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LETTERS TO THE EDITOR

A STUDY ON NON-LINEAR FREE VIBRATION OF CANTILEVER BEAMS

B. P. PATEL AND M. GANAPATHI
Institute of Armament Technology, Pune-411 025, *India*

AND

M. TOURATIER
Laboratoire de Modelisation et Mecanique des Structures, URA-1776, *CNRS-UPMC, Paris* 6—*ENSAM-ENS, Cachan*, 151 *Bd de l'Hopital*, 75013, *Paris, France*

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1. INTRODUCTION

Non-linear flexural vibration of beams with various boundary conditions such as hinged–hinged, clamped–clamped and clamped–hinged has been extensively studied using analytical approaches and numerical methods like FEM. A detailed review has been brought out by Rosenberg [1] and Satyamoorthy [2]. However, the investigation pertaining to the cantilever case, which is also of practical interest, is treated sparsely in the literature [3–9]. All these works are based on continuum approaches wherein either linear mode shape or truncation in series solution has been introduced for analysing the nonlinear vibration characteristics of the cantilever beams, except references [7–9]. This, in turn, may lead to underestimation of energy terms which may affect the actual prediction of type/degree of non-linearity. Furthermore, there is not much information available in the literature on how the changes in mode shapes at different amplitudes affect the type/degree of non-linearity.

The FEM has been established as a powerful numerical tool that provides solutions to many complicated engineering problems and does not involve an *a priori* assumption of the actual nature of the mode shape. Analysis of large amplitude/non-linear vibration behaviour of cantilever beams by the finite element method appears to be lacking in the literature. Owing to the difficulty associated with obtaining accurate analytical solution to the cantilever problems due to the existence of non-linearity in both inertia and curvature terms, modelling and analysis of such a problem by the FEM become important. Hence, the study reported here has been carried out using the finite element approach.

2. FINITE ELEMENT FORMULATION

For an inextensional uniform beam of length L, moment of inertia I and Young's modulus E, the longitudinal and transverse displacement u, w are related through the expression:

$$u = \int_0^x \left[\left\{ 1 - \left(\frac{\mathrm{d}w}{\mathrm{d}x} \right)^2 \right\}^{1/2} - 1 \right] \mathrm{d}x \tag{1}$$

where x is measured along the deformed axis of the beam.

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The strain energy, neglecting shear deformation, for ith element due to bending of the beam is written as

$$V = EI/2 \int_0^l \frac{1}{R^2}\,\mathrm{d}\,s \tag{2}$$

where R is the radius of curvature of the neutral axis of the bent beam and s is measured along the deformed local axis of the element.

Using the curvature–transverse displacement relationship, equation (2) is rewritten as

$$V = EI/2 \int_0^l \left(\frac{\mathrm{d}^2 w}{\mathrm{d}s^2}\right)^2 \left[1 - \left(\frac{\mathrm{d}w}{\mathrm{d}s}\right)^2\right]^{-1} \mathrm{d}s \tag{3}$$

The kinetic energy for ith element is written as

$$T = \frac{\rho A}{2} \int_0^l \left(\dot{w}\right)^2 \mathrm{d}s + \frac{\rho A}{2} \int_0^l \left(\sum_{j=1}^{i-1} \int_0^l \left\{1 - \left(\frac{\mathrm{d}w}{\mathrm{d}s}\right)^2\right\}^{-1/2} \frac{\mathrm{d}w}{\mathrm{d}s}\frac{\mathrm{d}\dot{w}}{\mathrm{d}s}\,\mathrm{d}s\right.$$

$$\left. + \int_0^s \left\{1 - \left(\frac{\mathrm{d}w}{\mathrm{d}s}\right)^2\right\}^{-1/2} \frac{\mathrm{d}w}{\mathrm{d}s}\frac{\mathrm{d}\dot{w}}{\mathrm{d}s}\,\mathrm{d}s\right)^2 \mathrm{d}s \tag{4}$$

The dot over the variable represents the derivative with respect to time. l, ρ and A are element length, denisty of material and cross-sectional area of the beam. Taking variation and assuming cubic transverse displacement distribution over the element, the element stiffness matrices ($\boldsymbol{K}_L$ linear, $\boldsymbol{K}_{NL}$ non-linear), mass matrices ($\boldsymbol{M}_L$ linear, $\boldsymbol{M}_{NL}$ non-linear) and non-linear damping matrix ($\boldsymbol{C}_{NL}$) can be evaluated. After the usual assembly procedure, the governing equation of equilibrium is given as

$$[[\boldsymbol{M}_L] + [\boldsymbol{M}_{NL}]]\{\ddot{\boldsymbol{\delta}}\} + [\boldsymbol{C}_{NL}]\{\dot{\boldsymbol{\delta}}\} + [[\boldsymbol{K}_L] + [\boldsymbol{K}_{NL}]]\{\boldsymbol{\delta}\} = \{\boldsymbol{0}\} \tag{5}$$

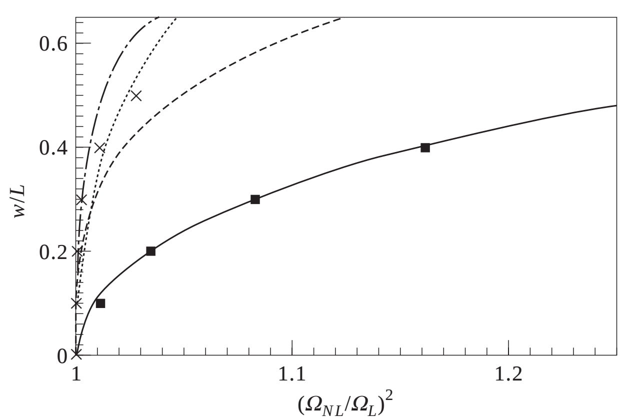

Figure 1. Amplitude-frequency relationship for cantilever beam. Key :—, present (without inplane inertia); . . . . . , references 8 and 9 (with inplane inertia); ----, present (one term in inplane inertia expression); –-–-–, present (10 terms in inplane inertia expression); ×, reference 6 (with inplane inertia); ■, reference 7 (without inplane inertia).

From equation (5), the frequency–amplitude relation for free vibration problem is evaluated using eigenvalue formulation. To solve the non-linear eigenvalue problem, an iterative procedure is employed. The iteration starts from a corresponding initial normalised mode shape obtained from linear analysis, with amplitude scaled up by a desired value. Based on this initial mode shape, the non-linear matrices are formed and then an eigenvalue and its corresponding eigenvector are obtained. This eigenvector is then scaled up again and the iteration continues until the required convergence criteria [10] are achieved, within the specific tolerance limit of less than 0·1%.

3. NUMERICAL RESULTS AND DISCUSSION

Based on progressive mesh refinement, 20 elements idealisation is found to be adequate to model the full beam for the present analysis. The first two non-dimensional frequencies $\Omega_L(=\omega^2(\rho AL^4/EI)$, ω is the linear frequency) obtained for linear vibration of the slender cantilever beam using the present formulation are 3·5160 and 22·0345, which are in excellent agreement with existing literature.

The solutions for the problem associated with the slender beam considering non-linear inplane inertia and bending effects are evaluated, and they are shown as the amplitude-frequency relationship in Figure 1 along with the available results in the literature. In Figure 1 the present results pertaining to with and without non-linear inplane inertia (due to inextensibility assumption of beam) are compared with the available results based on analytical methods coupled with numerical techniques given in references [6–9]. The present results obtained by neglecting the inplane inertia effect match very well with those of reference [7]. Further, the effect of inplane inertia on frequency considering one term as well as more term approximation in the inplane inertia expression is also brought out in Figure 1 and compared with the available results [6, 8, 9]. The analysis in reference [6] is based on the linear vibration mode whereas updated/actual mode shapes corresponding to different amplitudes are used in references [7–9] for non-linear vibration study. It is seen from Figure 1 that, for low vibration amplitudes, the present results with one term approximation in the non-linear inertia expression agree very well with those of the available solutions [6, 8, 9]. For large amplitudes, the accuracy in mode shape is important, as emphasized in references [8, 9]. For the amplitude ratio considered in the present analysis, converged results (mode shapes and non-linear frequencies) are obtained by retaining 10 terms in the non-linear inertia expression. It can be observed from Figure 1 that the present results considering the inplane inertia effect are in close agreement with references [8, 9].

The mode shapes evaluated at different amplitudes are presented in Table 1. It can be noted from Table 1 that the mode shape obtained pertaining to axial direction at lower amplitudes is negligible in comparison with the transverse mode shape whereas it is significant at moderate amplitudes. Hence, the effect of inplane non-linear inertia, in addition to bending non-linearity, is important in determining the type or strength/degree of non-linearity of the cantilever beam. Also, a comparison of the present mode shapes with reference [8] is made and they are in very good agreement.

It can be concluded, in general, that the effect of inplane inertia is to reduce the degree of non-linearity significantly with an increase in amplitudes of vibration. However, the inclusion/exclusion of inplane non-linear inertia effect does not change the type of non-linearity as reported in the case of immovable type of supports. This study is useful for carrying out similar work considering the relaxation of inextensibility assumption, and the inclusion of shear deformation and rotary inertia for studying problems like the dynamic response of flexible multibody systems.

TABLE 1

The mode shapes at different amplitudes

	Low amplitudes (present)				Moderate amplitudes (present)					
							u/L		w/L	
x/L	u/L	w/L	u/L	w/L	u/L	w/L	Present	Ref. 8	Present	Ref. 8
0·0000	0·0000E + 00	0·0000E + 00	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·2000	−1·4025E − 16	6·3871E − 09	−0·0001	0·0064	−0·0012	0·0191	−0·0033	−0·0012	0·0316	0·0319
0·4000	−8·4735E − 16	2·2988E − 08	−0·0008	0·0230	−0·0075	0·0683	−0·0206	−0·0143	0·1127	0·1133
0·6000	−2·1967E − 15	4·6113E − 08	−0·0022	0·0460	−0·0194	0·1362	−0·0535	−0·0524	0·2232	0·2239
0·8000	−3·9485E − 15	7·2547E − 08	−0·0039	0·0724	−0·0349	0·2134	−0·0962	−0·0917	0·3477	0·3456
1·0000	−5·8398E − 15	9·9999E − 08	−0·0058	0·0998	−0·0516	0·2933	−0·1422	−0·1429	0·4763	0·4763

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