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LETTERS TO THE EDITOR

VIBRATIONS OF A BEAM OF NON-UNIFORM CROSS-SECTION TRAVERSED BY A TIME VARYING CONCENTRATED FORCE

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1. INTRODUCTION

The analysis of the dynamic behavior of structural elements traversed by moving forces or masses is certainly a classical problem which has attracted the attention of many researchers. The reader is referred to comprehensive listing of references available in well known textbooks and papers [1–8]. In general the investigations have been motivated by the necessity of evaluating the dynamic behavior of a bridge travelled by a car or of rails travelled by a train.

The present paper, considerable more modest in its scope, deals with the approximate determination of the transverse response of the structural system depicted in Figure 1 as a load

$$P(t) = P_0\,\mathrm{e}^{-\alpha t} \tag{1}$$

which travels at constant speed, v, along the beam. The beam, of constant width b, is characterized by a parabolically varying thickness given by

$$h(\bar{x}) = h_m f(\bar{x}) = h_m[4(\gamma - 1)(\bar{x}^2/L^2 - \bar{x}/L) + \gamma], \tag{2}$$

where

$$\gamma = h(0)/h_m.$$

Consequently, the cross-sectional area of the beam is

$$A(\bar{x}) = bh_m f(\bar{x}) = A_m f(\bar{x}), \qquad A_m = bh_m, \tag{3a}$$

and the moment of inertia is given by

$$I(\bar{x}) = bh_m^3 f^3(\bar{x})/12 = I_m f^3(\bar{x}), \qquad I_m = bh_m^3/12. \tag{3b}$$

Three arrangements of boundary conditions will be considered: simply supported at both ends, clamped, simply supported and clamped at both ends.

Neglecting inertia effects of the load itself the governing differential system is

$$E\frac{\partial^2}{\partial\bar{x}^2}(I(\bar{x})w_{\bar{x}^2}) + \rho A(\bar{x})w_{t^2} = \delta(\bar{x} - vt)P_0\,\mathrm{e}^{-\alpha t}, \qquad \alpha > 0, \tag{4}$$

$$w(\bar{x}, 0) = w_t(\bar{x}, 0) = 0, \tag{5}$$

where shear and rotatory inertia effects have been disregarded since a first order approximation is being sought.

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The following two sections deal (1) with an exact solution of equations (4) and (5) in the case where the beam possesses a uniform thickness and (2) an approximate solution of the problem when $h(\bar{x})$ is given by equation (2).

2. EXACT ANALYTICAL SOLUTION

Making $\gamma = 1$ in equation (2) and substituting equations (2) and (3) in equation (4) results in

$$EI_m w_{\bar{x}^4} + \rho A_m w_{t^2} = \delta(\bar{x} - vt) P_0 \,\mathrm{e}^{-\alpha t}. \tag{6}$$

Let T be the time interval needed by the load $P(t)$ to traverse the beam. Accordingly,

$$\delta(\bar{x} - vt) = \delta(Lx - Lt/T) = \delta(L(x - t/T)) = \delta(x - t/T)/L. \tag{7}$$

Substituting equation (7) in equation (6) and introducing the dimensionless variable $x = \bar{x}/L$ one obtains

$$w_{x^4} + \beta_1 w_{t^2} = \beta_2 P_0 \delta(x - t/T)\,\mathrm{e}^{-\alpha t}, \tag{8}$$

where

$$\beta_1 = \rho A_m L^4/(EI_m), \qquad \beta_2 = L^3/(EI_m).$$

It is convenient to introduce the dimensionless displacement variable $u(x, t)$, where

$$u(x, t) = w(x, t)/w_e \tag{9}$$

and where w_e is the maximum displacement introduced by a static load P_0 acting at $\bar{x} = L/2$.
Accordingly

$$w_e = P_0 L^3/(\zeta EI_m). \tag{10}$$

Obviously the maximum static displacement occurs at $\bar{x} = L/2$ for the first and third type of boundary conditions and at $\bar{x} = L(1 - 1/\sqrt{5})$ when the beam is clamped–simply supported.

The values of ζ are $\zeta = 48$ (simply-supported case), $\zeta = 48\sqrt{5}$ (clamped–simply supported) and $\zeta = 192$ (clamped at both ends).

Substituting equation (9) in equation (8) results in

$$u_{x^4} + \beta_1 u_{t^2} = \zeta\delta(x - t/T)\,\mathrm{e}^{-\alpha t}. \tag{11}$$

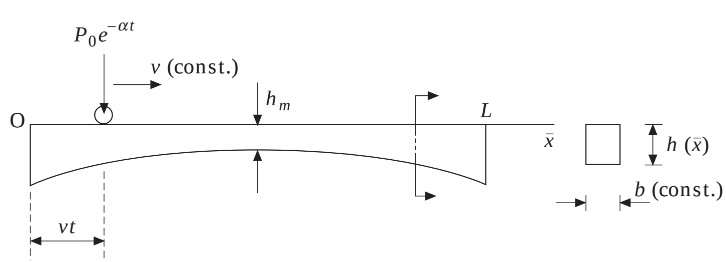

Figure 1. Vibrating system under study.

Making

$$u(x, t) = \sum_{n=1}^{\infty} \varphi_n(x)\psi_n(t), \tag{12}$$

where the $\varphi_n(x)$'s are the normal modes of the structure, and substituting equation (12) in equation (11) results in

$$\psi_n(t) + \omega_n^2\psi_n(t) = k_n\varphi_n(t/T)\,\mathrm{e}^{-\alpha t} \tag{13}$$

once usual normal mode operating procedures are performed and where the ω_n's are the natural circular frequencies of the system and

$$k_n = \zeta\bigg/\left(\beta_1 \int_0^1 \varphi_n^2(x)\,\mathrm{d}x\right). \tag{14}$$

Making

$$\mu_n^2 = \sqrt{\beta_1}\omega_n,$$

the normalized normal modes are, in the case of simply supported beams,

$$\varphi_n(x) = \sin\mu_n x, \qquad \mu_n = n\pi. \tag{15}$$

In the case of clamped–simply supported and clamped–clamped beams,

$$\varphi_n(x) = \cosh\mu_n x - \cos\mu_n x - c_n(\sinh\mu_n x - \sin\mu_n x), \tag{16}$$

with

$$C_n = (\cosh\mu_n - \cos\mu_n)/(\sinh\mu_n - \sin\mu_n).$$

Then, the particular solution of equation (13) is

$$\psi_{nP}(t) = \mathrm{e}^{-\alpha t}\left(k_1\cosh\frac{\mu_n t}{T} + k_2\cos\frac{\mu_n t}{T} + k_3\sinh\frac{\mu_n t}{T} + k_4\sin\frac{\mu_n t}{T}\right) \tag{17}$$

and the general solution of equation (13) results in

$$\psi_n(t) = C_{1n}\cos\omega_n t + C_{2n}\sin\omega_n t + \psi_{nP}(t), \tag{18}$$

where C_{1n} and C_{2n} are determined using the initial conditions $\psi_n(0) = \dot{\psi}_n(0) = 0$. If the beam is simply supported one has $k_1 = k_3 = 0$; for the other types of boundary conditions the four constants are different from zero.

Finally the solutions to the posed problem is

$$u(x, t) = \sum_{n=1}^{\infty} [C_{1n}\cos\omega_n t + C_{2n}\sin\omega_n t + \psi_{nP}(t)]\varphi_n(x). \tag{19}$$

3. APPROXIMATE SOLUTION: CASE OF A BEAM OF NON-UNIFORM CROSS-SECTION

Substituting equations (3) and (9) in equation (4) and introducing the dimensionless variables $x = \bar{x}/L$ and $\tau = t/T$ one obtains

$$f^3u_{x^4} + 6f^2f'u_{x^3} + 3(ff'^2 + f^2f'')u_{x^2} + \frac{\beta_1}{T^2}fu_{\tau^2} = \zeta\delta(x-\tau)\,\mathrm{e}^{-\alpha T\tau}, \tag{20}$$

subject to the governing boundary and initial conditions.

Making use of the Galerkin–Kantorovich method one expresses

$$u \simeq u_\alpha = \varphi(x)\psi(\tau), \tag{21}$$

where $\varphi(x)$ will be constructed in such a way as to satisfy the beam boundary conditions. Substituting equation (21) in equation (20) one obtains, after application of Galerkin's orthogonalization procedure,

$$J_1\ddot{\psi}(\tau) + J_2\psi(\tau) = \zeta\, \mathrm{e}^{-\alpha T\tau}\varphi(\tau), \tag{22}$$

where

$$J_1 = \frac{\beta_1}{T^2}\int_0^1 f\varphi^2\,\mathrm{d}x, \qquad J_2 = \int_0^1 [f^{\,3}\varphi^{IV} + 6f^{\,2}f'\varphi''' + 3(ff'^2 + f^{\,2}f'')\varphi'']\varphi\,\mathrm{d}x.$$

It is convenient to express $\varphi(x)$ in the form

$$\varphi(x) = x^4 + \alpha_3x^3 + \alpha_2x^2 + \alpha_1x, \tag{23}$$

where the α_i's are obtained by substituting equation (23) in the boundary conditions [9–11]. The particular solution of equation (22) is

$$\psi_P(\tau) = \mathrm{e}^{-\alpha T\tau}(k_4\tau^4 + k_3\tau^3 + k_2\tau^2 + k_1\tau + k_0) \tag{24}$$

and its general solution becomes

$$\psi(\tau) = C_1\cos\nu\tau + C_2\sin\nu\tau + \psi_P(\tau), \qquad \nu = \sqrt{J_2/J_1}, \tag{25}$$

where C_1 and C_2 are determined applying the boundary conditions.

4. RESULTS AND CONCLUSIONS

Numerical experiments have been performed making $\beta_1 = 0{\cdot}02$ and $0{\cdot}35$. With regards to the acting force $P(t) = P_0\,\mathrm{e}^{-\alpha t}$ three situations have been considered: constant value ($\alpha = 0$); $P(T)/P_0 = 0{\cdot}70$, $\alpha = 0{\cdot}118892$ ($T = 3$ s); $P(T)/P_0 = 0{\cdot}30$, $\alpha = 0{\cdot}401324$ ($T = 3$ s). The values of $u(x, t)$ have been plotted as a function of t at the beam center for the simply supported and clamped cases at both ends, and at $x = 1 - 1/\sqrt{5}$ for the clamped–simply supported situation.

Figures 2–5 deal with beams of uniform thickness. Figures 2 and 3 depict values of $u(x, t)$ for the simply supported case for $\beta_1 = 0{\cdot}02$ and $0{\cdot}35$, respectively, while Figures 4 and 5 show results for the clamped–simply supported and clamped–clamped situations, respectively, for $\beta_1 = 0{\cdot}02$. No significant variations were observed for the cases treated in Figures 4 and 5 when β_1 was taken equal to $0{\cdot}35$.

Good agreement is observed between the exact values and those obtained by means of the variational approach (six terms were employed when using the exact, normal mode approach).

Figures 6–9 deal with beams of non-uniform thickness ($\gamma = 1{\cdot}30$). All calculations have been performed for $T = 3$ s; $\beta_1 = 0{\cdot}02$ and $0{\cdot}35$. It is again observed, when performing the numerical determinations, that in the case of clamped–simply supported and clamped–clamped ends practically the same results are obtained for $\beta_1 = 0{\cdot}02$ and $\beta_1 = 0{\cdot}35$.

The proposed approach is quite simple and straightforward. The cases of ends elastically restrained against translation and rotation do not offer any conceptual and/or operational difficulties.

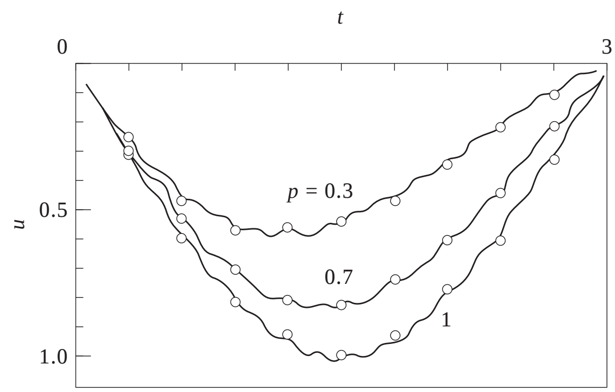

Figure 2. Plot of u (0·5, t) in the case of a simply supported beam ($\beta_1 = 0{\cdot}02$, $\gamma = 1$): ——, exact solution; ○, variational solution; $p = P(T)/P_0$.

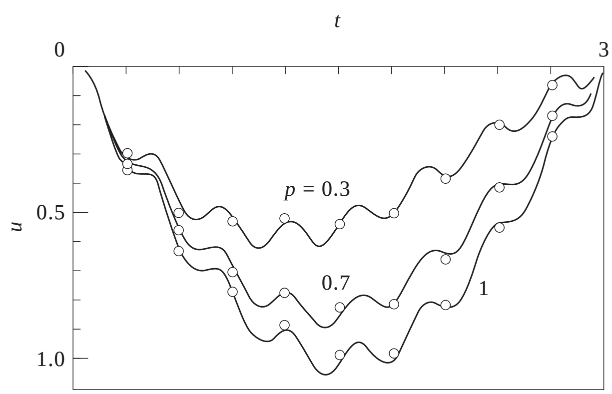

Figure 3. Plot of u (0·5, t) in the case of a simply supported beam ($\beta_1 = 0{\cdot}35$, $\gamma = 1$): ——, exact solution; ○, variational solution.

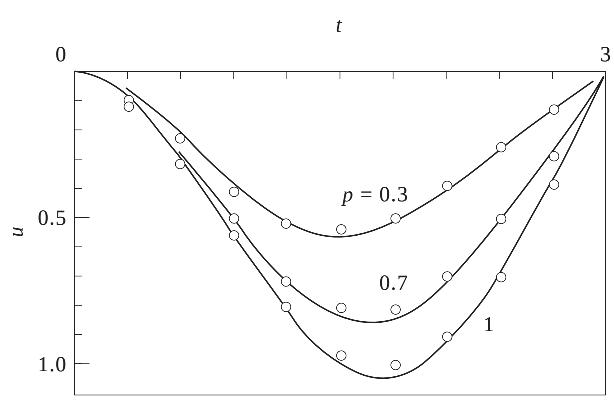

Figure 4. Plot of u $(1 - 1\sqrt{5}, t)$ in the case of a clamped–simply supported beam ($\beta_1 = 0{\cdot}02$, $\gamma = 1$): ——, exact solution; ○, variational approach.

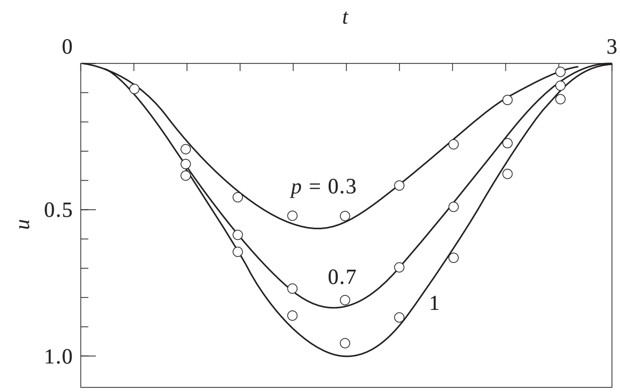

Figure 5. Plot of u (0·5, t) in the case of a clamped–clamped beam ($\beta_1 = 0{\cdot}02$, $\gamma = 1$): ——, exact solution; ○, variational solution.

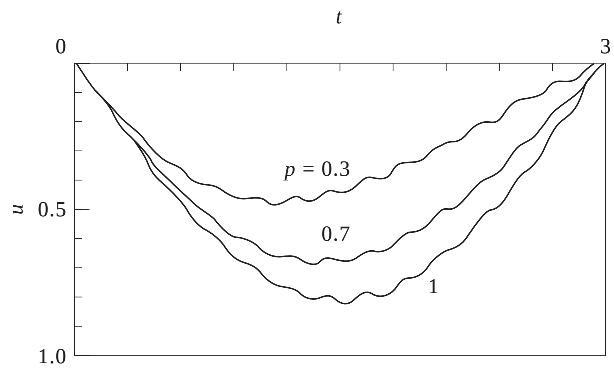

Figure 6. Plot of u (0·5, t) in the case of a simply supported beam of non-uniform thickness ($\beta_1 = 0{\cdot}021$, $\gamma = 1{\cdot}30$).

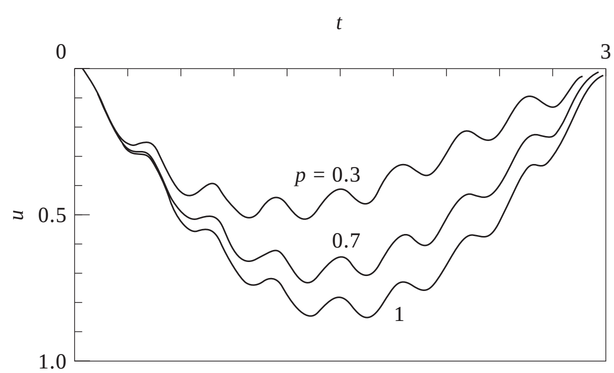

Figure 7. Plot of u (0·5, t) in the case of a simply supported beam of non-uniform thickness ($\beta_1 = 0{\cdot}35$, $\gamma = 1{\cdot}30$).

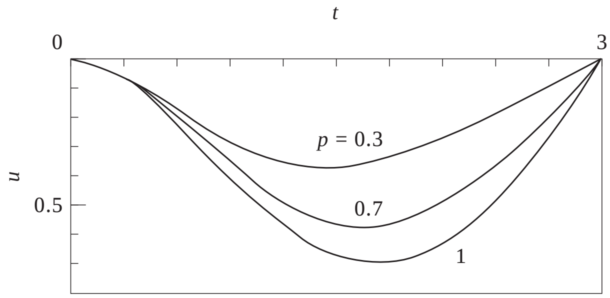

Figure 8. Plot of $u\,(1 - 1/\sqrt{5}, t)$ in the case of a clamped–simply supported beam of non-uniform thickness ($\beta_1 = 0{\cdot}02$, $\gamma = 1{\cdot}30$).

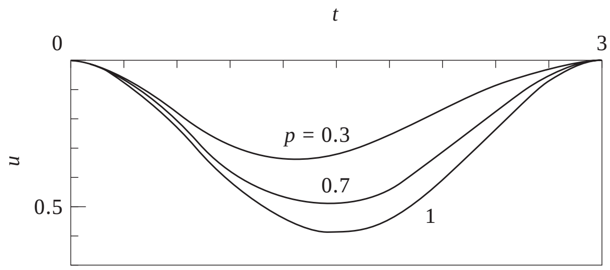

Figure 9. Plot of $u\,(0{\cdot}5, t)$ in the case of a clamped–clamped beam of non-uniform thickness ($\beta_1 = 0{\cdot}02$, $\gamma = 1{\cdot}30$).

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