20 \times 10^{10}$ N/m^2, $T = 700$ N, $\rho = 7{\cdot}8 \times 10^3$ kg/m^3, cycle = 5·917 ms. (i) Initial condition; (ii) 0·2 cycles; (iii) 1·2 cycles; (iv) 2·2 cycles; (v) 3·2 cycles; (vi) 3·2 cycles for $Y = 0$.

If one examines the finite amplitude equation of motion one has the concept of an instantaneous crest velocity:

$$\pm[c_1^2 + \tfrac{1}{2}c_2^2(\partial y/\partial x)^2]^{1/2}.$$

For a string without initial tension, i.e., $T = 0$, one will have a notional speed

$$(1/\sqrt{2})c_2\left(\frac{\partial y}{\partial x}\right).$$

It can be seen, for example, that if the gradient of the V notch were unity a speed of $c_2/\sqrt{2}$ would be implied. This could then be regarded as a localized condition with the string being strongly stretched and distorted in the process. It could be compared with a transverse wave produced by tension alone. In the D'Alembert model this would be equivalent to $c_1 = c_2/\sqrt{2}$: that is, a string tension of $YA/2$. For a steel wire of diameter 1 mm this would be an incredible and unbelievable 157 000 N. This can be compared with a practical tension in a piano wire of 1000 N for T. Whether such a wire could survive such an albeit momentary deformation is open to question. Certainly string stiffness would militate against its happening in the first place. But at least the equation for a finite displacement is aware of the problem which the D'Alembert solution chooses to ignore.

The shape of the localized distortion is clearly a problem. The gradient of any part of it could easily violate the implied conditions. A square-sided Dirac type impulse deformation is such an example. Even the neglected term equation would be inadequate for the purpose.

The highly dispersive nature of an elastically formed V notch can perhaps be better appreciated if one concedes localized stretching in the vicinity of the point of impact at a, 0. One could then allow a standing wave solution within the V notch itself. The V notch would become a finite energy packet momentarily held between $(a - b)$, 0 and $(a + b)$, 0 at $t = 0$. If one treats this as a string of height g and length $2b$ plucked at its centre its frequencies will all take the same value: $c_2 g/\pi b^2\sqrt{6}$. The individual wave velocities will show negative dispersion with the speed of the nth component given by $4c_2 g/n\pi b\sqrt{6}$. The packet will therefore burst after $t = 0$ with pairs of waves travelling in opposite directions along the string. This can be contrasted with the D'Alembert string where only a single pair of shapes will be produced.

It should be noted that the ratio of the V notch height to its base width sets the velocities which will be of the same order as longitudinal compression waves if $g/2b$ is a sizeable fraction. A possible role for these high velocity elastic transverse waves should not be discounted following an impact.

This solution also possesses soliton properties: e.g., the velocities are proportional to the wave crest amplitudes. The failure to retain the basic shape could be compensated by also considering string stiffness.

It should be emphasized that the simple V notch considered would be of limited value in practical cases such as piano hammer interactions. The mass of the hammer is usually of the same order as that of the string itself and contact times are relatively large but localized distortions have nevertheless been observed.

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