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LETTERS TO THE EDITOR

AN ASYMPTOTIC EDGE EFFECT IN THIN RECTANGULAR VIBRATING PLATES

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1. Introduction

It is well known [1], that at a free side, e.g., $x = 0$, of a rectangular plate $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, $-h/2 \leqslant z \leqslant h/2$, the bending moment M_{xc}, the twisting moment $H_c (H_c = -M_{xy}$ in the notation of reference [1]) and the shear force Q_{xc} of a classic plate theory solution cannot satisfy the three Poisson conditions, $M_{xc} = 0$, $H_c = 0$ and $Q_{xc} = 0$, but instead of them the two Kirchhoff conditions

$$M_{xc} = 0, \qquad V_x = Q_{xc} + \partial H_c/\partial y = 0, \tag{1a, b}$$

where V_x is the Kirchhoff shear force, are used at the edge. The condition (1b) was originally obtained from the variational equation; later it was confirmed by the Kelvin-Tait transformation [2] and, recently, on the basis of the reciprocity theorem [3]. The transformation of H_c and Q_{xc} into V_x is formally accompanied by the appearance of shearing forces of magnitudes of H_c concentrated at the corner points $x = 0$, $y = 0$ and $x = 0$, $y = b$. A recent discussion of the classic theory of thin plates, reviewed in reference [4], showed the lack of a generally accepted opinion as to whether the concentrated shear forces are real or imaginary.

To clarify this question, an asymptotic analysis of the exact shear edge effect solution near a free side of a rectangular isotropic linear elastic plate is carried out in what follows. One thus obtains an explicit asymptotic distribution of the shear stresses in the edge effect zone and, in particular, a rapid increase of the distributed shear forces acting on the areas perpendicular to the free side; when an adjacent side is simply supported these show themselves as the real shear forces concentrated at the corner points.

2. An asymptotic edge effect solution near the free side

The displacements u, v and w in the X-, Y- and Z-directions and the stresses of the exact shear edge effect solution for the plate are presented in the forms [5]

$$Gu = \sum_{m=0}^{\infty} Z_m \partial\psi_m/\partial y, \qquad Gv = -\sum_{m=0}^{\infty} Z_m \partial\psi_m/\partial x, \qquad w = 0,$$

$$\sigma_x = -\sigma_y = 2\sum_{m=0}^{\infty} Z_m \partial^2\psi_m/\partial x\partial y, \qquad \sigma_z = 0,$$

$$\tau_{xy} = \sum_{m=0}^{\infty} Z_m (\partial^2\psi_m/\partial y^2 - \partial^2\psi_m/\partial x^2), \qquad \tau_{xz} = \sum_{m=0}^{\infty} Z'_m \partial\psi_m/\partial y,$$

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$$\tau_{yz} = -\sum_{m=0}^{\infty} Z'_m \partial\psi_m/\partial x, \tag{2}$$

where G is the shear modulus, $(\,)' = \partial(\,)/\partial z$,

$$Z_m = \sin(\alpha_m z), \qquad \alpha_m = \pi(1+2m)/h, \qquad m = 0, 1, 2, \ldots,$$

and the functions $\psi_m(x, y)$ are solutions of the equations

$$\Delta\psi_m - \alpha_m^2\psi_m = 0, \tag{3}$$

where Δ is the Laplace operator.

It is assumed that the edge effect near the free side $x = 0$ holds only in the X-direction; then equation (3) may be written in the following asymptotic form:

$$\partial^2\psi_m/\partial x^2 - \alpha_m^2\psi_m = 0. \tag{4}$$

Under this condition the above edge effect solution (2) allows one to satisfy not only the three boundary conditions at the free side,

$$M_{xc} + M_x = 0, \qquad H_c + H = 0, \qquad Q_{xc} + Q_x = 0, \tag{5a–c}$$

where M_x, H and Q_x are the resultants of the stresses (2), but also makes asymptotically possible the following, more general than (5b), equation,

$$\tau_{xyc} + \tau_{xy} = 0, \tag{6}$$

where, in accordance with the linear distribution of τ_{xyc} over z and accounting for equation (4), one has

$$\tau_{xyc} = H_c z/I, \qquad \tau_{xy} = -\sum_{m=0}^{\infty} Z_m \partial^2\psi_m/\partial x^2 = -\sum_{m=0}^{\infty} \alpha_m^2 Z_m \psi_m, \tag{7}$$

and $I = h^3/12$. Really, for the asymptotic orders of the displacements and of the stresses of the edge effect solution, and of the classic plate theory solution, one has

$$u \sim \varepsilon^3, \qquad v \sim \varepsilon^2, \qquad \sigma_x = -\sigma_y \sim \varepsilon^2, \qquad \tau_{xy} \sim \varepsilon, \qquad \tau_{xz} \sim \varepsilon^2, \qquad \tau_{yz} \sim \varepsilon, \tag{8}$$

$$w_c \sim 1, \quad u_c \sim \varepsilon, \quad v_c \sim \varepsilon, \quad \sigma_{xc} \sim \varepsilon, \quad \sigma_{yc} \sim \varepsilon, \quad \tau_{xyc} \sim \varepsilon, \quad \tau_{xzc} \sim \varepsilon^2, \quad \tau_{yz} \sim \varepsilon^2,$$

where $\varepsilon = h/\max\{a, b\}$ and so one can conclude that the displacements u and v and the stresses σ_x and σ_y are small in comparison with u_c, v_c, σ_{xc} and σ_{yc}, respectively; therefore the condition (5a) is asymptotically fulfilled independently of the edge effect solution. In consequences of equations (1b), (2) and (7) and the equalities

$$\langle Z'_m \rangle = \alpha_m^2 \langle Z_m z \rangle,$$

$$Q_x = \langle \tau_{xz} \rangle = \sum_{m=0}^{\infty} \langle Z'_m \rangle \partial\psi_m/\partial y = \sum_{m=0}^{\infty} \alpha_m^2 \langle Z_m z \rangle \partial\psi_m/\partial y = -\partial\langle \tau_{xy} z \rangle/\partial y = -\partial H/\partial y,$$

where $\langle\;\rangle$ is an integral over z from $-h/2$ to $h/2$, the conditions (5b) and (5c) are also satisfied:

$$H_c + H = \langle(\tau_{xyc} + \tau_{xy})z\rangle = 0, \qquad Q_{xc} + Q_x = -\partial(H_c + H)/\partial y = 0.$$

Equation (6) allows one to formulate the boundary conditions at the free side $x = 0$ for the solutions ψ_m of equations (4) as

$$\psi_m(0, y) = 2H_c(0, y)\langle Z_m z\rangle/Ih\alpha_m^2 = 48H_0(-1)^m(h\alpha_m)^{-4},$$

where $H_0(y) = H_c(0, y)$, and to write the asymptotic solution of equation (4), the shear stresses and the shear forces in the following final forms:

$$\psi_m(x, y) = 48H_0(-1)^m(h\alpha_m)^{-4}\exp(-\alpha_m x),$$

$$\tau_{xy} = -48H_0h^{-4}\sum_{m=0}^{\infty}(-1)^m\alpha_m^{-2}\exp(-\alpha_m x)\sin\alpha_m z,$$

$$\tau_{xz} = 48(\partial H_0/\partial y)h^{-4}\sum_{m=0}^{\infty}(-1)^m\alpha_m^{-3}\exp(-\alpha_m x)\cos\alpha_m z,$$

$$\tau_{yz} = 48H_0h^{-4}\sum_{m=0}^{\infty}(-1)^m\alpha_m^{-2}\exp(-\alpha_m x)\cos\alpha_m z,$$

$$Q_x = \langle\tau_{xz}\rangle = 96(\partial H_0/\partial y)h^{-4}\sum_{m=0}^{\infty}\alpha_m^{-4}\exp(-\alpha_m x),$$

$$Q_y = \langle\tau_{yz}\rangle = 96H_0h^{-4}\sum_{m=0}^{\infty}\alpha_m^{-3}\exp(-\alpha_m x).$$

Note that in accordance with equations (2) and the asymptotic evaluations (8), the right-hand parts in the free vibration equations,

$$\partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y + \partial\tau_{xz}/\partial z = \rho\ddot{u}, \qquad \partial\tau_{xy}/\partial x + \partial\sigma_y/\partial y + \partial\tau_{yz}/\partial z = \rho\ddot{v},$$
$$\partial\tau_{xz}/\partial x + \partial\tau_{yz}/\partial y + \partial\sigma_z/\partial z = \rho\ddot{w},$$

where ρ is the density and $(\dot{\ }) = \partial(\ )/\partial t$, are usually negligible; therefore the above elastostatic edge effect solution in which the time t is a parameter may be also used in the dynamic problems.

3. THE EDGE EFFECT SHEAR STRESSES NEAR THE FREE SIDE AND THE CORNER FORCES

The shear stress distribution near the free side in the X-direction is illustrated in Table 1, in which values of the dimensionless resulting shear stress $\bar{\tau}$ acting upon the plane $y = \text{constant}$,

$$\bar{\tau} = h^2\sqrt{(\tau_{xyc} + \tau_{xy})^2 + \tau_{yz}^2}/6H_0,$$

are presented. One can see that the maximum value $\bar{\tau}_{\max} = 1$ corresponds to the classic theory magnitude $\tau_{xyc} = 6H_0/h^2$, which is realized beyond the edge effect zone. Thus, introduction of the edge effect solution does not lead to an increase of the maximum value of the shear stresses near the free side; this is given by classic plate theory.

Equations (1), (2) and (5)–(7) show that at the side $x = 0$ the edge effect shear stress τ_{xz} is distributed quadratically over z,

$$\partial\tau_{xz}/\partial z = \sum_{m=0}^{\infty} Z''_m \partial\psi_m/\partial y = -\sum_{m=0}^{\infty} \alpha_m^2 Z_m \partial\psi_m/\partial y = \partial\tau_{xy}/\partial y$$

$$= -(z/I)\partial H_0/\partial y = -(z/I)Q_x;$$

the distribution of the shear stress τ_{yz} (see the column $x/h = 0$ in Table 1) is also close to the parabolic in z approximation,

$$\tilde{\tau}_{yz} = 3Q_y[1 - (2z/h)^2]/2h,$$

and for $z = 0$ the difference $\tilde{\tau}_{yz} - \tau_{yz}$ equals 9·7%. These results are in agreement with the quadratic distribution of the shear edge effect stresses adopted in improved plate theories, e.g., in Reissner's plate theory [6].

Using the equality [7]

$$\sum_{m=0}^{\infty} \alpha_m^{-4} = h^4/96,$$

one can see that the integral R of Q_y over x is

$$R(y, t) = \int_0^{\infty} Q_y \mathrm{d}x = H_0(y, t).$$

For the edge effect forces $Q_y \sim \varepsilon^2$, while $R \sim \varepsilon^3$; in the framework of classic plate theory the forces Q_y distributed near the free side $x = 0$ should be regarded as the forces R concentrated at the side. Thus, introduction of the shear edge effect solution permits one to eliminate asymptotically both the shear forces Q_{xc} and the twisting moments H_c remaining in a classic plate theory solution at the free side; however, this off-loading is accompanied by the appearance of the concentrated shear forces R acting upon the areas perpendicular to the side.

TABLE 1

The dimensionless resulting shear stress $\bar{\tau} = h^2\sqrt{(\tau_{xyc} + \tau_{xy})^2 + \tau_{yz}^2}/6H_0$ *in the edge effect zone near the free side* $x = 0$

	x/h					
z/h	0·0	0·4	0·8	1·2	1·6	2·0
0·0	0·742	0·229	0·066	0·019	0·005	0·002
0·05	0·737	0·235	0·111	0·099	0·099	0·100
0·1	0·722	0·254	0·190	0·195	0·198	0·200
0·15	0·697	0·285	0·276	0·292	0·298	0·299
0·2	0·660	0·326	0·365	0·389	0·397	0·399
0·25	0·610	0·376	0·456	0·487	0·496	0·499
0·3	0·547	0·436	0·548	0·585	0·596	0·599
0·35	0·466	0·506	0·642	0·683	0·695	0·699
0·4	0·363	0·584	0·738	0·782	0·795	0·799
0·45	0·226	0·671	0·835	0·882	0·895	0·899
0·5	0·000	0·767	0·934	0·981	0·995	0·998

If an adjacent side, e.g., $y = 0$, is simply supported, then the distributed edge effect forces Q_y appearing at the supported side may be also regarded as the concentrated force $R(0, t)$ applied at the corner point $x = 0$, $y = 0$. However, if the adjacent side is clamped, then at the corner point the twisting moment of classic plate theory and, consequently, the corresponding corner force, equals zero. If the adjacent side is free, then also $H_c = 0$ at the corner point and therefore the edge effects near the side $x = 0$ and near the side $y = 0$ may be regarded independently, and the corner force equals zero too. Finally, if each of the adjacent sides either is simply supported or is clamped, then the edge effects near them do not arise; consequently, in these cases the concentrated force at the corner point is imaginary.

4. CONCLUSIONS

1. The exact elasticity shear edge effect solution near the free side of thin plates appropriate both in static and in free vibration problems has been presented in an explicit asymptotic form.

2. The edge effect solution does not change the maximum value of the shear stresses near the free side of a thin plate; this magnitude is obtained in the scope of classic plate theory.

3. The asymptotic analysis confirms the parabolic in z distribution of the edge effect shear stresses τ_{xz} and τ_{yz}.

4. The asymptotic analysis reveals the way of formation of the concentrated corner force and predicts that in thin plates the concentrated shear force appears only at the corner point where a free side is adjacent to a simply supported one.

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