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A FURTHER NOTE ON CLOSED-FORM FORMULAS FOR FUNDAMENTAL VIBRATION FREQUENCY OF BEAMS UNDER OFF-CENTRE LOAD

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1. INTRODUCTION

The frequency of the three respective classical beams carrying a mass at *various* positions were obtained by using Rayleigh's quotient [1],

$$\omega^2 = EI\int\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2\mathrm{d}x\Bigg/\left[\int\rho Ay^2\,\mathrm{d}x + My^2|_{x=a}\right], \tag{1}$$

where ρ is the density of the beam without the concentrated mass M. For the shape function y, a trigonometric function and the three static deflection curves involving the mass/load were substituted, one at a time, into equation (1) to obtain the fundamental frequency of the loaded beam [1]. The closed-form frequency expression of each case was then presented in terms of mass ratio $\alpha(=m/M)$ and position parameter $\zeta(=x/l)$.

Four types of deflection curves [1] are again considered here: (1) y_w, involving only the concentrated mass, (2) y_m in terms of only the distributed beam mass, (3) y_c, containing the combined terms, $y_w + y_m$, and (4) an appropriate trigonometric function y_t [1].

In this note, an alternate form for the frequency of each case will be written in terms of M/m (defined as β) as it is the familiar form presented in the literature by virtue of $\beta = 0$ instead of $\alpha = \infty$ for the unloaded case with $M = 0$. In the second part of this note, the results obtained from the closed-form expressions will be compared with those given by solving the transcendental equation [2].

2. CLOSED-FORM EXPRESSIONS FOR FUNDAMENTAL FREQUENCY

Let one now define ω_w, ω_m, ω_c, and ω_t as the associated frequency obtained from equation (1) by using y_w, y_m, y_c, and y_t, respectively. The frequency obtained by using Rayleigh method can be expressed symbolically as

$$\omega^2 = KEI/(ml^3)(C_\beta + C)/(D_\beta + D), \tag{2}$$

in which the parameters C_β and D_β are functions of both the mass ratio ($\beta = M/m$) and the load's position ($\zeta = x/l$), whereas C and D are independent of β.

By virtue of equation (1), the parameters for each respective case are obtained as

(a) for simply supported conditions:

$$\omega_w\text{:}\qquad K = 48;\quad C_\beta = 0,\quad C = 1;\quad D_\beta = 168\beta\zeta^2(\zeta^2 - 2\zeta + 1),$$

$$D = (16/105)(3\zeta^4 - 6\zeta^3 - \zeta^2 + 4\zeta + 2).$$

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$$\omega_m: \quad K = 48; \quad C_\beta = 0, \quad C = 1;$$

$$D_\beta = 10\beta\zeta^2(\zeta^6 - 4\zeta^5 + 4\zeta^4 + 2\zeta^3 - 4\zeta^2 + 1), \quad D = 31/63.$$

$$\omega_c: \quad K = 48; \quad C_\beta = \beta^2\zeta^2(\zeta^2 - 2\zeta + 1) + (\beta\zeta/4)(\zeta^3 - 2\zeta^2 + 1),$$

$$C = 1/40; \quad D = 31/2520,$$

$$D_\beta = 16\beta^3\zeta^4(\zeta^4 - 4\zeta^3 + 6\zeta^2 - 4\zeta + 1) + \beta^2\zeta^2[(156/35)\zeta^6 - (624/35)\zeta^5 + (332/15)\zeta^4 - 4\zeta^3 - (136/15)\zeta^2 + 4\zeta + (32/105)] + \beta\zeta[(9/35)\zeta^7 - (36/35)\zeta^6 + \zeta^5 + (3/5)\zeta^4 - \zeta^3 - (1/5)\zeta^2 + (1/4)\zeta + (17/140)].$$

$$\omega_t: \quad K = \pi^4; \quad C_\beta = 0, \quad C = 1; \quad D_\beta = 2\beta[1 - \cos^2(\pi\zeta)], \quad D = 1.$$

(b) For fixed–fixed conditions:

$$\omega_w: \quad K = 192; \quad C_\beta = 0, \quad C = 1; \quad D = (16/35)\zeta(\zeta^3 - 2\zeta^2 - 2\zeta + 3),$$

$$D_\beta = -64\beta\zeta^3(\zeta^3 - 3\zeta^2 + 3\zeta - 1).$$

$$\omega_m: \quad K = 192; \quad C_\beta = 0, \quad C = 1; \quad D_\beta = (6615/4)\beta\zeta^4(\zeta^4 - 4\zeta^3 + 6\zeta^2 - 4\zeta + 1),$$

$$D = 21/8.$$

$$\omega_c: K = 192; \quad C = 1/8, \quad C_\beta = -30\beta^2\zeta^3(\zeta^3 - 3\zeta^2 + 3\zeta - 1) + (15/2)\beta\zeta^2(\zeta^2 - 2\zeta + 1);$$

$$D_\beta = 1920\beta^3\zeta^6(\zeta^6 - 6\zeta^5 + 15\zeta^4 - 20\zeta^3 + 15\zeta^2 - 6\zeta + 1) + (2/7)\beta[\zeta^2(108\zeta^6 - 432\zeta^5 + 644\zeta^4 - 420\zeta^3 + 105\zeta^2 - 14\zeta + 9) + \beta\zeta^4(-1728\zeta^6 + 8640\zeta^5 - 17136\zeta^4 + 16704\zeta^3 - 7776\zeta^2 + 1152\zeta + 144)],$$

$$D = 1/21.$$

$$\omega_t: \quad K = \pi^4; \quad C_\beta = 0, \quad C = 1; \quad D_\beta = (\beta/8)[\cos^2(2\pi\zeta) - 2\cos(2\pi\zeta) + 1],$$

$$D = 3/16.$$

(c) For clamped–free conditions:

$$\omega_w: \quad K = 3; \quad C_\beta = 0, \quad C = 1; \quad D_\beta = \beta\zeta^3,$$

$$D = -\zeta[(1/70)\zeta^3 - (1/4)\zeta^2 + (3/4)\zeta - (3/4)].$$

$$\omega_m: \quad K = 3; \quad C_\beta = 0, \quad C = 1; \quad D = 13/54,$$

$$D_\beta = \beta\zeta^4[(5/48)\zeta^4 - (5/6)\zeta^3 + (35/12)\zeta^2 - 5\zeta + (15/4)].$$

$$\omega_c: \quad K = 3; \quad C_\beta = (\beta\zeta^2/4)(4\beta\zeta + \zeta^2 - 4\zeta + 6), \quad C = 3/20;$$

$$D_\beta = \beta^3\zeta^6 + (3/4)\beta^2\zeta^4[(11/35)\zeta^3 - \zeta^2 + \zeta + 1] + \beta\zeta^2[(9/560)\zeta^6 - (9/70)\zeta^5 + (9/20)\zeta^4 - (3/4)\zeta^3 + (9/16)\zeta^2 - (3/20)\zeta + (13/40)],$$

$$D = 13/360.$$

$$\omega_t: \quad K = \pi^5; \quad C_\beta = 0, \quad C = 1; \quad D = 16(3\pi - 8),$$

$$D_\beta = 32\pi\beta[\cos^2(\pi\zeta/2) - 2\cos(\pi\zeta/2) + 1].$$

It is worth mentioning that the parameters C and D are in general more concise than those given in terms of $\alpha(=m/M)$ [1].

TABLE 1

Fundamental frequency of loaded beams for different mass ratios and locations

β	ζ	$\bar{\omega}$ [2]	$\bar{\omega}_w$	$\bar{\omega}_m$	$\bar{\omega}_c$	$\bar{\omega}_t$
1	0·0, 1·0	9·8695	12·5499	9·8767	9·8767	9·8696
	1·0, 0·9	8·9962	9·86805	9·0328	9·0432	9·0437
	0·2, 0·8	7·4541	7·63423	7·5749	7·4575	7·5898
	0·3, 0·7	6·3946	6·43678	6·4953	6·3958	6·4951
	0·4, 0·6	5·8468	5·85753	5·9026	5·8482	5·8887
	0·5	5·6795	5·68399	5·7170	5·6809	5·6982
10	0·0, 1·0	9·8695	12·5499	9·8767	9·8767	9·8696
	0·1, 0·9	5·3322	5·37843	5·7448	5·3409	5·7858
	0·2, 0·8	3·2598	3·26237	3·4918	3·2599	3·5093
	0·3, 0·7	2·5279	2·52832	2·6283	2·5279	2·6293
	0·4, 0·6	2·2252	2·22527	2·2659	2·2252	2·2589
	0·5	2·1395	2·13955	2·1632	2·1395	2·1537

3. COMPARISON WITH THOSE OF TRANSCENDENTAL FUNCTIONS

In reference [2], transcendental equations were solved to obtain the frequencies for five different sets of boundary conditions. To compare the models developed here with those obtained from the transcendental functions, the equation for the simply supported beam given in reference [2] is reproduced to

$$2 \tanh \gamma \tan \gamma + \beta\gamma[\tanh \gamma \sin \gamma\zeta \,(\sin \gamma\zeta - \tan \gamma \cos \gamma\zeta) + \tan \gamma \sinh \gamma\zeta \,(\tanh \gamma \cosh \gamma\zeta - \sinh \gamma\zeta)] = 0. \tag{3}$$

Rewriting equation (2) as $\omega^2 = \bar{\omega}^2 EI/(ml^3)$, where $\bar{\omega} = \gamma^2$, Table 1 compares the fundamental frequencies obtained from different models. Several points are worth noting from Table 1.

(1) The frequencies obtained from the shape function y_c, which involves both the distributed beam mass $m(=wl/g)$ and the concentrated mass $M(=W/g)$, are closer to those evaluated by the transcendental equation (3) [2].

(2) As expected, the frequencies are unchanged with respect to different weights at the ends ($\zeta = 0$ and 1) due to the zero displacement.

(3) All models give similar results for cases with the weight placed near the beam's centre. The same finding is concluded in reference [3].

(4) In general, the frequencies obtained by using y_m and y_t are higher than others.

(5) As stated in reference [4], the model generated by using y_w (weight only) must not be used if the weight is placed near the beam's ends as the frequencies obtained are quite high and inaccurate.

It is seen that the model, by using $y_c(=y_w + y_m)$, can be used to *quickly* obtain the fundamental frequency of loaded beams, owing to its simple algebraic expressions. Nevertheless, the transcendental expression (3) is particularly useful for cases of higher mode frequencies.

4. CONCLUDING REMARKS

Alternate expressions for the fundamental frequency of beams carrying a mass at various positions have been written in terms of mass ratio $\beta(=M/m)$. Beams with end conditions of simply supported, fixed–fixed and clamped–free are considered. For the shape functions, a trigonometric function and the three deflection curves involving the mass/load are used. Although the function y_w is commonly used, it is suggested that $y_c(=y_w + y_m)$ be used for every case unless the beam's mass is negligible. Moreover, the model via y_c is highly recommended owing to the fast numerical solving of its simple algebraic functions when compared to that with the transcendental equations. One can thus quickly and accurately predict the fundamental frequency of the off-centre loaded beams by substituting the corresponding mass ratio and position parameter into the respective closed-form expression of ω_c.

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