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QUINTIC SPLINES IN THE STUDY OF TRANSVERSE VIBRATIONS OF NON-UNIFORM ORTHOTROPIC RECTANGULAR PLATES

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An analysis and numerical results are presented for free transverse vibrations of orthotropic rectangular plates of linearly varying thickness along one direction and resting on an elastic foundation of the Winkler type on the basis of classical plate theory. Following the Lévy approach i.e., two parallel edges being simply supported, the fourth order differential equation governing the motion of such plates has been solved by using the quintic splines interpolation technique for three different combinations of clamped, simply supported and free boundary conditions at the other two edges. The effect of the elastic foundation together with the orthotropy, aspect ratio and thickness variation on the natural frequencies of vibration is illustrated for the first three modes of vibration. Normalized displacements are presented for two different values of the taper constant keeping other plate parameters fixed for all the three boundary conditions. A comparison of the results with those available in literature is presented.

INTRODUCTION

A considerable amount of work dealing with natural frequencies of isotropic rectangular plates of variable thickness has appeared in the literature [1–4] to mention a few, of which [3, 4] are comprehensive survey papers. Recently there has been increasingly great interest in high strength materials for structural components used in mechanical, aerospace, ocean engineering, electronic and optical equipments. Composite materials such as glass-epoxy, boron-epoxy, Kevalar and graphites etc., are lighter, stiffer and stronger than any other material used earlier. These materials have a wide range of operating temperatures besides high damping and resistance to corrosion. Plates fabricated out of modern composites together with thickness variation are not only reduced in size and weight but also meet the desirability of high strength. This has necessitated the study of vibrational characteristics of orthotropic plates [5–16]. Further the problem of plates resting on an elastic foundation has achieved importance in modern technological and foundation engineering [17–20].

The present paper deals with the effect of Winkler type foundation on the natural frequencies of orthotropic rectangular plates with thickness varying linearly in one direction only. The two parallel edges ($y = 0$, $y = b$) are assumed to be simply supported while the other two edges are differently restrained (clamped, simply supported or free). The fourth order linear differential equation with variable coefficients which governs the motion has been solved by the method of quintic splines. This method of solution is preferred over other methods for the reasons (a) a chain of lower order approximations may yield a better accuracy than a global higher order approximation [21, 22] and

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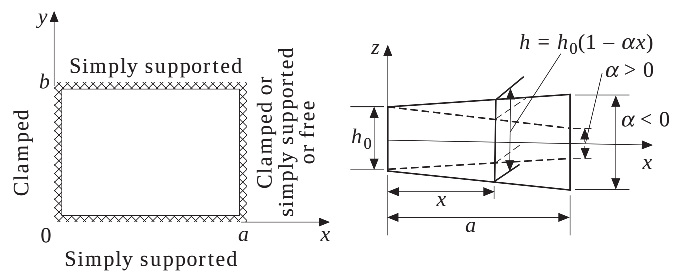

Figure 1. Boundary conditions and vertical cross-section of the plate.

(b) natural boundary conditions can be considered easily. Frequencies for the first three modes of vibration have been computed for various plate parameters. A five ply maple plywood has been taken as an example of a rectangular orthotropic material.

2. MATHEMATICAL FORMULATION

Consider a rectangular plate of length a, breadth b, thickness $h = h(x, y)$, density ρ and resting on a Winkler type foundation of foundation modulus k_f. The plate is referred to a system of rectangular Cartesian co-ordinates (x, y, z), the middle surface being $z = 0$, and the origin at one of the corners of the plate. The x- and y-axes are taken along the principal directions of orthotropy and the axis of z is perpendicular to the xy-plane (Figure 1). The differential equation which governs the transverse free vibrations of such plates is given by

$$D_x \frac{\partial^4 w}{\partial x^4} + D_y \frac{\partial^4 w}{\partial y^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + 2 \frac{\partial H}{\partial x} \frac{\partial^3 w}{\partial x\, \partial y^2} + 2 \frac{\partial H}{\partial y} \frac{\partial^3 w}{\partial y\, \partial x^2} + 2 \frac{\partial D_x}{\partial x} \frac{\partial^3 w}{\partial x^3}$$
$$+ 2 \frac{\partial D_y}{\partial y} \frac{\partial^3 w}{\partial y^3} + \frac{\partial^2 D_x}{\partial x^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 D_y}{\partial y^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 D_1}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 D_1}{\partial x^2} \frac{\partial^2 w}{\partial y^2}$$
$$+ 4 \frac{\partial^2 D_{xy}}{\partial x\, \partial y} \frac{\partial^2 w}{\partial y\, \partial x} + \rho h \frac{\partial^2 w}{\partial t^2} + k_f w = 0, \quad (1)$$

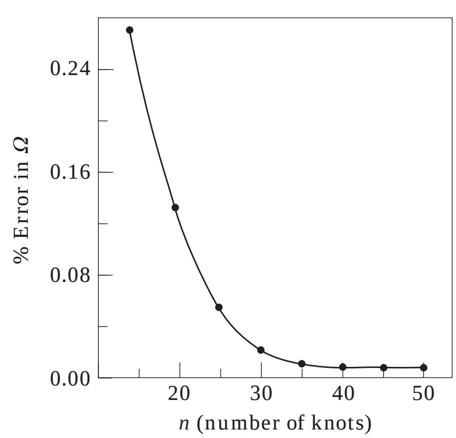

Figure 2. Percentage error in Ω for C–C plate for $a/b = 1{\cdot}0$, $K = 0{\cdot}02$ and $\alpha = -0{\cdot}5$. Percentage error $= [(\Omega_n - \Omega_{40})/\Omega_{40}] \times 100$; $n = 15(5)50$.

TABLE 1

Values of frequency parameter Ω, for C–C plate

Mode	K	α −0·5	−0·3	−0·1	0·0	0·1	0·3	0·5
	$a/b = 0{\cdot}5$							
I	0·00	28·1105	26·0053	23·8272	22·7046	21·5545	19·1562	16·5686
	0·01	29·7901	27·9497	26·1171	25·2092	24·3117	22·5730	20·9859
	0·02	31·3796	29·7673	28·2218	27·4864	26·7859	25·5356	24·6166
II	0·00	76·9075	71·1660	65·2160	62·1452	58·9976	52·4147	45·2923
	0·01	77·5359	71·9004	66·0871	63·1033	60·0598	53·7617	47·1101
	0·02	78·1665	72·6277	66·9470	64·0470	61·1035	55·0761	48·8619
III	0·00	150·5127	139·2896	127·6517	121·6422	115·4798	102·5826	88·6122
	0·01	150·8371	139·6666	128·0989	122·1344	116·0260	103·2781	89·5592
	0·02	151·1609	140·0426	128·5448	122·6247	116·5699	103·9691	90·4965
	$a/b = 1{\cdot}0$							
I	0·00	29·8960	27·6522	25·3333	24·1394	22·9180	20·3716	17·6311
	0·01	31·4809	29·4884	27·4980	26·5087	25·5277	23·6138	21·8378
	0·02	32·9894	31·2168	29·5044	28·6829	27·8942	26·4608	25·3496
II	0·00	78·6036	72·7311	66·6479	63·5093	60·2931	53·5693	46·3002
	0·01	79·2221	73·4501	67·5006	64·4471	61·3328	54·8881	48·0796
	0·02	79·8359	74·1619	68·3427	65·3715	62·3551	56·1761	49·7972
III	0·00	152·2119	140·8583	129·0871	123·0097	116·7784	103·7395	89·6205
	0·01	152·5328	141·2311	129·5294	123·4966	117·3188	104·4275	90·5567
	0·02	152·8530	141·6030	129·9704	123·9815	117·8565	105·1110	91·4838

TABLE 2

Values of frequency parameter Ω, for C–S plate

Mode	K	α −0·5	−0·3	−0·1	0·0	0·1	0·3	0·5
	$a/b = 0{\cdot}5$							
I	0·00	18·6482	17·5741	16·4497	15·8644	15·2604	13·9827	12·5734
	0·01	21·0281	20·2936	19·6012	19·2789	18·9788	18·4754	18·1869
	0·02	23·1643	22·6891	22·3120	22·1738	22·0796	22·0698	22·4244
II	0·00	61·5457	57·2418	52·7817	50·4801	48·1213	43·1895	37·8590
	0·01	62·3284	58·1483	53·8525	51·6550	49·4206	44·8282	40·0546
	0·02	63·1015	59·0408	54·9024	52·8038	50·6867	46·4095	42·1405
III	0·00	128·9620	119·6347	109·9660	104·9752	99·8595	89·1609	77·5925
	0·01	129·3395	120·0724	110·4845	105·5453	100·4915	89·9639	78·6830
	0·02	129·7158	120·5987	111·0005	106·1123	101·1194	90·7597	79·7593
	$a/b = 1{\cdot}0$							
I	0·00	21·3082	19·9359	18·5186	17·7888	17·0422	15·4846	13·8038
	0·01	23·4223	22·3718	21·3677	20·8912	20·4386	19·6360	19·0606
	0·02	25·3601	24·5670	23·8792	23·5891	23·3458	23·0498	23·1410
II	0·00	63·5939	59·1037	54·4556	52·0594	49·6052	44·4808	38·9533
	0·01	64·3513	59·9816	55·4941	53·1994	50·8668	46·0737	41·0903
	0·02	65·0999	60·8471	56·5134	54·3155	52·0977	47·6138	43·1257
III	0·00	130·8933	121·4008	111·5643	106·4887	101·2870	90·4132	78·6630
	0·01	131·2651	121·8322	112·0755	107·0507	101·9102	91·2051	79·7388
	0·02	131·6359	122·2620	112·5842	107·6096	102·5294	91·9902	80·8009

where

$$D_x = E_x^* h^3/12, \qquad D_y = E_y^* h^3/12, \qquad D_{xy} = G_{xy} h^3/12, \qquad D_1 = E^* h^3/12,$$

$$H = D_1 + 2D_{xy}, \qquad (E_x^*, E_y^*) = (E_x, E_y)/(1 - \nu_x \nu_y), \qquad E^* = \nu_y E_x^* = \nu_x E_y^*,$$

$w(x, y, t)$ is the transverse deflection, t is the time, ρ is the mass density and E_x, E_y, ν_y, ν_x and G_{xy} are material constants in proper directions.

Further, it is assumed that the two opposite edges of the plate $y = 0$ and $y = b$ are simply supported and that the thickness varies in the x-direction only i.e., $h = h(x)$. For a harmonic solution, the deflection w (Lévy approach) is assumed to be

$$w(x, y, t) = \bar{w}(x) \sin(p\pi y/b)\, e^{i\omega t} \tag{2}$$

where p is a positive integer and ω is the radian frequency.

Introducing the non-dimensional variables

$$X = x/a, \qquad Y = y/b, \qquad \bar{h} = h/a, \qquad W = \bar{w}/a \tag{3}$$

equation (1) reduces to

$$\begin{aligned} &\bar{h}^3 W^{iv} + 6\bar{h}^2\bar{h}' W''' + [3\{\bar{h}^2\bar{h}'' + 2\bar{h}\bar{h}'^2\} - 2(\eta^*/E_x^*)\bar{h}^3\lambda^2]W'' \\ &-6(\eta^*/E_x^*)\bar{h}^2\bar{h}'\lambda^2 W' + [(E_y^*/E_x^*)\bar{h}^3\lambda^4 - 3(E^*/E_x^*) \\ &\{\bar{h}^2\bar{h}'' + 2\bar{h}\bar{h}'^2\}\lambda^2 - 12(\rho a^2\omega^2/E_x^*)\bar{h} + 12K]W = 0, \end{aligned} \tag{4}$$

TABLE 3

Values of frequency parameter Ω, for C–F *plate*

Mode	K	α −0·5	−0·3	−0·1	0·0	0·1	0·3	0·5
	$a/b = 0·5$							
I	0·00	6·2732	5·8362	5·4196	5·2229	5·0369	4·7106	4·4804
	0·01	11·2235	11·4608	11·8588	12·1359	12·4784	13·4208	14·8639
	0·02	14·5774	15·1193	15·8708	16·3486	16·9124	18·3737	20·4418
II	0·00	27·7286	26·0856	24·4016	23·5411	22·6663	20·8658	18·9790
	0·01	29·3671	27·9771	26·6215	25·9650	25·3302	24·1663	23·2969
	0·02	30·9199	29·7492	28·6702	28·1812	27·7396	27·0740	26·9915
III	0·00	76·2075	71·0087	65·6350	62·8689	60·0402	54·1519	47·8453
	0·01	76·8383	71·7389	66·4979	63·8162	61·0885	55·4783	49·6388
	0·02	77·4639	72·4618	67·3497	64·7495	62·1192	56·7743	51·3742
	$a/b = 1·0$							
I	0·00	11·7439	10·7327	9·7046	9·1852	8·6637	7·6234	6·6221
	0·01	15·0359	14·6119	14·3464	14·2958	14·3171	14·6532	15·5800
	0·02	17·7246	17·6575	17·8171	18·0103	18·2998	19·2595	20·9359
II	0·00	34·1337	31·6098	29·0511	27·7572	26·4525	23·8073	21·1071
	0·01	35·4714	33·1832	30·9672	29·8405	28·7703	26·7557	25·0710
	0·02	36·7609	34·6856	32·7149	31·7877	30·9150	29·4151	28·5351
III	0·00	81·0645	75·2520	69·2680	66·1985	63·0674	56·5796	49·6823
	0·01	81·6546	75·9392	70·0853	67·0986	64·0673	57·8536	51·4177
	0·02	82·2406	76·6205	70·8933	67·9869	65·0516	59·1008	53·1008

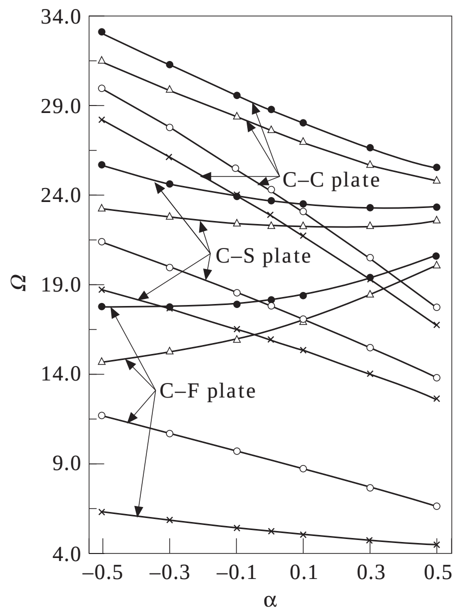

Figure 3. Natural frequencies of C–C, C–S and C–F plates for the first mode of vibration. $a/b = 0{\cdot}5$: –x–x–, $K = 0{\cdot}0$; –△–△–, $K = 0{\cdot}02$. $a/b = 1{\cdot}0$: –○–○–, $K = 0{\cdot}0$; –●–●–, $K = 0{\cdot}02$.

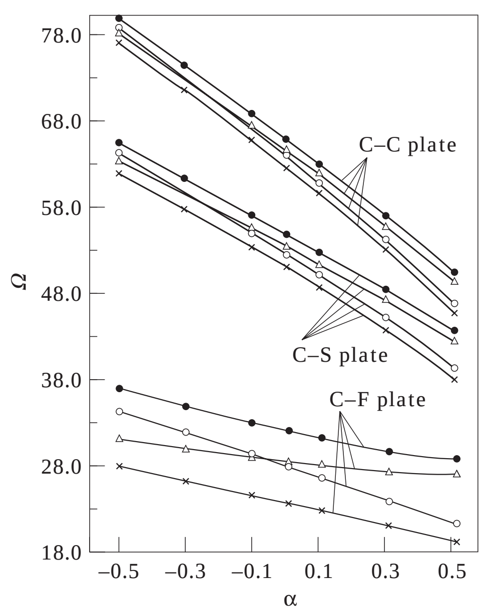

Figure 4. Natural frequencies of C–C, C–S and C–F plates for the second mode of vibration. Key as Figure 3.

where

$$\lambda^2 = p^2\pi^2 a^2/b^2, \qquad K = ak_f/E_x^*, \qquad \eta^* = E^* + 2G_{xy}.$$

For a linear thickness variation in the x direction given by

$$\bar{h} = h_0(1 - \alpha X), \tag{5}$$

where h_0 is the thickness of the plate at $X = 0$ and α is the taper parameter. Equation (4) now reduces to

$$A_0 W^{iv} + A_1 W''' + A_2 W'' + A_3 W' + A_4 W = 0, \tag{6}$$

where

$$A_0 = (1 - \alpha X)^3, \qquad A_1 = -6\alpha(1 - \alpha X)^2, \qquad A_2 = 6\alpha^2(1 - \alpha X) - 2\lambda^2(\eta^*/E_x^*)(1 - \alpha X)^3,$$

$$A_3 = 6\alpha\lambda^2(\eta^*/E_x^*)(1 - \alpha X)^2, \qquad \Omega^2 = 12\rho a^2\omega^2/E_x^* h_0^2$$

$$A_4 = \lambda^4(E_y^*/E_x^*)(1 - \alpha X)^3 - 6\alpha^2\lambda^2(E^*/E_x^*)(1 - \alpha X) - \Omega^2(1 - \alpha X) + 12(K/h_0^3),$$

and primes denote differentiation with respect to X.

The solution of equation (6) together with the boundary conditions at the edges $X = 0$ and $X = 1$ constitutes a well defined boundary value problem which has been solved by a quintic splines interpolation technique.

3. SOLUTION BY QUINTIC SPLINES

According to the spline technique [20], one divides the interval [0, 1] into n equal subintervals ΔX by means of points X_i, $i = 0, 1, 2, \ldots, n$. The quintic spline takes the form

$$W(X) = a_0 + \sum_{j=1}^{4} a_j (X - X_0)^j + \sum_{i=0}^{n-1} b_i (X - X_i)_*^5, \tag{7}$$

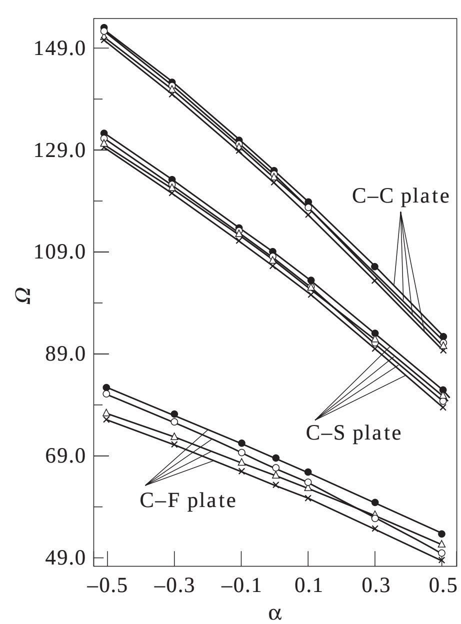

Figure 5. Natural frequencies of C–C, C–S and C–F plates for the third mode of vibration. Key as Figure 3.

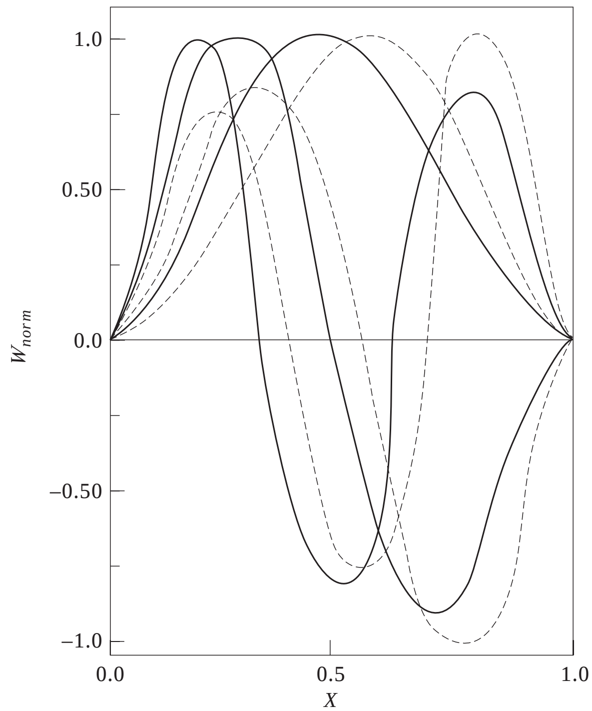

Figure 6. Normalized displacements of C–C plate for the first three modes of vibration. ----, $\alpha = 0{\cdot}5$; ——, $\alpha = -0{\cdot}5$; $a/b = 1{\cdot}0$; $K = 0{\cdot}01$.

where

$$(X - X_i)_* = \begin{cases} 0 & \text{if } X \leqslant X_i \\ X - X_i & \text{if } X > X_i \end{cases},$$

$\Delta X = 1/n$ and $a_0, \ldots, a_4, b_0, \ldots, b_{n-1}$ are $(n+5)$ unknown constants.

Substitution for $W(X)$ and its derivatives into equation (6) gives, for satisfaction at the m^{th} knot,

$$\begin{aligned} &A_4 a_0 + [A_4(X_m - X_0) + A_3]a_1 + [A_4(X_m - X_0)^2 + 2A_3(X_m - X_0) + 2A_2]a_2 \\ &\quad + [A_4(X_m - X_0)^3 + 3A_3(X_m - X_0)^2 + 6A_2(X_m - X_0) + 6A_1]a_3 \\ &\quad + [A_4(X_m - X_0)^4 + 4A_3(X_m - X_0)^3 + 12A_2(X_m - X_0)^2 + 24A_1(X_m - X_0) + 24A_0]a_4 \\ &\quad + \sum_{i=0}^{n-1} [A_4(X_m - X_i)^5 + 5A_3(X_m - X_i)^4 + 20A_2(X_m - X_i)^3 \\ &\quad + 60A_1(X_m - X_i)^2 + 120A_0(X_m - X_i)]b_i = 0. \end{aligned} \tag{8}$$

For $m = 0(1)n$, one obtains a set of $(n+1)$ homogeneous equations having $(n+5)$ unknowns a_i, $i = 0(1)4$, b_j, $j = 0, 1, \ldots, (n-1)$, which can be represented by the matrix equation

$$[\mathbf{A}]\{\mathbf{B}\} = \{\mathbf{0}\}, \tag{9}$$

where $\mathbf{A}$ is a matrix of order $(n+1) \times (n+5)$ and, $\{\mathbf{B}\}$ and $\{\mathbf{0}\}$ are column vectors.

4. BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

The three sets of boundary conditions namely C–C, C–S, C–F have been considered in which the first symbol represents the condition at the edge $X = 0$ and second symbol at the edge $X = 1$ and C, S, F stand for clamped, simply supported and free edge, respectively. The relations which should be satisfied at clamped, simply supported and free edge are $W = \mathrm{d}W/\mathrm{d}X = 0$; $W = (\mathrm{d}^2W/\mathrm{d}X^2) - (E^*/E_x^*)\lambda^2 W = 0$ and $(\mathrm{d}^2W/\mathrm{d}X^2) - (E^*/E_x^*)\lambda^2 W = (\mathrm{d}^3W/\mathrm{d}X^3) - \{(E^* + 4G_{xy})/E_x^*\}\lambda^2(\mathrm{d}w/\mathrm{d}X) = 0$, respectively.

Applying the boundary condition C–C to the displacement function (7), one obtains a set of four homogeneous equations in terms of unknown constants a_i, $i = 0(1)4$, b_j, $j = 0(1)(n-1)$ which can be written as

$$[\mathbf{B}_{CC}]\{\mathbf{B}\} = \{\mathbf{0}\}, \tag{10}$$

where $[\mathbf{B}_{CC}]$ is a $4 \times (n+5)$ matrix.

The equations (10) together with the field equations (9) give a complete set of $(n+5)$ equations in $(n+5)$ unknowns which can be denoted by

$$[\mathbf{A}/\mathbf{B}_{CC}]\{\mathbf{B}\} = \{\mathbf{0}\}. \tag{11}$$

For a non-trivial solution of equation (11), the frequency determinant must vanish and hence,

$$|\mathbf{A}/\mathbf{B}_{CC}| = 0. \tag{12}$$

Similarly for C–S and C–F plates, the frequency determinants can be written as

$$|\mathbf{A}/\mathbf{B}_{CS}| = 0, \qquad |\mathbf{A}/\mathbf{B}_{CF}| = 0, \tag{13, 14}$$

respectively.

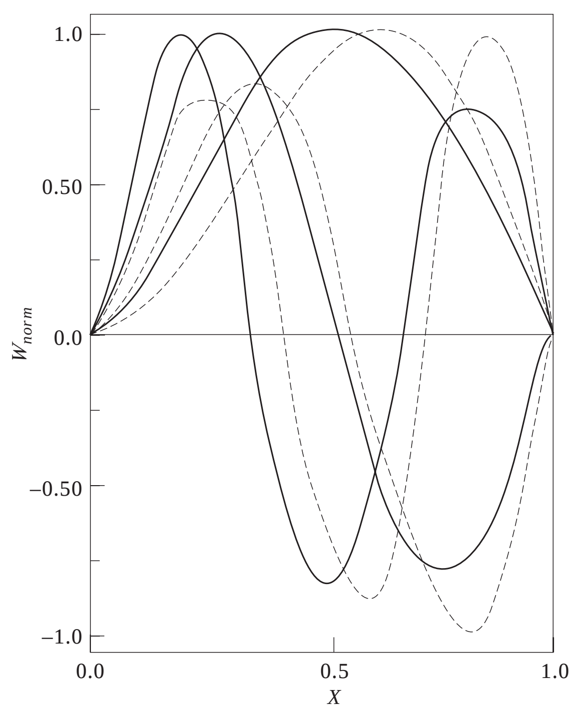

Figure 7. Normalized displacements of C–S plate for the first three modes of vibration. Key as Figure 6.

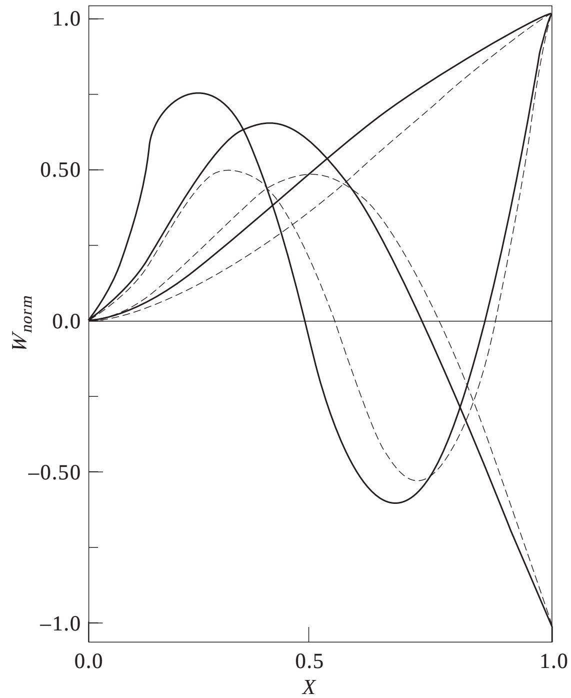

Figure 8. Normalized displacements of C–F plate for the first three modes of vibration. Key as Figure 6.

5. NUMERICAL RESULTS AND DISCUSSION

The frequency equations (12), (13) and (14) provide the values of the frequency parameter Ω for various values of plate parameters. In the work reported here, numerical results have been computed for the first three modes of vibration, for various values of foundation parameter $K(=ak_f/E_x^*) = 0{\cdot}00,\ 0{\cdot}01,\ 0{\cdot}02$ and taper constant $\alpha = 0{\cdot}0,\ -0{\cdot}5$ (0·2) 0·5 on the natural frequencies for two values of aspect ratio $a/b = 0{\cdot}5,\ 1{\cdot}0$ for three boundary conditions C–C, C–S and C–F. This is achieved by writing $p = 1$ in the frequency equations and determining the first three values of Ω. The values of elastic constants used for the plate material are $E_x = 1{\cdot}87 \times 10^6$ p.s.i. $(=1{\cdot}3147 \times 10^5$ kg/cm$^2)$, $E_y = 0{\cdot}60 \times 10^6$ p.s.i. $(= 0{\cdot}4218 \times 10^5$ kg/cm$^2)$, $G_{xy} = 0{\cdot}159 \times 10^6$ p.s.i. $(=0{\cdot}1118 \times 10^5$ kg/cm$^2)$ and $\nu_x = 0{\cdot}12$ (5-ply maple plywood [25]). The thickness h_0 at the origin has been taken as 0·1.

To choose the appropriate interval ΔX, the computer program developed for the evaluation of frequency parameter Ω was run for 15(5)50. The numerical values showed a consistent improvement with the increase in number of knots. In all the above computations, $n = 40$ has been fixed, since further increase in n does not improve the results except in fourth ($\alpha < -0{\cdot}3$) or fifth ($\alpha \geqslant -0{\cdot}3$) decimal place (Figure 2). Double precision arithmetic has been used for the computation of results.

The results are presented in Tables 1–3 and Figures 3–8. It is found that the frequency parameter Ω for a C–S plate is greater than that for a C–F plate but less than that for a C–C plate for the same set of values of plate parameters a/b, α and K.

Figure 3 shows the behaviour of the frequency parameter Ω for varying values of taper constant α, foundation parameter K and the aspect ratio a/b for a plate vibrating in fundamental mode. The frequency parameter is found to decrease with the increase in α for all the boundary conditions i.e., C–C, C–S and C–F when $K = 0{\cdot}0$. However, in the presence of an elastic foundation i.e., $K > 0$, the frequency parameter Ω is found to decrease with the increasing values of α for C–C and C–S plates but in the case of C–F boundary the behaviour is different. In this case for $a/b = 0{\cdot}5$ the frequency parameter is found to increase with increasing value of α while for $a/b = 1{\cdot}0$ it first decreases and then increases. The frequency parameter Ω is found to increase with the increasing values of the foundation parameter K, all other parameters being fixed.

When the plate is vibrating in the second mode (Figure 4), the frequency parameter Ω is found to decrease with increasing values of α for all the three boundary conditions. The frequency parameter Ω is found to increase with the increasing value of K (other

TABLE 4

Comparison of frequency parameter Ω for isotropic plates of uniform thickness: $K = 0{\cdot}0$, $\nu = 0{\cdot}3$

	Boundary conditions			
	C–C	C–S	C–F	C–F
Mode	$a/b = 0{\cdot}5$			$a/b = 1{\cdot}0$
I	23·830	17·339	5·702	12·678
	23·816*	17·332*	5·704*	12·687*
				12·68**
				12·83†
II	63·709	52·231	24·953	33·060
	63·535*	52·098*	24·944*	33·065*
III	123·766	106·857	64·578	72·541
	122·929*	106·479*	64·402*	72·398*

* Values taken from [23].
** Values calculated by finite element method [24].
† Values calculated by optimized Kantorovich method[24].

TABLE 5

Values of fundamental frequencies $\Omega(=12\rho a^2\omega^2/E_y^* h_0^2)$ for orthotropic plates of uniform thickness: $K = 0{\cdot}0$, $a/b = 2{\cdot}0$ ($D_x/D_y = 3{\cdot}117$, $D_1/D_y = \nu_x = 0{\cdot}120$, $H/D_y = 0{\cdot}648$)

Boundary conditions	Present study	Reference [25]	
		Exact	Rayleigh method
SFSF: S|(F, F)|S	17·391	17·39	17·42
SFSS: S|(S, F)|S	20·652	20·65	20·70
SFSC: S|(C, F)|S	26·057	26·06	26·22
SSSS: S|(S, S)|S	48·653	48·65	48·65
SCSS: S|(S, C)|S	68·517	68·52	68·53
SCSC: S|(C, C)|S	94·556	94·56	94·57

Symbols: C, clamped; S, simply supported; F, free.

TABLE 6

Values of frequency parameter $\Omega(=12\rho a^2\omega^2/E_y^* h_0^2)$ *for a uniform orthotropic SFSC plate (5-ply maple plywood)*: $K = 0{\cdot}0$, $a/b = 2{\cdot}0$

p	Mode	Present study	Reference [25] Exact	Reference [25] Rayleigh method
1	1	26·057	26·06	26·22
	2	97·671	97·68	97·70
	3	254·675	254·68	254·65
	4	490·986	490·98	491·00
3	1	161·722	161·72	162·67
	2	212·016	212·04	213·67
5	1	439·736	439·74	441·14

parameters being fixed). However, the rate of increase of Ω is less when compared to that for the fundamental mode.

As far as the behaviour of the plate vibrating in the third mode (Figure 5) is concerned, it is the same as for the second mode with the difference that the rate of decrease of frequency parameter Ω with taper constant α is much higher when compared to that for the first two modes.

For a square plate ($a/b = 1{\cdot}0$), the frequency parameter Ω for $K \geqslant 0$ is found to decrease with the increase in α for all the three plates in all the modes considered here, except for C–F plate, vibrating in its fundamental mode and $K > 0$. In this case the frequencies first decrease and then increase with increasing values of α.

The results (Tables 1–3, Figures 3–5) show that presence of an elastic foundation increases the frequency parameter in all the cases. The increase becomes more and more pronounced as α increases i.e., as the plate gets thinner and thinner towards the edge $X = 1$, for all the boundary conditions considered here. Further, this increase becomes less and less as the aspect ratio a/b increases. The same is true with increasing number of modes. This can be attributed to the fact that an increase in a/b and α amounts to an increase in the stiffness of the plate.

Mode shapes have been computed for $\alpha = \pm 0{\cdot}5$, $K = 0{\cdot}01$ and $a/b = 1{\cdot}0$ for all the boundary conditions. Normalized displacements $W_{norm}(=W/W_{max})$ are shown in Figures 6–8, for the first three modes of vibration. From Figures 6 and 7, it is evident that the transverse deflection for $\alpha = 0{\cdot}5$ is less towards the edge $X = 0$ and greater towards the edge $X = 1$ than the corresponding deflection for $\alpha = -0{\cdot}5$ for C–C and C–S plates. However, in the case of a C–F plate (Figure 8), the behaviour is not of this kind. In this case the transverse deflection for $\alpha = -0{\cdot}5$ is greater than the corresponding deflection for $\alpha = 0{\cdot}5$ towards both the edges. The nodal lines are seen to shift towards the edge $X = 0$ as the edge $X = 1$ increases in thickness.

Table 4 shows a comparison of the present results for isotropic plates (obtained by taking $\eta^*/E_x^* = 1$, $E^*/E_x^* = \nu$, $E_y^*/E_x^* = 1$ and $E_y = E_x = E$ in equation (4)) of uniform thickness ($\alpha = 0$) with those of [23] obtained by using Chebyshev polynomials for $K = 0{\cdot}0$, $a/b = 0{\cdot}5$ and $\nu = 0{\cdot}3$. The only available fundamental frequency for a square plate ($a/b = 1{\cdot}0$) and C–F boundary obtained by two methods i.e., by the finite element method and by optimized Kantorovich method [24] has been reported. The results show a very good agreement.

To compare the frequencies for orthotropic plates with those of Hearmon [25], as also given by Leissa [1] in Tables 9.8 and 9.10, p.257–258 for uniform thickness, results have been computed for all the six similar combinations of boundary conditions with aspect ratio $a/b = 2{\cdot}0$ and $K = 0{\cdot}0$ for the same orthotropic material (5-ply maple plywood) and are given in Table 5 for the fundamental mode. For higher modes and different values of $p = 1, 3, 5$ for a uniform orthotropic plate of the same material with $a/b = 2{\cdot}0$, $K = 0{\cdot}0$ with a SFSC boundary condition, the results are reported in Table 6. A remarkable agreement of the present results with those of exact frequencies for uniform orthotropic plates in Tables 5 and 6 show the computational accuracy of the present technique.

6. CONCLUSION

The vibration of orthotropic rectangular plates of linearly varying thickness on an elastic foundation has been studied on the basis of classical plate theory. The numerical results show that the frequency parameter for a C–S plate is always greater than C–F plate but less than C–C plate, keeping all other parameters fixed. The effect of an elastic foundation is found to increase the frequency for all three boundary conditions. The above analysis also shows that a desired frequency can be obtained by a proper choice of plate parameters which will be helpful to design engineers. Further the numerical results for plates with uniform and non-uniform thickness suggested that an accurate natural frequency can be obtained from the characteristic equation for any combination of the boundary conditions at the two edges which speaks of the versatility of this technique.

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