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CAN AN UNDAMPED OSCILLATOR DISSIPATE ENERGY?

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1. INTRODUCTION

It can readily be established that the power dissipated by a linear oscillator which is subjected to wide band excitation is independent of the oscillator loss factor η. The underlying reason for this behaviour is that the mean squared oscillator velocity $\sigma_{\dot{x}}^2$ is inversely proportional to η while the power dissipated is proportional to $\eta\sigma_{\dot{x}}^2$. This raises the curious point that, mathematically at least, power would appear to be dissipated for the case $\eta = 0$, so that power is lost through an undamped system. This apparent anomaly is resolved in the following section by considering the non-stationary behaviour of the oscillator. As discussed in section 3, this work has some connection with fuzzy structure theory [1, 2], wherein the energy dissipated through a set of "fuzzy" oscillators is found to be independent of the oscillator loss factor.

2. POWER DISSIPATED BY AN OSCILLATOR

The equation of motion of a single-degree-of-freedom linear oscillator subjected to white noise excitation can be written in the form

$$\ddot{x} + \eta\omega_n\dot{x} + \omega_n^2 x = F, \tag{1}$$

where x is the oscillator displacement, η is the loss factor, ω_n is the natural frequency, and F is the force per unit mass. It is well known that equation (1) has the stationary solution

$$\sigma_x^2 = \pi S_0/2\eta\omega_n^3, \qquad \sigma_{\dot{x}}^2 = \pi S_0/2\eta\omega_n, \tag{2, 3}$$

where σ_x^2 and $\sigma_{\dot{x}}^2$ are the mean squared displacement and velocity respectively, and S_0 is the (single sided) spectral level of the excitation. The expected value of the power dissipated is given by

$$P = E[\eta\omega_n\dot{x}^2] = \pi S_0/2, \tag{4}$$

which is clearly independent of the system loss factor. Mathematically at least, equation (4) is valid for $\eta \to 0$, which leads to the curious anomaly that a system with no damping appears to dissipate energy. It is shown in what follows that equation (4) is in fact correct for $\eta = 0$, although the correct physical interpretation of this result requires a consideration of the non-stationary, rather than the stationary, response of the system.

If equation (1) is multiplied by $\dot{x}$ and then averaged (in an ensemble sense rather than a temporal sense), then the following power balance equation results:

$$E[\dot{x}\ddot{x} + \eta\omega_n\dot{x}^2 + \omega_n^2\dot{x}x] = E[\dot{x}F]. \tag{5}$$

The term on the right of this equation is the power input P to the system; the terms on the left can be re-arranged to yield

$$P = \eta\omega_n E[\dot{x}^2] + (\mathrm{d}/\mathrm{d}t)E[\dot{x}^2/2 + \omega_n^2 x^2/2], \tag{6}$$

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where the terms on the right can now be identified as the power dissipated by damping and the rate of increase of the system energy (kinetic plus potential). If the system is excited from rest, so that $x = \dot{x} = 0$ at $t = 0$, then the non-stationary mean squared response has the following well known form [3, 4]:

$$E[x^2] = \frac{\pi S_0}{2\eta\omega_n^3}\left\{1 - \exp(-\eta\omega_n t)\left[1 + \left(\frac{\eta\omega_n}{2\omega_1}\right)\sin 2\omega_1 t + \left(\frac{\eta^2\omega_n^2}{2\omega_1^2}\right)\sin^2\omega_1 t\right]\right\}, \tag{7}$$

$$E[\dot{x}^2] = \frac{\pi S_0}{2\eta\omega_n}\left\{1 - \exp(-\eta\omega_n t)\left[1 - \left(\frac{\eta\omega_n}{2\omega_1}\right)\sin 2\omega_1 t + \left(\frac{\eta^2\omega_n^2}{2\omega_1^2}\right)\sin^2\omega_1 t\right]\right\}. \tag{8}$$

Here $\omega_1^2 = \omega_n^2(1 - \eta^2/4)$. Clearly, providing $\eta \neq 0$, these results tend to the stationary solution, equations (2) and (3), as $t \to \infty$. Once the stationary state is achieved then the second term on the right of equation (6) becomes zero, and the power dissipated is given by equation (4). For zero damping however ($\eta = 0$) a stationary state can never be achieved, and in this case equations (7) and (8) become

$$E[x^2] = (\pi S_0/2\omega_n^2)[t - (1/2\omega_n)\sin 2\omega_n t], \tag{9}$$

$$E[\dot{x}^2] = (\pi S_0/2)[t + (1/2\omega_n)\sin 2\omega_n t]. \tag{10}$$

The use of equations (9) and (10) in equation (6) leads to equation (4), but the physics which underlies the result is clearly quite different from that for the case $\eta \neq 0$: for $\eta \neq 0$ the (steady state) power dissipation is due to the oscillator damping, whereas for $\eta = 0$ a steady state condition does not occur, and the power P is associated with a build-up of the oscillator energy.

For completeness it can be noted that the results derived thus far regarding P are actually a subset of a much more general result: it can be shown that the result $P = \pi S_0/2$ holds for all η, for all times t, and for all possible starting conditions. To demonstrate this, the solution of equation (1) can be written in the form

$$\mathbf{x} = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{F}(\tau)\,d\tau + \mathbf{x}_0, \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -\eta\omega_n \end{pmatrix}, \tag{11, 12}$$

where $\mathbf{x} = (x\ \ \dot{x})^T$, $\mathbf{F} = (0\ \ F)^T$, and $\mathbf{x}_0$ contains the initial conditions. If equation (11) is multiplied by $\mathbf{F}^T(t)$ and then averaged, then the following result ensues:

$$P = E\left[\int_0^t \mathbf{F}^T(t)\, e^{\mathbf{A}(t-\tau)}\mathbf{F}(\tau)\,d\tau\right]. \tag{13}$$

Now the excitation $F(t)$ is white noise, and thus it follows that $E[F(t)F(\tau)] = \pi S_0\delta(t-\tau)$ where δ is the Dirac delta function. By making use of this result, it then follows from equation (12) that $P = \pi S_0/2$ in all cases, although it can be noted that the time $t = 0$ is somewhat problematical. Strictly, equation (13) yields $P = 0$ at $t = 0$, followed by $P = \pi S_0/2$ at all subsequent non-zero times—the discontinuous nature of P can be traced to the delta function correlation function of the white noise excitation: band limited excitation would smooth this behaviour.

3. CONCLUSIONS

It has been shown that the power dissipated by a linear oscillator subjected to white noise excitation is given by equation (4), a result which is independent of the oscillator loss factor η. Curiously, equation (4) remains (mathematically) valid for $\eta = 0$, and this is explained by the fact that the oscillator never reaches steady state for this case, but rather undergoes a steady increase in energy. Clearly equations (9) and (10) will eventually become invalid for any practical system, since the linearity of the system will cease beyond a certain level of response.

The present work has some relation to recent developments in fuzzy structure theory. In particular it has been found that the effective damping produced by a set of "fuzzy" oscillators depends only on the number of oscillators and not on the oscillator loss factor [1, 2]. The basic physics behind this result is encapsulated in equation (4); each oscillator is effectively subjected to broad band excitation arising from the "master" structure. It should be noted however that this result is only valid providing the fuzzy oscillators have a high modal overlap—if this is not the case then the master structure may not have a smooth response spectrum, and the oscillators can no longer be considered to be subjected to wide band excitation. Clearly a high modal overlap cannot occur for $\eta = 0$, and thus the present results regarding $\eta = 0$ are not directly applicable to fuzzy structures; rather, the present work has highlighted a curious result in random vibration theory which has arisen as an aside to fuzzy structure theory. In this regard it can be noted that the case of the transient response an undamped fuzzy system (subjected to an impulse) has recently been studied in some detail by Weaver [5].

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