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GEOMETRIC STIFFNESS AND STABILITY OF RIGID BODY MODES

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The objective of this study is to examine the effect of geometric stiffness forces on the stability of elastic and rigid body modes. A simple rotating beam model is used to demonstrate the effect of axial forces and dynamic coupling between the modes of displacement on the rigid body motion. The effect of longitudinal deformation due to bending is systematically introduced to the dynamic equations using the principle of virtual work. The effect of higher order terms in the inertia forces as the result of including longitudinal displacement caused by bending deformation is examined using several models. One of these models is a linear model in which the effect of longitudinal displacement due to bending is neglected in formulating the inertia forces, but this effect is considered when the elastic forces are formulated. This model shows unstable behavior at high values of the angular velocity of the beam. Three different beam models are then developed in order to examine the effect of geometric stiffness forces. In the first model, called the *consistent complete model* (CCM), the effect of longitudinal displacement caused by bending is included in formulating both the inertia and elastic forces. In the second model, called the *consistent incomplete model* (CIM), the effect of longitudinal displacement due to bending is neglected in formulating both the elastic and inertia forces. In the third model, the *second inconsistent model* (SIM), the effect of longitudinal displacement due to bending is included in formulating the inertia forces, but this effect is neglected when the elastic forces are formulated. Numerical results obtained in this investigation demonstrate that the three models lead to a stable solution at high values of angular velocities. These results also demonstrate that including the effect of longitudinal displacement due to bending in the inertia forces is not the only approach that can be used to maintain the beam stability at high values of angular velocity. The effect of geometric stiffness forces on the stability of rigid body modes of a translating and rotating beam model is also examined in this paper.

1. INTRODUCTION

The effect of the instability of elastic modes on the stability of rigid body modes was recently examined [1]. A three dimensional beam model, in which the coupling between the axial, in-plane, and out-of-plane bending deformation is considered, was developed. It was demonstrated using this model that the neglect of the geometric stiffness forces can lead to unstable solution for the elastic modes, and this in turn causes the instability of the rigid body modes due to the dynamic coupling. The results obtained in this study also demonstrated that rotating and translating beam models, in which the effect of

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geometric stiffness is neglected, can have stable solutions at some ranges of relatively high speeds, and the stability limit depends on the ratio between the bending and axial stiffness coefficients of the beam. While the effect of geometric stiffness on the dynamics of rotating beams has been examined in several investigations [2–7], no attempt has been made to develop a more complete non-linear model that can be used to examine the effect of longitudinal displacement caused by bending deformation on the stability of rigid body modes.

In these investigations, the geometric effect of longitudinal displacement due to bending is examined and its effect on the stability of the rigid body modes of a translating and rotating beam is discussed. The effect of the longitudinal displacement due to bending is systematically introduced to the dynamic equations of the rotating beam using the principle of virtual work. Using this approach, all the non-linear terms that result from introducing the effect of the longitudinal displacement caused by bending deformation can be systematically formulated and their effects can be numerically examined. Using the results of this preliminary numerical study, three different beam models have been developed. In the first model, the effect of the longitudinal displacement due to bending in both the inertia and elastic forces is included. This model, which is called in this investigation the CCM, leads to the definition of the geometric stiffening force which is required to maintain the stability of the rotating beams when the angular velocity increases.

Using the co-ordinate transformation suggested by Mayo *et al*. [5], a simple form for the strain energy is obtained despite the fact that a non-linear strain displacement relationship is used. In the second model, which is referred to as CIM, the effect of longitudinal displacement caused by bending is neglected when both the inertia and elastic forces are formulated. It is demonstrated that this model also leads to a stable solution at high values of the angular velocity of the rotating beam. A third model that leads to a stable solution for the rotating beam has also been developed in this study. In this model, the effect of longitudinal displacement caused by bending is neglected in formulating the elastic forces and this effect is considered in formulating the inertia forces. The effect of longitudinal displacement due to bending on the stability of the rigid body modes of a translating and rotating beam model is also examined in this investigation.

2. AXIAL AND BENDING DEFORMATIONS

In this section, the two-dimensional rotating flexible beam model shown in Figure 1(a) is first considered. The beam is assumed to rotate about the Z-axis with an angular

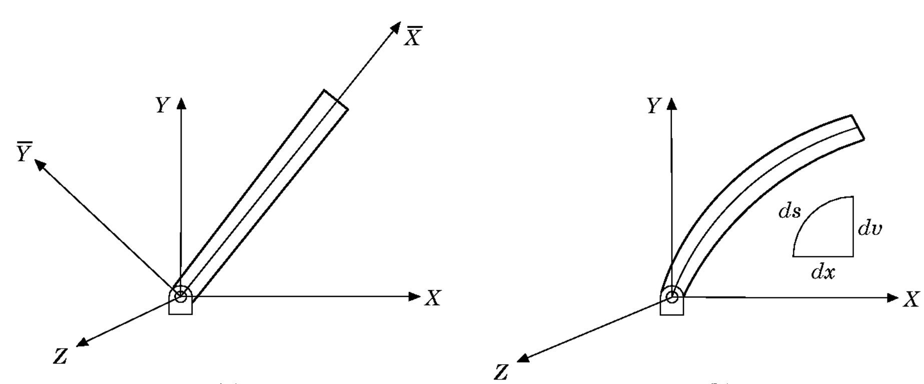

Figure 1. (a) Undeformed rotating beam, (b) deformed rotating beam.

velocity $\dot{\theta}$. The origin of the beam co-ordinate system is assumed to be fixed, and therefore, the beam does not undergo a translational motion. The global position vector of an arbitrary point on the beam can be written as

$$\mathbf{r} = \mathbf{A}\bar{\mathbf{u}}, \tag{1}$$

where $\mathbf{A}$ is the planar rotation matrix defined as

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{2}$$

and $\bar{\mathbf{u}}$ is the local position vector of the arbitrary point on the beam, which is defined as

$$\bar{\mathbf{u}} = \bar{\mathbf{u}}_o + \bar{\mathbf{u}}_f. \tag{3}$$

In this equation, $\bar{\mathbf{u}}_o = [x \;\; 0]^{\mathrm{T}}$ is the position vector of an arbitrary point on the beam in the undeformed configuration, $\bar{\mathbf{u}}_f = [u_t \;\; v]^{\mathrm{T}}$ is the deformation vector, v is the transverse displacement, and u_t is the total longitudinal displacement which is defined as [5]

$$u_t = u + u_r + u_g. \tag{4}$$

The displacement u is the longitudinal deformation due to the axial elastic motion of the points on the center line of the beam, u_g is the longitudinal displacement caused by the transverse deflection of the beam, and $u_r = -y\,\partial v/\partial x$ is the result of the rotation of the cross-section. In the case of a slender beam, the effect of the displacement u_r can be neglected. In order to evaluate the displacement u_g, one observes from Figure 1(b) that

$$ds - dx = \sqrt{(dx)^2 + (dv)^2} - dx = \sqrt{1 + (\partial v/\partial x)^2}dx - dx = \tfrac{1}{2}(\partial v/\partial x)^2 \mathrm{d}x,$$

where ds is an infinitesimal arc length. It follows from the preceding equation that

$$u_g = -\int_0^x (ds - dx) = -\frac{1}{2}\int_0^x \left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x. \tag{5}$$

The axial and bending displacements can be expressed in terms of space dependent shape functions and time dependent co-ordinates as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{S}\mathbf{q}_f = \begin{bmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 \end{bmatrix}\begin{bmatrix} \mathbf{q}_{f1} \\ \mathbf{q}_{f2} \end{bmatrix}, \tag{6}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the rows of the shape function matrix $\mathbf{S}$ associated with the axial and in-plane directions respectively, and $\mathbf{q}_{f1}$ and $\mathbf{q}_{f2}$ are the axial and in-plane transverse components of the time dependent elastic co-ordinate vector $\mathbf{q}_f$.

The position vector $\mathbf{r}$ of equation (1) can be written more explicitly as

$$\mathbf{r} = \mathbf{A}(\bar{\mathbf{u}}_o + \bar{\mathbf{u}}_f). \tag{7}$$

A virtual change of this vector can be written as

$$\delta\mathbf{r} = \mathbf{A}_\theta\bar{\mathbf{u}}\;\delta\theta + \mathbf{A}\;\delta\bar{\mathbf{u}}_f, \tag{8}$$

where $\mathbf{A}_\theta$ is the derivative of $\mathbf{A}$ with respect to θ. The velocity and acceleration vectors can be obtained by differentiating equation (7) as

$$\dot{\mathbf{r}} = \dot{\theta}\mathbf{A}_\theta\bar{\mathbf{u}} + \mathbf{A}\dot{\bar{\mathbf{u}}}_f, \qquad \ddot{\mathbf{r}} = \ddot{\theta}\mathbf{A}_\theta\bar{\mathbf{u}} - \dot{\theta}^2\mathbf{A}\bar{\mathbf{u}} + 2\dot{\theta}\mathbf{A}_\theta\dot{\bar{\mathbf{u}}}_f + \mathbf{A}\ddot{\bar{\mathbf{u}}}_f. \tag{9, 10}$$

3. VIRTUAL WORK AND STRAIN ENERGY

The virtual work of the inertia force can be written as

$$\delta W_i = \int_V \rho \ddot{\mathbf{r}}^{\mathrm{T}}\, \delta \mathbf{r}\, \mathrm{d}V. \tag{11}$$

Substituting equations (8) and (10) into equation (11), one obtains

$$\delta W_i = \delta W_l + \delta W_g, \tag{12}$$

where δW_l is the virtual work of the inertia forces which does not include the effect of u_g, while δW_g is the change in the virtual work of the inertia forces as the result of including the displacement u_g. These two components of the virtual work are defined as

$$\delta W_l = \delta W_{l\theta} + \delta W_{lu} + \delta W_{lv}, \qquad \delta W_g = \delta W_{g\theta} + \delta W_{gu} + \delta W_{gv} + \delta W_{gu_g}, \tag{13, 14}$$

where the components of the virtual work δW_l are defined as

$$\delta W_{l\theta} = \int_V \rho[\ddot{\theta}\{(x+u)^2 + v^2\} + 2\dot{\theta}\{\dot{u}(x+u) + \dot{v}v\} + \ddot{v}(x+u) - \ddot{u}v]\, \delta\theta\, \mathrm{d}V, \tag{15}$$

$$\delta W_{lu} = \int_V \rho[\ddot{u} - 2\dot{\theta}\dot{v} - \dot{\theta}^2(x+u) - \ddot{\theta}v]\, \delta u\, \mathrm{d}V, \tag{16}$$

$$\delta W_{lv} = \int_V \rho[\ddot{v} + 2\dot{\theta}\dot{u} - \dot{\theta}^2 v + \ddot{\theta}(x+u)]\, \delta v\, \mathrm{d}V, \tag{17}$$

and the components of the virtual work that appear in δW_g are

$$\delta W_{g\theta} = \int_V \rho[\ddot{\theta}\{2u_g(x+u) + u_g^2\} + 2\dot{\theta}\{\dot{u}_g(x+u) + \dot{u}u_g + \dot{u}_g u_g\} + \{\ddot{v}u_g - v\ddot{u}_g\}]\, \delta\theta\, \mathrm{d}V, \tag{18}$$

$$\delta W_{gu} = \int_V \rho(\ddot{u}_g - \dot{\theta}^2 u_g)\, \delta u\, \mathrm{d}V, \qquad \delta W_{gv} = \int_V \rho(\ddot{\theta}u_g + 2\dot{\theta}\dot{u}_g)\, \delta v\, \mathrm{d}V, \tag{19, 20}$$

$$\delta W_{gu_g} = \int_V \rho[-\ddot{\theta}v - \dot{\theta}^2\{(x+u) + u_g\} - 2\dot{\theta}\dot{v} + (\ddot{u} + \ddot{u}_g)]\, \delta u_g\, \mathrm{d}V. \tag{21}$$

3.1. STRAIN ENERGY

In calculating the strain energy for a two-dimensional beam, the contribution from the shearing strains will be neglected. Thus, only the effect of the normal strains will be considered. In this case, the strain energy can be written as

$$U = \int_V \left[\int_0^{\varepsilon_{xx}} \sigma_x d\varepsilon_{xx}\right] \mathrm{d}V. \tag{22}$$

For the case of isotropic linear material,

$$\sigma_x = E\varepsilon_{xx}, \tag{23}$$

where E is the modulus of elasticity. Substituting equation (23) into equation (22) and integrating leads to

$$U = \frac{1}{2}\int_0^l EA\epsilon_{xx}^2 \, \mathrm{d}x, \tag{24}$$

where A is the cross-sectional area of the beam and l is the length. Using a non-linear strain–displacement relationship, the normal strain ϵ_{xx} can be written as

$$\epsilon_{xx} = \partial u_i/\partial x - y\,\partial^2 v/\partial x^2 + \tfrac{1}{2}(\partial v/\partial x)^2, \tag{25}$$

where $u_i = u + u_g$.

Substitute equation (25) into equation (24) the strain energy can be written as

$$U = \frac{1}{2}\int_0^l EA\left(\frac{\partial u_i}{\partial x}\right)^2 \mathrm{d}x + \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 v}{\partial x^2}\right)^2 \mathrm{d}x + \frac{1}{2}\int_0^l EA\,\frac{\partial u_i}{\partial x}\left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x + \frac{1}{2}\int_0^l \frac{EA}{4}\left(\frac{\partial v}{\partial x}\right)^4 \mathrm{d}x, \tag{26}$$

where I is the second moment of area of the beam cross-section.

3.2. CO-ORDINATE TRANSFORMATION

If the longitudinal displacement caused by the transverse deflection u_g is neglected and only that due to the axial deformation u is considered, the first two second order integrals in equation (26) will yield linear axial and transverse stiffness terms which appear when a linear elastic model is used. The third integral has a third order term and therefore will give rise to a second order elastic force or stiffness term which couples the axial and bending displacements. The fourth integral is a function of the non-linear strain only, and it has a fourth order term yielding a stiffness coefficient of third order [5].

Using the definition of u_g given by equation (5), the displacement u_i can be written as [5]

$$u_i = u - \frac{1}{2}\int_0^x \left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x. \tag{27}$$

It follows that

$$\partial u_i/\partial x = \partial u/\partial x - \tfrac{1}{2}(\partial v/\partial x)^2. \tag{28}$$

Substituting equation (28) into equation (26), one obtains

$$U = \frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x + \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 v}{\partial x^2}\right) \mathrm{d}x. \tag{29}$$

It is clear from this equation that the consideration of the longitudinal displacement caused by the transverse deflection u_g removes third and higher order terms from the strain energy expression. This will lead to a constant stiffness matrix if the strain energy is written in terms of u instead of u_i. In this case, the strain energy can simply be written as

$$U = \tfrac{1}{2}\mathbf{q}_f^{\mathrm{T}}\mathbf{K}_{ff}\mathbf{q}_f, \tag{30}$$

where $\mathbf{K}_{ff}$ is the conventional constant stiffness matrix which can be written as

$$\mathbf{K}_{ff} = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \tag{31}$$

and k_{11} and k_{22} are the axial and transverse stiffness coefficients, respectively.

3.3. SPECIAL CASE OF CONSTANT ANGULAR VELOCITY

If the angular velocity is specified and assumed to be constant, the virtual work components δW_l and δW_g reduce to

$$\delta W_l = \int_V \rho[\ddot{u} - 2\dot{\theta}\dot{v} - \dot{\theta}^2(x+u)]\,\delta u\,\mathrm{d}V + \int_V \rho[\ddot{v} + 2\dot{\theta}\dot{u} - \dot{\theta}^2 v]\,\delta v\,\mathrm{d}V, \tag{32}$$

$$\delta W_g = \int_V \rho(\ddot{u}_g - \dot{\theta}^2 u_g)\,\delta u\,\mathrm{d}V + \int_V 2\rho\dot{\theta}\dot{u}_g\,\delta v\,\mathrm{d}V$$
$$+ \int_V \rho[-\dot{\theta}^2\{(x+u)+u_g\} - 2\dot{\theta}\dot{v} + (\ddot{u}+\ddot{u}_g)]\,\delta u_g\,\mathrm{d}V. \tag{33}$$

If one further assumes that the effect of the axial displacement u on the transverse vibration is small such that it can be ignored, the virtual work components δW_l and δW_g can be further reduced to

$$\delta W_l = \int_V \rho[\ddot{v} - \dot{\theta}^2 v]\,\delta v\,\mathrm{d}V, \tag{34}$$

$$\delta W_g = \int_V 2\rho\dot{\theta}\dot{u}_g\,\delta v\,\mathrm{d}V + \int_V \rho[-\dot{\theta}^2(x+u_g) - 2\dot{\theta}\dot{v} + \ddot{u}_g]\,\delta u_g\,\mathrm{d}V. \tag{35}$$

Note that in equations (34) and (35), the effect of the axial displacement due to bending is considered. In later sections, the effect of the axial displacement u on the bending vibration of the rotating beam will also be examined. Let

$$v = \mathbf{S}_2\mathbf{q}_{f2}, \qquad u_g = \tfrac{1}{2}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}(x)\mathbf{q}_{f2}. \tag{36}$$

It follows that

$$\delta v = \mathbf{S}_2\,\delta\mathbf{q}_{f2}, \qquad \delta u_g = \mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}(x)\,\delta\mathbf{q}_{f2}, \tag{37}$$

where

$$\mathbf{B}(x) = -\int_0^x \left(\frac{\partial \mathbf{S}_2^{\mathrm{T}}}{\partial x}\right)\left(\frac{\partial \mathbf{S}_2}{\partial x}\right)\mathrm{d}x. \tag{38}$$

Substituting the preceding equations into the expressions for the virtual work, one obtains

$$\delta W_i = \delta W_l + \delta W_g = \ddot{\mathbf{q}}_{f2}^{\mathrm{T}}[\mathbf{M}_l + \mathbf{M}_g]\,\delta\mathbf{q}_{f2} - [(\mathbf{Q}_v)_l^{\mathrm{T}} + (\mathbf{Q}_v)_g^{\mathrm{T}}]\,\delta\mathbf{q}_{f2}, \tag{39}$$

in which

$$\mathbf{M}_l = \int_V \rho \mathbf{S}_2^{\mathrm{T}} \mathbf{S}_2 \, \mathrm{d}V, \qquad (\mathbf{Q}_v)_l = \dot{\theta}^2 \int_V \rho \mathbf{S}_2^{\mathrm{T}} \mathbf{S}_2 \, \mathrm{d}V \mathbf{q}_{f2} = \dot{\theta}^2 \mathbf{M}_l \mathbf{q}_{f2}, \tag{40}$$

$$\mathbf{M}_g = \int_V \mathbf{B}(x) \mathbf{q}_{f2} \mathbf{q}_{f2}^{\mathrm{T}} \mathbf{B}(x) \, \mathrm{d}V, \tag{41}$$

$$(\mathbf{Q}_v)_g^{\mathrm{T}} = -2\dot{\theta}\dot{\mathbf{q}}_{f2}^{\mathrm{T}} \int_V \rho \{\mathbf{B}^{\mathrm{T}}(x)\mathbf{q}_{f2}\mathbf{S}_2 - \mathbf{S}_2^{\mathrm{T}}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}(x)\} \, \mathrm{d}V$$
$$+ \dot{\theta}^2 \mathbf{q}_{f2}^{\mathrm{T}} \int_V \rho \{x\mathbf{B}^{\mathrm{T}}(x) + \tfrac{1}{2}\mathbf{B}^{\mathrm{T}}(x)\mathbf{q}_{f2}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}(x)\} \, \mathrm{d}V - \dot{\mathbf{q}}_{f2}^{\mathrm{T}} \int_v \rho \mathbf{B}^{\mathrm{T}}(x)\dot{\mathbf{q}}_{f2}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}(x) \, \mathrm{d}V. \tag{42}$$

The vector $(\mathbf{Q}_v)_g$ can also be written as

$$(\mathbf{Q}_v)_g = \dot{\theta}^2 \mathbf{G}_1 \mathbf{q}_{f2} + \dot{\theta} \mathbf{G}_2 \dot{\mathbf{q}}_{f2} + \mathbf{G}_3 \dot{\mathbf{q}}_{f2}, \tag{43}$$

where

$$\mathbf{G}_1 = \int_V \rho \{x\mathbf{B}(x) + \tfrac{1}{2}\mathbf{B}(x)\mathbf{q}_{f2}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}(x)\} \, \mathrm{d}V, \tag{44}$$

$$\mathbf{G}_2 = -2\int_V \rho \{\mathbf{S}_2^{\mathrm{T}}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}(x) - \mathbf{B}(x)\mathbf{q}_{f2}\mathbf{S}_2\} \, \mathrm{d}V, \qquad \mathbf{G}_3 = -\int_V \rho \mathbf{B}(x)\mathbf{q}_{f2}\dot{\mathbf{q}}_{f2}^{\mathrm{T}}\mathbf{B}(x) \, \mathrm{d}V. \tag{45, 46}$$

Note that $\mathbf{G}_2$ is a skew symmetric matrix, and $\mathbf{G}_3$ depends on the velocity.

4. DIFFERENT BEAM FORMULATIONS

It is important to point out that while the strain energy of equation (29) is formulated in terms of the displacement u, this displacement component does not represent the total axial deformation of the beam. The total displacement is defined by equation (4). With regard to the definition of the axial displacement, there are several important observations which are discussed in this section. These observations can be summarized as follows.

(1) It is important to note that the use of the expression of u_g as defined by equation (5) automatically eliminates the non-linear term $\frac{1}{2}(\partial v/\partial x)^2$ in the strain–displacement relationship of equation (25). As a result, the use of u_g and the transformation of equation (27) leads to the simple form of the strain energy of equation (29). This strain energy expression leads to a constant stiffness matrix. If the displacement u_g is neglected, the strain–displacement relationship is non-linear, and in this case the strain energy takes the complex form given by equation (26).

(2) In view of the first observation, it is clear that when the effect of the axial displacement due to bending is considered, the mass matrix becomes complex while the stiffness matrix takes a simple form. This is the case of a CCM in which the effect of u_g is considered in formulating both the inertia and elastic forces.

(3) At relatively high values of the angular velocity of the beam, the use of an inconsistent model may lead to unstable solution. One of the inconsistent models is to neglect the effect of u_g in formulating the inertia force, but to consider such an effect

in formulating the elastic forces. This model leads to a constant mass matrix and also will lead to the simple expression of strain energy as defined by equation (29). Therefore, this model has a constant stiffness matrix as well as a constant mass matrix. It is important to remember that such a simple inconsistent model is developed using the non-linear strain displacement relationship of equation (25). This model produces a form of the equations of motion similar to the case in which the effect of u_g is neglected when both the inertia and elastic forces are formulated and at the same time a linear strain–displacement relationship is used. Therefore, this inconsistent model leads to the same results and the same stability problems as the linear model in which the equations are formulated without considering the effect of u_g.

(4) A CIM can be developed by neglecting the effect of u_g in formulating both the inertia and elastic forces, while using the non-linear strain–displacement relationship of equation (25). Note that if u_g is neglected in this case, one must use the complex form of the strain energy defined by equation (26). This leads to a non-linear stiffness matrix. This model will be used to prove that if the effect of the axial displacement due to bending is neglected in a consistent manner in a non-linear model, the rotating beam will not encounter the instability problem at high values of the angular velocity. This in turn will prove that it is not necessary to include the effect of the axial displacement caused by bending when formulating the equations of motion in order to obtain a stable behavior of the beam at relatively high values of the angular velocity.

(5) While including the effect of u_g in formulating the stiffness matrix and neglecting its effect in formulating the mass matrix leads to unstable behavior at relatively high values of the angular velocity, the neglect of this effect in formulating the stiffness matrix while including it in formulating the mass matrix leads to another inconsistent model that results in a stable solution at the relatively high values of the angular velocity. Note that such a model is an elastically non-linear model that includes elastic coupling between the axial and bending displacements.

Before several of these models and their effect on the dynamics and stability of rotating beams are discussed, a simple one-degree-of-freedom beam model is considered. This model will be used in the following two sections to examine the effect of higher order non-linear inertia terms that result from the use of the effect of the axial displacement caused by the bending deformation. The numerical results obtained using this simple model will be used as the basis for obtaining simpler, yet accurate models that can be used to study the effect of the coupling between the bending and the axial displacements.

5. A SIMPLE ONE-DEGREE-OF-FREEDOM BEAM MODEL

In this section, a simple one-degree-of-freedom rotating flexible beam model is considered. This model will be used to examine the effect of the longitudinal displacement u_g resulting from the bending on the dynamics of the rotating beam. In this model, the effect of the longitudinal displacement u is neglected. This effect will be considered in later sections. The case in which the transverse in-plane displacement is described by one mode of vibration is considered. In this case, the vector $\mathbf{q}_{f2}$ and the row vector $\mathbf{S}_2$ reduce to scalars, given as

$$\mathbf{q}_{f2} = q_{f2}, \qquad \mathbf{S}_2 = S_2. \tag{47}$$

In the analysis presented in this section and the following section, the effect of u_g is considered in formulating the stiffness coefficient. Since the transformation of equation (27) is employed, one obtains a constant stiffness coefficient despite the fact that a non-linear strain–displacement relationship is used. Note that if the effect of the axisl displacement u is neglected, $u_l = -\frac{1}{2}\int_0^x (\partial v/\partial x)^2\,\mathrm{d}x$ and the strain is $\epsilon_{xx} = -y\,\partial^2 v/\partial x^2$ which leads to a simple form for the strain energy. In this and the following sections, the effect of the inertia non-linearities that result from the use of the displacement u_g will be examined both analytically and numerically using the simple single-degree-of-freedom model for the rotating beam.

In the model used in this investigation, the transverse deformation of the beam is described by the shape function

$$S_2 = 3\xi^2 - \xi^3, \tag{48}$$

where $\xi = x/l$. Substituting this value of S_2 into equation (38), one obtains the function $\mathbf{B}(x)$, which is a scalar in this case, defined as

$$B(x) = -(9/l)(\tfrac{4}{3}\xi^3 - \xi^4 + \tfrac{1}{5}\xi^5). \tag{49}$$

By substituting the values of S_2 and $B(x)$ into equations (40), (41), (44–46) one obtains

$$M_l = 33m/35, \qquad (Q_v)_l = \dot{\theta}^2 M_l q_{f2}, \tag{50}$$

$$M_g = (1704m/385)(q_{f2}/l)^2, \qquad G_1 = m\{-\tfrac{81}{70} + \tfrac{852}{385}(q_{f2}/l)^2\}, \tag{51}$$

$$G_2 = 0, \qquad G_3 = -\frac{1704m}{385}\left(\frac{\dot{q}_{f2}q_{f2}}{l^2}\right), \tag{52}$$

where m is the total mass of the beam. Using equations (40), (43) and (52) with equation (39), one obtains the virtual work δW_i as

$$\delta W_i = [\ddot{q}_{f2}(M_l + M_g) - \dot{\theta}^2(M_l + G_1)q_{f2} - G_3\dot{q}_{f2}]\,\delta q_{f2}. \tag{53}$$

If the external forces are neglected, the virtual work of the inertia forces is equal to the virtual work of the elastic forces,

$$\delta W_i = \delta W_s, \tag{54}$$

where δW_i is defined by equation (53) and δW_s is defined as

$$\delta W_s = -\delta U = -q_{f2}k_{22}\,\delta q_{f2}, \tag{55}$$

where $k_{22} = 12EI/l^3$ is the in-plane bending stiffness coefficient. Substituting equations (53) and (55) into equation (54), the equation of motion of the single-degree-of-freedom rotating beam can be obtained as

$$(M_l + M_g)\ddot{q}_{f2} + \{k_{22} - \dot{\theta}^2(M_l + G_1)\}q_{f2} - G_3\dot{q}_{f2} = 0. \tag{56}$$

This non-linear differential equation, which describes the in-plane transverse vibration of the beam, includes all the non-linear terms resulting from u_g. This equation is a homogeneous second order differential equation and it is non-linear in the system co-ordinate.

6. EFFECT OF NON-LINEAR TERMS IN THE SIMPLE BEAM MODEL

In this section, the simple beam model developed in the preceding section is used to examine the effect of considering the axial displacement u_g on the beam stability. By examining the equation of motion presented in the preceding section, it becomes clear that the non-linearity in the beam equation due to the displacement u_g is represented by the matrices $\mathbf{M}_g$, $\mathbf{G}_1$, $\mathbf{G}_2$, and $\mathbf{G}_3$. It can also be noticed that the matrix $\mathbf{G}_1$ can be written as

$$\mathbf{G}_1 = \mathbf{G}_{1l} + \mathbf{G}_{1n}, \tag{57}$$

where $\mathbf{G}_{1l}$ does not contribute to the non-linearity, and $\mathbf{G}_{1l}$ and $\mathbf{G}_{1n}$ can be defined as

$$\mathbf{G}_{1l} = \int_V \rho x \mathbf{B}(x)\,\mathrm{d}V, \qquad \mathbf{G}_{1n} = \int_V \rho \mathbf{B}(x)\mathbf{q}_{f2}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}(x)\,\mathrm{d}V. \tag{58}$$

In this section, several cases will be considered in order to better understand the effect of the displacement u_g on the stability of the transverse vibration of the rotating beam.

Case 1: As previously mentioned in section 4, considering the effect of u_g leads to a simple form of the strain energy when the co-ordinate transformation of equation (27) is used. Therefore, if the effect of u_g is taken into consideration in formulating the strain energy and if all the terms that result from the use of this effect are neglected when the inertia forces are formulated, one obtains constant mass and stiffness coefficients. This is the case of the first inconsistent model (FIM). Note that in this model, a non-linear strain–displacement relationship is used despite the fact that the stiffness coefficient is constant. Using this model, it can be shown that the equation of motion of the single-degree-of-freedom rotating beam reduces to

$$\ddot{q}_{f2} = -\beta^2 q_{f2}, \qquad \beta^2 = k_{22}/M_l - \dot{\theta}^2. \tag{59}$$

This equation clearly demonstrates that if the angular velocity is greater than $\sqrt{k_{22}/M_l}$, the in-plane transverse mode of vibration becomes unstable. For $\dot{\theta} < \sqrt{k_{22}/M_l}$, the vibration will be oscillatory and the solution is defined as

$$q_{f2} = c\cos(\beta t + \phi), \tag{60}$$

where c and ϕ are constants that depend on the initial conditions.

It is interesting to note that the model described by equation (59) is similar to the model obtained when the effect of u_g is neglected and a linear strain–displacement relationship is used. This case also leads to constant mass and stiffness coefficients, and such a model also produces unstable solutions. One must note, however, that the definition of the displacements used in this model is different from the one used in the FIM where the displacement u_g is introduced.

Case 2: In this case, all the non-linear inertia terms resulting from the use of u_g are neglected and only the linear terms are retained. Using this approximation, one obtains a linear equation for the beam motion defined as

$$M_l\ddot{q}_{f2} + \{k_{22} - \dot{\theta}^2(M_l + G_{1l})\}q_{f2} = 0, \tag{61}$$

which can also be written in the simple form

$$\ddot{q}_{f2} = -\beta^2 q_{f2}, \qquad \beta^2 = [k_{22} - \dot{\theta}^2(M_l + G_{1l})]/M_l. \tag{62}$$

Examining the values of G_{1l} and M_l using equations (51) and (52), it is clear that the in-plane transverse vibration is stable regardless of the value of the angular velocity. The solution of this equation is in a form similar to equation (60) for all values of the angular velocity $\dot{\theta}$. This solution is oscillatory and its amplitude is dependent on the initial displacement.

Case 3: In this case only the term that contains $G_3\dot{q}_{f2}$ in the inertia forces is neglected, and as a result, the beam equation reduces to

$$(M_l + M_g)\ddot{q}_{f2} + \{k_{22} - (M_l + G_1)\}q_{f2} = 0. \tag{63}$$

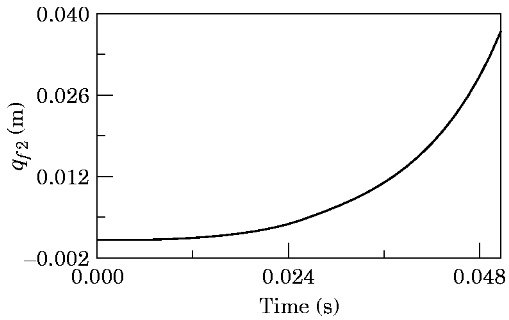

Figure 2. In-plane transverse deformation obtained using case 1 and angular velocity $\dot{\theta} = 200$ rad/s.

This non-linear equation can be solved using direct numerical integration methods (Runge–Kutta). The numerical solution of this equation will be presented later in this section.

Case 4: In this case, one considers all the non-linear terms resulting from considering the axial displacement u_g. This is the case in which the equation of motion is given by equation (56). The solution of this non-linear equation can be obtained using direct numerical integration methods (Runge–Kutta).

6.1. NUMERICAL RESULTS

The numerical results obtained using the four beam models described in this section will be compared in order to better understand the effect of the displacement u_g on the transverse vibration of the rotating beam. The model considered consists of a beam with length $l = 0{\cdot}5$ m and a circular cross-section area A which has a diameter $d = 0{\cdot}01$ m. This beam is made of steel with a modulus of elasticity $E = 2 \times 10^{11}$ N/m^2 and a mass density $\rho = 7800$ kg/m^3. The beam is assumed to rotate with a constant angular velocity $\dot{\theta}$ with an initial in-plane transverse displacement of 0·001 m. The numerical results presented in this section are obtained for different values of the angular velocity.

Figure 2 shows how the linear formulation (59) leads to an unstable solution if the angular velocity exceeds the stability limit which is defined, in this example, as $\dot{\theta} = 180{\cdot}6$ rad/s. The solution of case 2, in which the higher order terms $(q_{f2}/l)^2$ and $(\dot{q}_{f2}/l)^2$, are neglected, is shown in Figure 3 using angular velocities of 50, 100 and 150 rad/s.

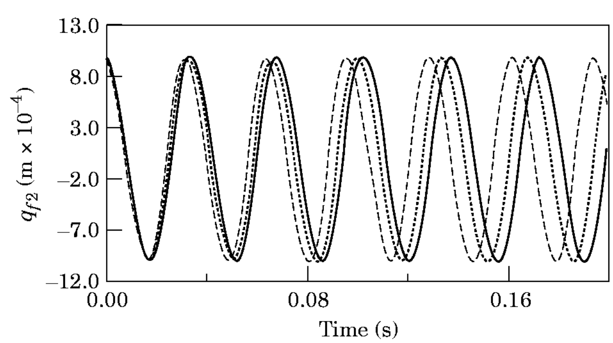

Figure 3. In-plane transverse deformation obtained using case 2. $\dot{\theta}$ values (rad/s): ——, 50; · · · ·, 100; – – –, 150.

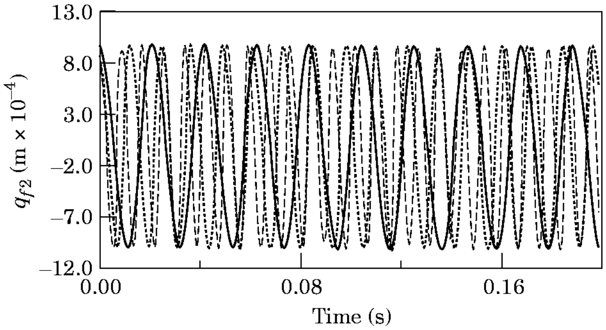

Figure 4. In-plane transverse deformation obtained using case 2. $\dot{\theta}$ values (rad/s): ——, 1000; · · · ·, 1500; –––, 2000.

Figure 4 shows that the vibration remains stable for higher angular velocities of 500, 1000, and 15 000 rad/s. From the results presented in these two figures, it is clear that the vibration is stable regardless of the value of the angular velocity, and the oscillation frequency increases as the angular velocity increases. This despite the fact that the amplitude remains equal to the initial displacement.

It is clear from equations (51), (53) and (56) that the order of magnitudes of the terms that include $(q_{f2}/l)^2$ and $(\dot{q}_{f2}/l)^2$ are small compared to other terms in equation (56). As a consequence, the solutions obtained for cases 3 and 4 are approximately the same as the solution of case 2. Figure 5 shows the in-plane transverse vibration obtained using cases 1, 2, 3 and 4 when the angular velocity of 150 rad/s is used. For $\dot{\theta} = 1000$ rad/s, case 1 leads to an unstable solution while the three non-linear cases have, approximately, the same solution as demonstrated by the results presented in Figure 6. The results shown in Figure 5 also demonstrate that the solution obtained using case 1 can be significantly different from that obtained using the other cases at lower values of the angular velocity. It is clear from the results presented in this section that the non-linear terms resulting from u_g in the inertia force do not have a significant effect on the accuracy and stability of the simple rotating beam model examined in this investigation. Therefore, it is justified to neglect the effect of these non-linear terms.

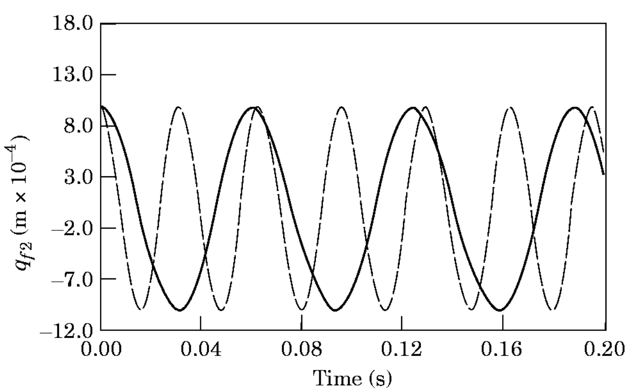

Figure 5. In-plane transverse deformation obtained using angular velocity $\dot{\theta} = 150$ rad/s; ——, Case 1; · · · ·, Case 2; –––, Case 3; –·–·, Case 4.

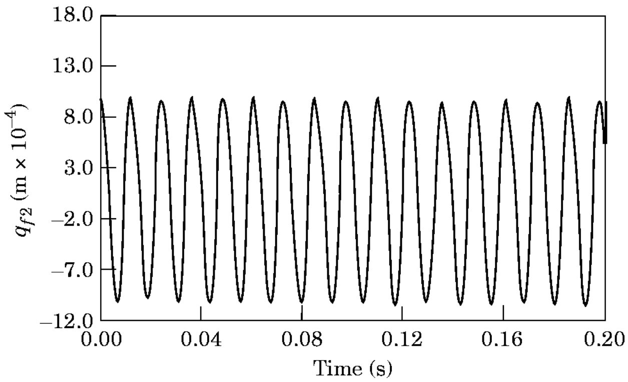

Figure 6. In-plane transverse deformation obtained using angular velocity $\dot{\theta} = 1000$ rad/s; ——, Case 2; · · · · , Case 3; –––, Case 4.

7. EFFECT OF THE AXIAL DISPLACEMENT

In the analysis presented in the preceding sections, the effect of the axial displacement u was neglected. In this section, the effect of the axial deformation is introduced and the effect of its coupling with the transverse deformation on the dynamics of the rotating beam is examined. To this end, one uses the displacement field

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{q}_{f1} \\ \mathbf{q}_{f2} \end{bmatrix}, \tag{64}$$

where $\mathbf{S}_1$ is the shape function row vector associated with the axial vibration and $\mathbf{S}_2$ is the shape function row vector associated with the in-plane mode of vibration. In this case, the virtual work of the inertia forces is represented by equations (32) and (33). Using these equations and equation (64), one obtains the equations for $\mathbf{M}_l$, $\mathbf{M}_g$, $(\mathbf{Q}_v)_l$, and $(\mathbf{Q}_v)_g$ as follows:

$$\mathbf{M}_l = \int_V \rho \begin{bmatrix} \mathbf{S}_1^{\mathrm{T}}\mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2^{\mathrm{T}}\mathbf{S}_2 \end{bmatrix} \mathrm{d}V, \qquad \mathbf{M}_g = \int_V \rho \begin{bmatrix} \mathbf{0} & \mathbf{S}_1\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}(x) \\ \mathbf{B}(x)\mathbf{q}_{f2}\mathbf{S}_1 & \mathbf{B}(x)\mathbf{q}_{f2}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}(x) \end{bmatrix} \mathrm{d}V, \tag{65}$$

$$(\mathbf{Q}_v)_l = \begin{bmatrix} (\mathbf{Q}_v)_{lf1} \\ (\mathbf{Q}_v)_{lf2} \end{bmatrix} = \dot{\theta}^2(\mathbf{L}_0 + \mathbf{M}_l\mathbf{q}_f) + \dot{\theta}\mathbf{L}_1\dot{\mathbf{q}}_f, \tag{66}$$

$$(\mathbf{Q}_v)_g = \begin{bmatrix} (\mathbf{Q}_v)_{gf1} \\ (\mathbf{Q}_v)_{gf2} \end{bmatrix} = \dot{\theta}^2(\mathbf{G}_{01} + \mathbf{G}_{11})\mathbf{q}_f + \dot{\theta}\mathbf{G}_{22}\dot{\mathbf{q}}_f + \mathbf{G}_{33}\dot{\mathbf{q}}_f, \tag{67}$$

where $\mathbf{B}(x)$ is defined by equation (38) and

$$\mathbf{L}_0 = \int_V \rho \begin{bmatrix} x\mathbf{S}_1^{\mathrm{T}} \\ \mathbf{0} \end{bmatrix} \mathrm{d}V, \qquad \mathbf{L}_1 = 2\int_V \rho \begin{bmatrix} \mathbf{0} & \mathbf{S}_1^{\mathrm{T}}\mathbf{S}_2 \\ -\mathbf{S}_2^{\mathrm{T}}\mathbf{S}_1 & \mathbf{0} \end{bmatrix} \mathrm{d}V, \tag{68}$$

$$\mathbf{G}_{01} = \int_V \rho \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & x\mathbf{B}(x) \end{bmatrix} \mathrm{d}V, \qquad \mathbf{G}_{11} = \int_V \rho \begin{bmatrix} \mathbf{0} & \frac{1}{2}\mathbf{S}_1^{\mathrm{T}}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}(x) \\ \mathbf{B}(x)\mathbf{q}_{f2}\mathbf{S}_1 & \frac{1}{2}\mathbf{B}(x)\mathbf{q}_{f2}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}(x) \end{bmatrix} \mathrm{d}V, \tag{69}$$

$$\mathbf{G}_{22} = -2\int_V \rho \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2^{\mathrm{T}}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}(x) - \mathbf{B}(x)\mathbf{q}_{f2}\mathbf{S}_2 \end{bmatrix} \mathrm{d}V, \tag{70}$$

$$\mathbf{G}_{33} = \int_V \rho \begin{bmatrix} \mathbf{0} & -\mathbf{S}_1^{\mathrm{T}}\dot{\mathbf{q}}_{f2}\mathbf{B}(x) \\ \mathbf{0} & -\mathbf{B}(x)\mathbf{q}_{f2}\dot{\mathbf{q}}_{f2}^{\mathrm{T}}\mathbf{B}(x) \end{bmatrix} \mathrm{d}V. \tag{71}$$

In the following three sections, three different beam models are considered. The first model is the CCM in which the effect of longitudinal displacement due to transverse vibration is considered in evaluating both the inertia and elastic forces. This model leads to a constant stiffness matrix when the transformation of equation (27) is used. The second model is the CIM in which the effect of longitudinal displacement caused by bending is neglected in evaluating both the stiffness and inertia forces. This model leads to a non-linear stiffness matrix since a non-linear strain–displacement relationship is used and the effect of u_g is neglected. The third model is the *second inconsistent model* (SIM) in which the effect of longitudinal displacement due to bending is considered in formulating the inertia forces and neglected in formulating the stiffness forces. The results obtained using these three models clearly demonstrate that including the effect of the geometric stiffening resulting from the longitudinal displacement caused by bending is not the only approach to obtain a stable solution for the rotating beams at higher values of the angular velocity. These results demonstrate that if such an effect is consistently neglected, a stable behavior of the beam can be obtained when a non-linear model is used.

8. CONSISTENT COMPLETE MODEL

In the CCM, the effect of longitudinal displacement caused by bending deformation is considered in formulating both the inertia and elastic forces. Using the co-ordinate transformation of equation (27) [5], a constant stiffness matrix can be obtained despite the fact that a non-linear strain–displacement relationship is used.

In order to examine the effect of including the axial displacement on the beam dynamic behavior, one considers the case of a single mode of vibration for both axial and in-plane transverse motion. In this case of a two-degree-of-freedom system, the deformation co-ordinates $\mathbf{q}_{f1}$ and $\mathbf{q}_{f2}$ will be represented by scalar quantities

$$\mathbf{q}_{f1} = q_{f1}, \qquad \mathbf{q}_{f2} = q_{f2}, \tag{72}$$

and also the shape function row vectors $\mathbf{S}_1$ and $\mathbf{S}_2$ will be defined by the scalars

$$S_1 = \xi, \qquad S_2 = 3\xi^2 - \xi^3. \tag{73}$$

The matrix $\mathbf{B}(x)$ reduces to the function $B(x)$ which is defined by equation (49). Substituting these functions into equations (65), (68–71), one obtains

$$\mathbf{M}_l = m\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{33}{35} \end{bmatrix}, \qquad \mathbf{M}_g = m\begin{bmatrix} 0 & -\frac{81}{70}(q_{f2}/l) \\ -\frac{81}{70}(q_{f2}/l) & \frac{1704}{385}(q_{f2}/l)^2 \end{bmatrix}, \tag{74}$$

$$\mathbf{L}_0 = \frac{ml}{3}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{L}_1 = \frac{11m}{10}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{75}$$

$$\mathbf{G}_{01} = -\frac{81m}{70}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{G}_{11} = m\left(\frac{q_{f2}}{l}\right)\begin{bmatrix} 0 & -\frac{81}{140} \\ -\frac{81}{70} & \frac{852}{385}(q_{f2}/l) \end{bmatrix}, \tag{76}$$

$$\mathbf{G}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{G}_{33} = m\left(\frac{\dot{q}_{f2}}{l}\right)\begin{bmatrix} 0 & \frac{81}{70} \\ 0 & -\frac{1704}{385}(q_{f2}/l) \end{bmatrix}. \tag{77}$$

The equation of motion can be obtained by equating the virtual work of the inertia forces and the virtual work of the elastic force. This leads to the following matrix equation for the two-degree-of-freedom beam model:

$$[\mathbf{M}_l + \mathbf{M}_g]\ddot{\mathbf{q}}_f - [(\mathbf{Q}_v)_l + (\mathbf{Q}_v)_g] = -\mathbf{K}_{ff}\mathbf{q}_f, \tag{78}$$

where, in this case, the stiffness matrix $\mathbf{K}_{ff}$ is written as

$$\mathbf{K}_{ff} = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} = \begin{bmatrix} EA/l & 0 \\ 0 & 12EI/l^3 \end{bmatrix}. \tag{79}$$

Substituting equations (66) and (67) into equation (78), taking into consideration that in this case $\mathbf{G}_{22} = \mathbf{0}$, the equation of motion of the two-degree-of-freedom beam model can be written as

$$[\mathbf{M}_l + \mathbf{M}_g]\ddot{\mathbf{q}}_f = -\mathbf{K}_{ff}\mathbf{q}_f + \dot{\theta}^2[\mathbf{L}_0 + (\mathbf{M}_l + \mathbf{G}_{01} + \mathbf{G}_{11})\mathbf{q}_f] + \dot{\theta}\mathbf{L}_1\dot{\mathbf{q}}_f + \mathbf{G}_{33}\dot{\mathbf{q}}_f. \tag{80}$$

Note that in this equation the inertia matrix depends on the beam co-ordinates, while the stiffness matrix is constant. This is the result of including the effect of u_g in formulating both the inertia and stiffness forces.

9. CONSISTENT INCOMPLETE MODEL

If the effect of longitudinal displacement caused by bending is neglected in formulating the strain energy, the stiffness matrix of the two-degree-of-freedom model considered in this section becomes non-linear. Also, if the effect of this displacement is neglected in the formulation of the inertia forces, we obtain the CIM in which the effect of u_g is consistently neglected. In this case, the equation of motion of the rotating beam can be written as

$$\mathbf{M}_l\ddot{\mathbf{q}}_f + (\mathbf{K}_{ff} + \mathbf{K}_G + \mathbf{K}_H)\mathbf{q}_f = (\mathbf{Q}_v)_l + \mathbf{Q}_G, \tag{81}$$

where $\mathbf{M}_l$ is the linear mass matrix which is defined as

$$\mathbf{M}_l = \int_V \rho \begin{bmatrix} \mathbf{S}_1^{\mathrm{T}}\mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2^{\mathrm{T}}\mathbf{S}_2 \end{bmatrix} \mathrm{d}V, \tag{82}$$

$\mathbf{K}_{ff}$ is the constant stiffness matrix, $\mathbf{K}_G$ is the second order non-linear stiffness matrix due to the third term of equation (26), $\mathbf{K}_H$ is the third order non-linear stiffness matrix resulting from the fourth term of equation (26), $(\mathbf{Q}_v)_l$ is the Coriolis and centrifugal force vector defined by equation (66), and $\mathbf{Q}_G$ is the non-linear elastic force vector which is defined as

$$\mathbf{Q}_G = -\frac{1}{2}\begin{bmatrix} \mathbf{q}_f^{\mathrm{T}}(\partial(\mathbf{K}_G + \mathbf{K}_H)/\partial \mathbf{q}_{f1})\mathbf{q}_f \\ \mathbf{q}_f^{\mathrm{T}}(\partial(\mathbf{K}_G + \mathbf{K}_H)/\partial \mathbf{q}_{f2})\mathbf{q}_f \end{bmatrix}. \tag{83}$$

If the assumption of a single mode of vibration of both axial and in-plane transverse directions is used with the shape functions previously defined in the preceding section, one obtains

$$\mathbf{M}_l = m\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{33}{35} \end{bmatrix}, \tag{84}$$

$$\mathbf{K}_G = \frac{EA}{l}\left(\frac{q_{f1}}{l}\right)\begin{bmatrix} 0 & 0 \\ 0 & \frac{24}{5} \end{bmatrix}, \qquad \mathbf{K}_H = \frac{EA}{l}\left(\frac{q_{f2}}{l}\right)^2\begin{bmatrix} 0 & 0 \\ 0 & \frac{288}{35} \end{bmatrix}, \tag{85}$$

$$\mathbf{Q}_G = -\frac{12EA}{5}\left(\frac{q_{f2}}{l}\right)^2\begin{bmatrix} 1 \\ \frac{24}{7}\,(q_{f_2}/l)) \end{bmatrix}, \tag{86}$$

and $(\mathbf{Q}_v)_l$ and $\mathbf{K}_{ff}$ are defined by equations (60) and (64), respectively. In this case, the equations of motion of the two-degree-of-freedom model can be written as

$$\mathbf{M}_l\ddot{\mathbf{q}}_f = -(\mathbf{K}_{ff} + \mathbf{K}_G + \mathbf{K}_H - \dot{\theta}^2\mathbf{M}_l)\mathbf{q}_f + \dot{\theta}\mathbf{L}_1\dot{\mathbf{q}}_f + \mathbf{Q}_G. \tag{87}$$

This equation is not the same as equation (80) because the effect of the axial displacement caused by the transverse bending is neglected in formulating the inertia and elastic forces. It will be demonstrated that the model described by equation (87) remains stable regardless of the value of the angular velocity, thereby demonstrating that the beam stability at higher values of the angular velocity is not dependent only on u_g, but it depends generally on the coupling between the axial and bending deformations.

10. SECOND INCONSISTENT MODEL

In this model, if the effect of the axial displacement caused by the transverse bending is taken into consideration in formulating the inertia forces but it is neglected in formulating the elastic forces, the non-linear strain–displacement relation is used. The equations of motion of the rotating beam for this model can be written as

$$(\mathbf{M}_l + \mathbf{M}_g)\ddot{\mathbf{q}}_f + (\mathbf{K}_{ff} + \mathbf{K}_G + \mathbf{K}_H)\mathbf{q}_f = (\mathbf{Q}_v)_l + (\mathbf{Q}_v)_g + \mathbf{Q}_G. \tag{88}$$

Using the shape functions defined in the preceding two sections, one can show that the equations of motion of the two-degree-of-freedom beam model can be written more explicitly as

$$\begin{aligned}(\mathbf{M}_l + \mathbf{M}_g)\ddot{\mathbf{q}}_f = {} & \dot{\theta}^2[\mathbf{L}_0 + (\mathbf{M}_l + \mathbf{G}_{01} + \mathbf{G}_{11})\mathbf{q}_f] + (\dot{\theta}\mathbf{L}_1 + \mathbf{G}_{33})\dot{\mathbf{q}}_f \\ & + \mathbf{Q}_G - (\mathbf{K}_{ff} + \mathbf{K}_G + \mathbf{K}_H)\mathbf{q}_f.\end{aligned} \tag{89}$$

The results obtained using this SIM will be examined in the following section.

11. COMPARATIVE STUDY

In this section, the results obtained using the three models (CCM, CIM, SIM) discussed in the preceding three sections are compared. The same beam model described in the preceding sections is also used in this section. Initial displacements of 0·001 m are assumed for both the axial and in-plane transverse deformation. A direct integration method is used to solve the equations of motion of the three non-linear models. The results obtained using these three non-linear models are also compared with the linear model (FIM) in which the effect of longitudinal displacement caused by bending deformation is neglected in formulating the inertia forces only. Figures 7 and 8 compare the results obtained for the axial displacement q_{f1} using the four models when the angular velocities

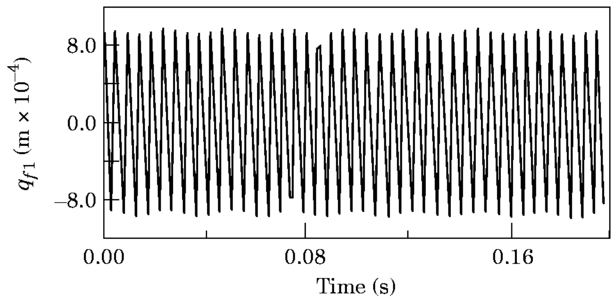

Figure 7. Axial deformation obtained using angular velocity $\dot{\theta} = 100$ rad/s; ——, FIM; · · · · , CCM; ----, CIM; – · – ·, SIM.

of 100 and 150 rad/s are used. It is clear from the results presented in these figures that the longitudinal vibration is approximately the same for the four models. This implies, for a low range of the angular velocity, that the FIM gives a good approximation for the longitudinal vibration as compared to the non-linear models. As the angular velocity increases, the longitudinal displacement remains stable but the vibration amplitude increases. Figures 9 and 10 show the in-plane transverse vibration obtained using the four models for angular velocities of 100 and 150 rad/s. By comparing the results presented in these two figures and the results presented in Figures 3 and 4, it becomes clear that including the axial displacement slightly increases oscillation amplitude. It is also clear that neglecting the effect of u_g in formulating the elastic forces tends to reduce the oscillation amplitude, and the SIM, in which the effect of longitudinal displacement caused by bending is neglected in formulating the elastic forces only, has the least amplitude. Figure 11 shows the maximum oscillation amplitude obtained using the four methods as function of the angular velocity in the low angular velocity range. It is clear from the results presented in this figure that the FIM has the highest maximum amplitude while the SIM has the lowest. Figure 12 shows that the axial displacements obtained using the three models (CCM, CIM, SIM) are the same when the angular velocity is increased to 1000 rad/s. Figure 13 shows the in-plane vibration predicted using these three models when the angular velocity is equal to 1000 rad/s. It is clear that the motion remains oscillatory and stable and the CCM has the highest amplitude when compared to the other two models. Figure 14 shows the maximum amplitude of the transverse vibration obtained using the three models for the high range of the angular velocity. It is clear from the results presented in the last two figures that, while all the three non-linear models lead to a stable oscillation at relatively high values of the angular velocity, the SIM in which the effect of the longitudinal displacement caused by bending is included in the inertia forces only, gives the lowest amplitude. The stability of the SIM depends on the sign of the term

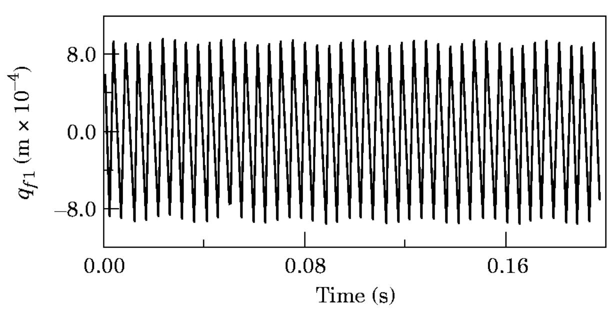

Figure 8. Axial deformation obtained using angular velocity $\dot{\theta} = 150$ rad/s; key as Figure 7.

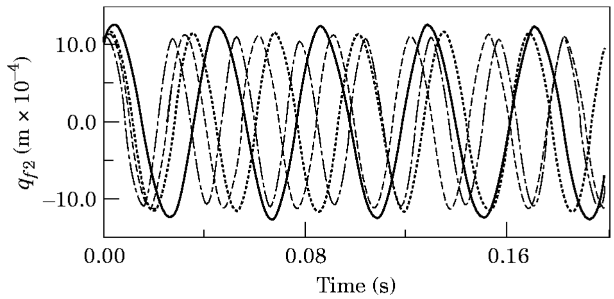

Figure 9. In-plane deformation obtained using angular velocity $\dot{\theta} = 100$ rad/s; key as Figure 7.

$M_l + G_{1l}$ (see equation (61)) which, in turns, depends on the assumed displacement field. For a given model, a negative sign of this term leads to a stable solution, while a positive sign leads to an unstable solution. Table 1 shows different assumed displacement fields and the coefficient $M_l + G_{1l}$ that result from the use of these fields when a single mode of vibration is used.

12. TRANSLATING AND ROTATING BEAM

In the preceding section, only the case of a rotating beam is considered. In this and the following sections one considers the case of a rotating and translating beam and examines the effect of the longitudinal displacement due to bending on the stability of the rigid body modes. First the equations that govern the rigid body motion are summarized.

12.1. RIGID BODY MOTION

In the analysis presented in this section, the beam is assumed to rotate with a constant angular velocity about an axis passing through one of its end points as shown in Figure 15. It is also assumed that the beam can have an arbitrary planar rigid body translation. In the case of rigid body motion, the position vector of an arbitrary point on the beam center line can be written as

$$\mathbf{r} = \mathbf{R} + \mathbf{A}\bar{\mathbf{u}}_o,$$

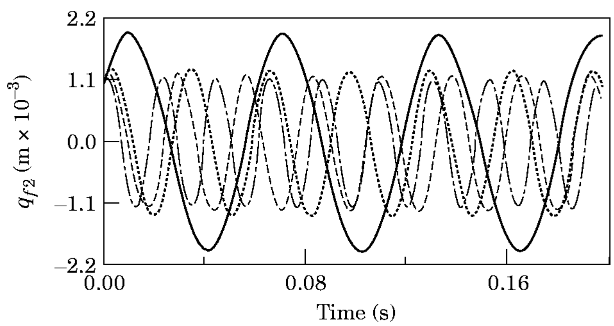

Figure 10. In-plane deformation obtained using angular velocity $\dot{\theta} = 150$ rad/s; key as Figure 7.

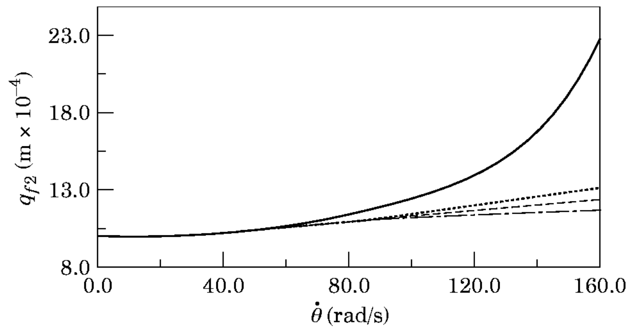

Figure 11. Maximum transverse displacement as a function of the angular velocity; key as Figure 7.

where $\mathbf{R} = [R_x \ R_y]^T$ is the position vector of the reference point, $\mathbf{A}$ is the planar transformation matrix defined as

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{90}$$

and $\bar{\mathbf{u}}_o = [x \ \ 0]^T$ is the position vector of the arbitrary point.

The velocity and acceleration vectors of an arbitrary point can be written as

$$\dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\theta}\mathbf{A}_\theta \bar{\mathbf{u}}_o, \qquad \ddot{\mathbf{r}} = \ddot{\mathbf{R}} - \dot{\theta}^2 \mathbf{A}\bar{\mathbf{u}}_o,$$

where $\mathbf{A}_\theta = \partial \mathbf{A}/\partial\theta$. The virtual work of the inertia force can be written as

$$\delta W_i = \int_V \rho \ddot{\mathbf{r}}^T \, \delta\mathbf{r} \, \mathrm{d}V,$$

where $\delta\mathbf{r}$ is the virtual change of the position vector, which is defined in the case of specified angular rotation as

$$\delta\mathbf{r} = \delta\mathbf{R}.$$

Substituting the values of $\ddot{\mathbf{r}}$ and $\delta\mathbf{r}$ into the virtual work expression, one obtains

$$\delta W_i = m\left\{\begin{bmatrix} \ddot{R}_x \\ \ddot{R}_y \end{bmatrix}^T - \frac{1}{2}\dot{\theta}^2 l \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}^T\right\}\begin{bmatrix} \delta R_x \\ \delta R_y \end{bmatrix}.$$

If there are no gravity or external forces acting on the beam, the virtual work of the inertia

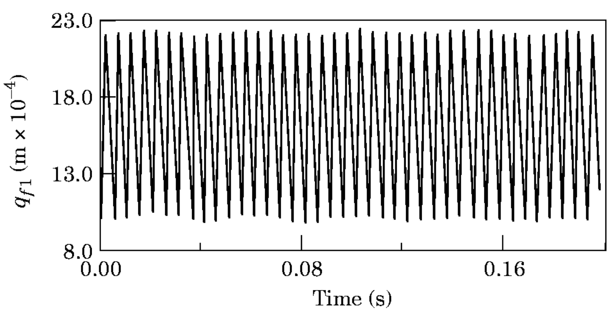

Figure 12. Axial deformation obtained using angular velocity $\dot{\theta} = 1000$ rad/s: ——, CCM; · · · ·, CIM; – – –, FIM.

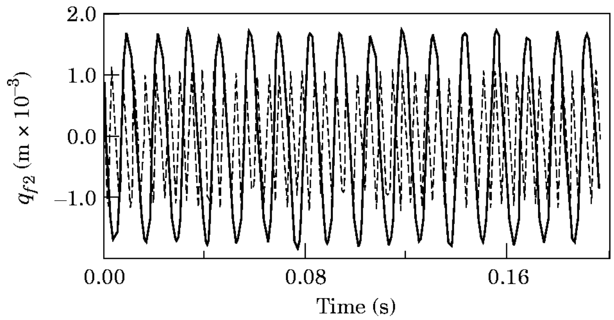

Figure 13. In-plane deformation due to angular velocity $\dot{\theta} = 1000$ rad/s; key as Figure 12.

force is identically equal to zero and the equations of the reference point motion can be written as

$$\ddot{R}_x = \tfrac{1}{2} l \dot{\theta}^2 \cos\theta, \qquad \ddot{R}_y = \tfrac{1}{2} l \dot{\theta}^2 \sin\theta.$$

Integrating these equations once and twice, assuming zero initial conditions, and keeping in mind that $\theta = \dot{\theta} t$, one obtains the reference point velocity and displacement as

$$\dot{R}_x = \tfrac{1}{2} l \dot{\theta} \sin\theta, \quad \dot{R}_y = \tfrac{1}{2} l \dot{\theta}(1 - \cos\theta), \quad R_x = (l/2)(1 - \cos\theta), \quad R_y = (l/2)(\dot{\theta} t - \sin\theta). \tag{91}$$

It is clear from the displacement equations that the motion in the x direction is oscillatory, while the motion in the y direction increases with time. The results obtained in this section for the simple rigid body model will be used to provide an explanation for some of the results obtained using the flexible body model.

12.2. FLEXIBLE BODY MODEL

In this section, the effect of the displacement due to bending on the dynamic equations of the rotating and translating flexible beam is examined. To this end, the planar beam model shown in Figure 15 is used. The beam is assumed to rotate with an angular velocity $\dot{\theta}$ and can have an arbitrary rigid body base translation. The global position vector of an arbitrary point on the beam can be written as

$$\mathbf{r} = \mathbf{R} + \mathbf{A}\bar{\mathbf{u}}, \tag{92}$$

where $\mathbf{R} = [R_x \; R_y]^{\mathrm{T}}$ is the global position vector of the reference point, $\mathbf{A}$ is the planar transformation matrix defined by equation (90), and $\bar{\mathbf{u}}$ is the local position vector of the

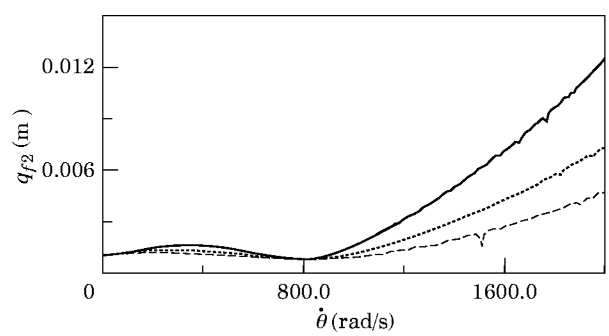

Figure 14. Maximum transverse displacement as a function of the angular velocity; key as Figure 12.

TABLE 1

Centrifugal stiffness coefficients for different mode shapes

Mode shape	Stiffness coefficient
Simply supported	$(M_l)_j + (G_{1l})_j = -(m(C_{ss})_j/2)[(n_j^2 l^2/3 - 1)$
$S_j = (C_{ss})_j \sin n_j x$	$- \frac{1}{4}\cos 2n_j l + (5/8n_j l)\sin 2n_j l]$
where $(C_{ss})_j$ is an arbitrary constant and $n_j = j\pi/l$	
Cantilever beam	$(M_l)_1 + (G_{1l})_1 = -0{\cdot}3588m(C_{cl})_1^2$
$S_j = (C_{cl})_j[\sin n_j x - \sinh n_j x + D_j(\cos n_j x - \cosh n_j x)]$	$(M_l)_2 + (G_{1l})_2 = -5{\cdot}2812m(C_{cl})_2^2$
where $D_j = (\cos n_j l + \cosh n_j l)/(\sin n_j l - \sinh n_j l)$	$(M_l)_3 + (G_{1l})_3 = -16{\cdot}8868m(C_{cl})_3^2$
$n_1 = 1{\cdot}875/l, \quad n_2 = 4{\cdot}694/l, \quad n_3 = 7{\cdot}855/l, \quad n_4 = 10{\cdot}996/l$	$(M_l)_4 + (G_{1l})_4 = -35{\cdot}0562m(C_{cl})_4^2$
Free ends	$(M_l)_2 + (G_{1l})_2 = -14{\cdot}9379m(C_{fe})_2^2$
$S_j = (C_{fe})_j[\sin n_j x + \sinh n_j x + D_j(\cos n_j x + \cosh n_j x)]$	$(M_l)_3 + (G_{1l})_3 = -32{\cdot}5762m(C_{fe})_4^2$
where $D_j = -(\sinh n_j l - \sin n_j l)/(\cosh n_j l - \cos n_j l)$	$(M_l)_4 + (G_{1l})_4 = -136{\cdot}2841m(C_{fe})_4^2$
$n_1 = 0, \quad n_2 = 4{\cdot}732/l, \quad n_2 = 7{\cdot}853/l, \quad n_4 = 10{\cdot}995/l, \quad n_4 = 14{\cdot}137/l$	$(M_l)_5 + (G_{1l})_5 = -219{\cdot}9443m(C_{fe})_5^2$
Fixed ends	$(M_l)_1 + (G_{1l})_1 = -0{\cdot}0639m(C_{ff})_1^2$
$S_j = (C_{ff})_j[\sin n_j x - \sinh n_j x + D_j(\cos n_j x - \cosh n_j x)]$	$(M_l)_2 + (G_{1l})_2 = -3{\cdot}1662m(C_{ff})_2^2$
where $D_j = -(\sin n_j l - \sinh n_j l)/(\cos n_j l - \cosh n_j l)$	$(M_l)_3 + (G_{1l})_3 = -14{\cdot}8930m(C_{ff})_2^2$
$n_1 = 1{\cdot}875/l, \quad n_2 = 4{\cdot}694/l, \quad n_3 = 7{\cdot}855/l, \quad n_4 = 10{\cdot}996/l$	$(M_l)_4 + (G_{1l})_4 = -33{\cdot}0554m(C_{ff})_4^2$

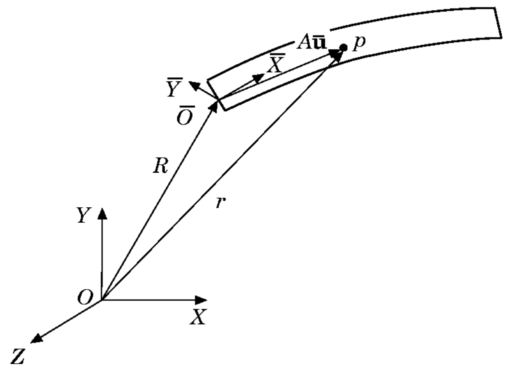

Figure 15. Rotating and translating beam model.

arbitrary point on the beam center line defined in the beam co-ordinate system. The vector $\bar{\mathbf{u}}$ can be written as

$$\bar{\mathbf{u}} = \bar{\mathbf{u}}_o + \bar{\mathbf{u}}_f, \tag{93}$$

where in the case of a slender beam $\bar{\mathbf{u}}_o = [x \;\; 0]^{\mathrm{T}}$, and $\bar{\mathbf{u}}_f$ is the deformation vector defined as

$$\bar{\mathbf{u}}_f = \begin{bmatrix} u + u_g \\ v \end{bmatrix}, \tag{94}$$

where u is the axial displacement, u_g is the longitudinal displacement resulting from the bending deformation, and v is the transverse displacement. These displacement components are defined as described in the preceding sections as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{q}_{f1} \\ \mathbf{q}_{f2} \end{bmatrix}, \qquad u_g = -\tfrac{1}{2}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}(x)\mathbf{q}_{f2}, \tag{95, 96}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the axial and bending shape functions row vectors, $\mathbf{q}_{f1}$ and $\mathbf{q}_{f2}$ are respectively the vectors of axial and bending time dependent co-ordinates, and $\mathbf{B}(x)$ is the matrix defined in the preceding sections.

13. INERTIA FORCES OF THE TRANSLATING AND ROTATING BEAM

If the beam is assumed to rotate with a constant angular velocity, the virtual change in the global position vector of an arbitrary point on the beam can be written as

$$\delta\mathbf{r} = \delta\mathbf{R} + \mathbf{A}\,\delta\bar{\mathbf{u}}_f. \tag{97}$$

The velocity and acceleration vectors are defined as

$$\dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\theta}\mathbf{A}_\theta\bar{\mathbf{u}} + \mathbf{A}\dot{\bar{\mathbf{u}}}_f, \qquad \ddot{\mathbf{r}} = \ddot{\mathbf{R}} - \dot{\theta}^2\mathbf{A}\bar{\mathbf{u}} + 2\dot{\theta}\mathbf{A}_\theta\dot{\bar{\mathbf{u}}}_f + \mathbf{A}\ddot{\bar{\mathbf{u}}}_f. \tag{98, 99}$$

The virtual work of the inertia forces can be defined as

$$\delta W_i = \delta W_l + \delta W_g = \int_V \rho\ddot{\mathbf{r}}^{\mathrm{T}}\,\delta\mathbf{r}\,\mathrm{d}V, \tag{100}$$

where δW_l is the virtual work which does not include the effect of u_g, while δW_g is the change of the virtual work as a result of including the axial displacement u_g. If equations (97) and (99) are substituted into equation (100), the two components of the virtual work can be defined as

$$\delta W_l = \delta W_{lR_x} + \delta W_{lR_y} + \delta W_{lu} + \delta W_{lv}, \tag{101}$$

$$\delta W_g = \delta W_{gR_x} + \delta W_{gR_y} + \delta W_{gu} + \delta W_{gv} + \delta W_{gu_g}, \tag{102}$$

where the components of the virtual work δW_l are defined as

$$\delta W_{lR_x} = \int_V \rho[\ddot{R}_x - \dot{\theta}^2\{(x+u)\cos\theta - v\sin\theta\} - 2\dot{\theta}\{\dot{u}\sin\theta + \dot{v}\cos\theta\} + \ddot{u}\cos\theta - \ddot{v}\sin\theta]\,\delta R_x\,\mathrm{d}V, \tag{103}$$

$$\delta W_{lR_y} = \int_V \rho[\ddot{R}_y - \dot{\theta}^2\{(x+u)\sin\theta + v\cos\theta\} - 2\dot{\theta}\{-\dot{u}\cos\theta + \dot{v}\sin\theta\} + \ddot{u}\sin\theta + \ddot{v}\cos\theta]\,\delta R_y\,\mathrm{d}V, \tag{104}$$

$$\delta W_{lu} = \int_V \rho[\ddot{R}_x\cos\theta + \ddot{R}_y\sin\theta - \dot{\theta}^2(x+u) - 2\dot{\theta}\dot{v} + \ddot{u}]\,\delta u\,\mathrm{d}V, \tag{105}$$

$$\delta W_{lv} = \int_V \rho[-\ddot{R}_x\sin\theta + \ddot{R}_x\cos\theta - \dot{\theta}^2 v + 2\dot{\theta}\dot{u} + \ddot{v}]\,\delta v\,\mathrm{d}V, \tag{106}$$

and the components of δW_g are defined as

$$\delta W_{gR_x} = \int_V \rho[-\dot{\theta}^2 u_g\cos\theta - 2\dot{\theta}\dot{u}_g\sin\theta + \ddot{u}_g\cos\theta]\,\delta R_x\,\mathrm{d}V, \tag{107}$$

$$\delta W_{gR_y} = \int_V \rho[-\dot{\theta}^2 u_g\sin\theta + 2\dot{\theta}\dot{u}_g\cos\theta + \ddot{u}_g\sin\theta]\,\delta R_y\,\mathrm{d}V, \tag{108}$$

$$\delta W_{gu} = \int_V \rho[\ddot{u}_g - \dot{\theta}^2 u_g]\,\delta u\,\mathrm{d}V, \qquad \delta W_{gv} = 2\int_V \rho\dot{\theta}\dot{u}_g\,\delta v\,\mathrm{d}V, \tag{109, 110}$$

$$\delta W_{gu_g} = \int_V \rho[\ddot{R}_x\cos\theta + \ddot{R}_y\sin\theta - \dot{\theta}^2(x+u+u_g) - 2\dot{\theta}\dot{v} + \ddot{u} + \ddot{u}_g]\,\delta u_g\,\mathrm{d}V. \tag{111}$$

If the effect of the longitudinal displacement due to bending is neglected, δW_g is identically equal to zero. One also notes that both δv and δu_g can be expressed in terms of the virtual change in the bending deformation generalized co-ordinates.

The virtual work components of equations (101) and (102) can be written as

$$\delta W_l = \ddot{\mathbf{q}}^{\mathrm{T}} \mathbf{M}_l \, \delta \mathbf{q} - (\mathbf{Q}_v)_l^{\mathrm{T}} \, \delta \mathbf{q}, \qquad \delta W_g = \ddot{\mathbf{q}}^{\mathrm{T}} \mathbf{M}_g \, \delta \mathbf{q} - (\mathbf{Q}_v)_g^{\mathrm{T}} \, \delta \mathbf{q}, \tag{112, 113}$$

where $\mathbf{q}$ is the vector of the generalized co-ordinates which is defined as

$$\mathbf{q}^{\mathrm{T}} = [R_x \quad R_y \quad \mathbf{q}_{f1}^{\mathrm{T}} \quad \mathbf{q}_{f2}^{\mathrm{T}}], \tag{114}$$

$\mathbf{M}_l$ and $\mathbf{M}_g$ are mass matrices, and $(\mathbf{Q}_v)_l$ and $(\mathbf{Q}_v)_g$ are the Coriolis and centrifugal force vectors.

The mass matrix $\mathbf{M}_g$ and the vector $(\mathbf{Q}_v)_g$ are mainly due to the effect of the longitudinal displacement caused by the bending deformation. The matrix $\mathbf{M}_l$ and the vector $(\mathbf{Q}_v)_l$ are the mass matrix and the centrifugal and Coriolis force vector that result if the effect of the longitudinal displacement u_g caused by the bending deformation is neglected. Using equations (103–113), it can be shown that

$$\mathbf{M}_l = \int_V \rho \begin{bmatrix} \mathbf{M}_{lRR} & \mathbf{M}_{lRf} \\ \mathbf{M}_{lfR} & \mathbf{M}_{lff} \end{bmatrix} = \int_V \rho \begin{bmatrix} 1 & 0 & \mathbf{S}_1 \cos\theta & -\mathbf{S}_2 \sin\theta \\ 0 & 1 & \mathbf{S}_1 \sin\theta & \mathbf{S}_2 \cos\theta \\ \mathbf{S}_1^{\mathrm{T}} \cos\theta & \mathbf{S}_1^{\mathrm{T}} \sin\theta & \mathbf{S}_1^{\mathrm{T}} \mathbf{S}_1 & \mathbf{0} \\ -\mathbf{S}_2^{\mathrm{T}} \sin\theta & \mathbf{S}_2^{\mathrm{T}} \cos\theta & \mathbf{0} & \mathbf{S}_2^{\mathrm{T}} \mathbf{S}_2 \end{bmatrix} \mathrm{d}V, \tag{115}$$

$$(\mathbf{Q}_v)_l = \begin{bmatrix} (Q_v)_{lR_x} \\ (Q_v)_{lR_y} \\ (\mathbf{Q}_v)_{lf1} \\ (\mathbf{Q}_v)_{lf2} \end{bmatrix} = \begin{bmatrix} (\mathbf{Q}_v)_{lR} \\ (\mathbf{Q})_{lf} \end{bmatrix} = \dot{\theta}^2 \begin{bmatrix} \mathbf{L}_{0R} + \mathbf{M}_{lRf} \mathbf{q}_f \\ \mathbf{L}_{0f} + \mathbf{M}_{lff} \mathbf{q}_f \end{bmatrix} + \dot{\theta} \begin{bmatrix} \mathbf{L}_{1R} \dot{\mathbf{q}}_f \\ \mathbf{L}_{1f} \dot{\mathbf{q}}_f \end{bmatrix}, \tag{116}$$

where

$$\mathbf{L}_{0R} = \int_V \rho \begin{bmatrix} x \cos\theta \\ x \sin\theta \end{bmatrix} \mathrm{d}V, \qquad \mathbf{L}_{1R} = 2 \int_V \rho \begin{bmatrix} \mathbf{S}_1 \sin\theta & \mathbf{S}_2 \cos\theta \\ -\mathbf{S}_1 \cos\theta & \mathbf{S}_2 \sin\theta \end{bmatrix} \mathrm{d}V, \tag{117}$$

$$\mathbf{L}_{0f} = \int_V \rho \begin{bmatrix} x\mathbf{S}_1^{\mathrm{T}} \\ \mathbf{0} \end{bmatrix} \mathrm{d}V, \qquad \mathbf{L}_{1f} = 2 \int_V \rho \begin{bmatrix} \mathbf{0} & \mathbf{S}_1^{\mathrm{T}} \mathbf{S}_2 \\ -\mathbf{S}_2^{\mathrm{T}} \mathbf{S}_1 & \mathbf{0} \end{bmatrix} \mathrm{d}V. \tag{118}$$

The non-linear matrix $\mathbf{M}_g$ and vector $(\mathbf{Q}_v)_g$ can also be written more explicitly as

$$\mathbf{M}_g = \begin{bmatrix} \mathbf{0} & \mathbf{M}_{gRf} \\ \mathbf{M}_{gfR} & \mathbf{M}_{gff} \end{bmatrix} = \int_V \rho \begin{bmatrix} 0 & 0 & \mathbf{0} & \mathbf{q}_{f2}^{\mathrm{T}} \mathbf{B} \cos\theta \\ 0 & 0 & \mathbf{0} & \mathbf{q}_{f2}^{\mathrm{T}} \mathbf{B} \sin\theta \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}_1^{\mathrm{T}} \mathbf{q}_{f2}^{\mathrm{T}} \mathbf{B} \\ \mathbf{B} \mathbf{q}_{f2} \cos\theta & \mathbf{B} \mathbf{q}_{f2} \sin\theta & \mathbf{B} \mathbf{q}_{f2} \mathbf{S}_1 & \mathbf{B} \mathbf{q}_{f2} \mathbf{q}_{f2}^{\mathrm{T}} \mathbf{B} \end{bmatrix} \mathrm{d}V, \tag{119}$$

$$(\mathbf{Q}_v)_g = \begin{bmatrix} (Q_v)_{gR_x} \\ (Q_v)_{gR_y} \\ (\mathbf{Q}_v)_{gf1} \\ (\mathbf{Q}_v)_{gf2} \end{bmatrix} = \begin{bmatrix} (\mathbf{Q}_v)_{gR} \\ (\mathbf{Q}_v)_{gf} \end{bmatrix} = \dot{\theta}^2 \begin{bmatrix} \mathbf{G}_{1R} \mathbf{q}_f \\ \mathbf{G}_{1f} \mathbf{q}_f \end{bmatrix} + \dot{\theta} \begin{bmatrix} \mathbf{G}_{2R} \dot{\mathbf{q}}_f \\ \mathbf{G}_{2f} \dot{\mathbf{q}}_f \end{bmatrix} + \begin{bmatrix} \mathbf{G}_{3R} \dot{\mathbf{q}}_f \\ \mathbf{G}_{3f} \dot{\mathbf{q}}_f \end{bmatrix}, \tag{120}$$

where

$$\mathbf{G}_{1R} = \frac{1}{2}\int_V \rho\begin{bmatrix} \mathbf{0} & \mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}\cos\theta \\ \mathbf{0} & \mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}\sin\theta \end{bmatrix}\mathrm{d}V, \qquad \mathbf{G}_{2R} = 2\int_V \rho\begin{bmatrix} \mathbf{0} & \mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}\sin\theta \\ \mathbf{0} & -\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B}\cos\theta \end{bmatrix}\mathrm{d}V, \tag{121}$$

$$\mathbf{G}_{3R} = \int_V \rho\begin{bmatrix} \mathbf{0} & -\dot{\mathbf{q}}_{f2}^{\mathrm{T}}\mathbf{B}\cos\theta \\ \mathbf{0} & -\dot{\mathbf{q}}_{f2}^{\mathrm{T}}\mathbf{B}\sin\theta \end{bmatrix}\mathrm{d}V, \tag{122}$$

$$\mathbf{G}_{1f} = \int_V \rho\begin{bmatrix} \mathbf{0} & \frac{1}{2}\mathbf{S}_1^{\mathrm{T}}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B} \\ \mathbf{B}\mathbf{q}_{f2}\mathbf{S}_1 & x\mathbf{B} + \frac{1}{2}\mathbf{B}\mathbf{q}_{f2}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B} \end{bmatrix}\mathrm{d}V, \tag{123}$$

$$\mathbf{G}_{2f} = -2\int_V \rho\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2^{\mathrm{T}}\mathbf{q}_{f2}^{\mathrm{T}}\mathbf{B} - \mathbf{B}\mathbf{q}_{f2}\mathbf{S}_2 \end{bmatrix}\mathrm{d}V, \qquad \mathbf{G}_{3f} = \int_V \rho\begin{bmatrix} \mathbf{0} & -\mathbf{S}_1^{\mathrm{T}}\dot{\mathbf{q}}_{f2}\mathbf{B} \\ \mathbf{0} & -\mathbf{B}\mathbf{q}_{f2}\dot{\mathbf{q}}_{f2}^{\mathrm{T}}\mathbf{B} \end{bmatrix}\mathrm{d}V. \tag{124}$$

Note the dependence of the matrix $\mathbf{M}_g$ and the vector $(\mathbf{Q}_v)_g$ on the matrix $\mathbf{B}$ that appears in the equation of the longitudinal displacement caused by the bending deformation.

14. DYNAMIC EQUATIONS OF THE TRANSLATING AND ROTATING BEAM

In this section, the dynamic equations of motion of the rotating and translating beam model used in this investigation are developed. This model will be used to examine the effect of the coupling between the rigid body and deformation modes on the beam dynamics and stability. As in the preceding sections, one considers a beam model in which one mode of vibration is used to describe the axial displacement, and one mode of vibration is used to describe the transverse deformation. This is in addition to two rigid body modes which describe the arbitrary translation of the beam co-ordinate system. This translation is described by the displacement R_x and R_y. Using this model, the vectors $\mathbf{q}_{f1}$ and $\mathbf{q}_{f2}$ reduce to the scalars q_{f1} and q_{f2}. One assumes that the shape functions associated with these two co-ordinates are, respectively, given by

$$S_1 = \xi, \qquad S_2 = 3\xi^2 - \xi^3. \tag{125}$$

In this case, the matrix $\mathbf{B}$ reduces also to the function B defined as

$$B = -(9/l)(\tfrac{4}{3}\xi^3 - \xi^4 + \tfrac{1}{5}\xi^5), \tag{126}$$

Using equations (125) and (126), it can be shown that the vectors and matrices presented in the preceding section can be written as

$$\mathbf{M}_l = \begin{bmatrix} \mathbf{M}_{lRR} & \mathbf{M}_{lRf} \\ \mathbf{M}_{lfR} & \mathbf{M}_{lff} \end{bmatrix} = m\begin{bmatrix} 1 & 0 & \frac{1}{2}\cos\theta & -\frac{3}{4}\sin\theta \\ 0 & 1 & \frac{1}{2}\sin\theta & \frac{3}{4}\cos\theta \\ \frac{1}{2}\cos\theta & \frac{1}{2}\sin\theta & \frac{1}{3} & 0 \\ -\frac{3}{4}\sin\theta & \frac{3}{4}\cos\theta & 0 & \frac{33}{35} \end{bmatrix}, \tag{127}$$

$$\mathbf{L}_{0R} = \frac{ml}{2}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad \mathbf{L}_{1R} = m\begin{bmatrix} \sin\theta & \frac{3}{2}\cos\theta \\ -\cos\theta & \frac{3}{2}\sin\theta \end{bmatrix}, \tag{128}$$

$$\mathbf{L}_{0f} = \frac{ml}{3}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{L}_{1f} = \frac{11m}{10}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{129}$$

$$\mathbf{M}_g = \begin{bmatrix} \mathbf{M}_{gRR} & \mathbf{M}_{gRf} \\ \mathbf{M}_{gfR} & \mathbf{M}_{gff} \end{bmatrix} = m\left(\frac{q_{f2}}{l}\right)\begin{bmatrix} 0 & 0 & 0 & -\frac{3}{2}\cos\theta \\ 0 & 0 & 0 & -\frac{3}{2}\sin\theta \\ 0 & 0 & 0 & -\frac{81}{70} \\ -\frac{3}{2}\cos\theta & -\frac{3}{2}\sin\theta & -\frac{81}{70} & \frac{1704}{385}(q_{f2}/l) \end{bmatrix}, \quad (130)$$

$$\mathbf{G}_{1R} = -\frac{3m}{4}\left(\frac{q_{f2}}{l}\right)\begin{bmatrix} 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}, \qquad \mathbf{G}_{2R} = 3m\left(\frac{q_{f2}}{l}\right)\begin{bmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \end{bmatrix}, \quad (131)$$

$$\mathbf{G}_{3R} = \tfrac{3}{2}m\left(\frac{\dot{q}_{f2}}{l}\right)\begin{bmatrix} 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}, \qquad \mathbf{G}_{1f} = m\begin{bmatrix} 0 & -\frac{81}{140}(q_{f2}/l) \\ -\frac{81}{70}(q_{f2}/l) & -\frac{81}{70} + \frac{852}{385}(q_{f2}/l)^2 \end{bmatrix}, \quad (132, 133)$$

$$\mathbf{G}_{2f} = m\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{G}_{3f} = m\begin{bmatrix} 0 & \frac{81}{70}(\dot{q}_{f2}/l) \\ 0 & -\frac{1704}{385}(\dot{q}_{f2}/l)(q_{f2}/l) \end{bmatrix}. \quad (134)$$

14.1. EQUATIONS OF MOTION

The virtual work of the elastic forces is defined as

$$\delta W_s = -\mathbf{q}^{\mathrm{T}}\mathbf{K}\,\delta\mathbf{q}, \quad (135)$$

where the stiffness matrix $\mathbf{K}$ is defined as

$$\mathbf{K} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{K}_{ff} \end{bmatrix}. \quad (136)$$

The matrix $\mathbf{K}_{ff}$ is the same matrix used for the consistent complete model discussed in the preceding sections. This matrix is constant despite the fact that a non-linear strain–displacement relationship is used to formulate the strain energy.

If one assumes that there are no external forces acting on the beam, the virtual work done by the inertia forces is equal to that done by the elastic forces. This will lead to the matrix equation of the beam motion, which can be written as

$$(\mathbf{M}_l + \mathbf{M}_g)\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = (\mathbf{Q}_v)_l + (\mathbf{Q}_v)_g. \quad (137)$$

Substituting equations (127) and (130) into equation (137) leads to the following acceleration equations:

$$\mathbf{M}_{lRR}\ddot{\mathbf{R}} + (\mathbf{M}_{lRf} + \mathbf{M}_{gRf})\ddot{\mathbf{q}}_f = (\mathbf{Q}_v)_{lR} + (\mathbf{Q}_v)_{gR}, \quad (138)$$

$$(\mathbf{M}_{lfR} + \mathbf{M}_{gfR})\ddot{\mathbf{R}} + (\mathbf{M}_{lff} + \mathbf{M}_{gff})\ddot{\mathbf{q}}_f = -\mathbf{K}_{ff}\mathbf{q}_f + (\mathbf{Q}_v)_{lf} + (\mathbf{Q}_v)_{gf}. \quad (139)$$

Equations (138) and (139) consist of four non-linear coupled second order ordinary differential equations which can be solved using direct numerical integration methods.

15. COUPLING BETWEEN RIGID BODY AND DEFORMATION MODES

The equations of motion of the rotating and translating beam presented in the preceding section are highly non-linear if all the non-linear terms resulting from including the effect of the longitudinal displacement caused by the bending deformation are considered. It was demonstrated, however, by the numerical results presented in the preceding sections that the effect of the higher order terms can be neglected when the inertia forces are formulated. In this case one obtains a set of linearized equations which can be used to obtain an accurate solution for the dynamics of the rotating beam. By neglecting the second and higher order non-linear terms, the equations of motion of the beam can be written in the form

$$\ddot{\mathbf{R}} = \begin{bmatrix} \ddot{R}_x \\ \ddot{R}_y \end{bmatrix} = \dot{\theta}^2\mathbf{H}_{1R} + (\dot{\theta}^2\mathbf{H}_{2R} + \mathbf{H}_{3R})\mathbf{q}_f + \dot{\theta}\mathbf{H}_{4R}\dot{\mathbf{q}}_f, \tag{140}$$

$$\ddot{\mathbf{q}}_f = \begin{bmatrix} \ddot{q}_{f1} \\ \ddot{q}_{f2} \end{bmatrix} = \dot{\theta}^2\mathbf{H}_{1f} + (\dot{\theta}^2\mathbf{H}_{2f} + \mathbf{H}_{3f})\mathbf{q}_f + \dot{\theta}\mathbf{H}_{4f}\dot{\mathbf{q}}_f, \tag{141}$$

where

$$\mathbf{H}_{1R} = (1/m)(\mathbf{L}_{0R} - \mathbf{M}_{lRf}\mathbf{H}_{1f}), \qquad \mathbf{H}_{2R} = (1/m)\mathbf{M}_{lRf}(\mathbf{I} - \mathbf{H}_{2f}), \tag{142}$$

$$\mathbf{H}_{3R} = -(1/m)\mathbf{M}_{lRf}\mathbf{H}_{3f}, \qquad \mathbf{H}_{4R} = (1/m)(\mathbf{L}_{1R} - \mathbf{M}_{lRf}\mathbf{H}_{4f}), \tag{143}$$

$$\mathbf{H}_{1f} = [\mathbf{M}_{lff} - (1/m)\mathbf{M}_{lfR}\mathbf{M}_{lRf}]^{-1}(\mathbf{L}_{0f} - (1/m)\mathbf{M}_{lfR}\mathbf{L}_{0R}), \tag{144}$$

$$\mathbf{H}_{2f} = \mathbf{I} + [\mathbf{M}_{lff} - (1/m)\mathbf{M}_{lfR}\mathbf{M}_{lRf}]^{-1}\mathbf{G}_{lf}, \tag{145}$$

$$\mathbf{H}_{3f} = -[\mathbf{M}_{lff} - (1/m)\mathbf{M}_{lfR}\mathbf{M}_{lRf}]^{-1}\mathbf{K}_{ff}, \tag{146}$$

$$\mathbf{H}_{4f} = [\mathbf{M}_{lff} - (1/m)\mathbf{M}_{lfR}\mathbf{M}_{lRf}]^{-1}(\mathbf{L}_{1f} - (1/m)\mathbf{M}_{lfR}\mathbf{L}_{1R}), \tag{147}$$

in which $\mathbf{I}$ is the identity matrix and the matrix $\mathbf{G}_{1f}$ is reduced in this case to

$$\mathbf{G}_{1f} = \int_V \rho \begin{bmatrix} 0 & 0 \\ 0 & xB \end{bmatrix} \mathrm{d}V.$$

The vectors and matrices in the preceding equation can be written more explicitly for the beam model used in this investigation as

$$\mathbf{H}_{1R} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{H}_{2R} = \begin{bmatrix} 0 & -\frac{162}{71}\sin\theta \\ 0 & \frac{162}{71}\cos\theta \end{bmatrix}, \tag{148}$$

$$\mathbf{H}_{3R} = \begin{bmatrix} (6k_{11}/m)\cos\theta & -\frac{140}{71}(k_{22}/m)\sin\theta \\ (6k_{11}/m)\sin\theta & \frac{140}{71}(k_{22}/m)\cos\theta \end{bmatrix}, \qquad \mathbf{H}_{4R} = \begin{bmatrix} \frac{22}{71}\sin\theta & -\frac{3}{5}\cos\theta \\ -\frac{22}{71}\cos\theta & -\frac{3}{5}\sin\theta \end{bmatrix}, \tag{149}$$

$$\mathbf{H}_{1f} = l\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{H}_{2f} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{145}{71} \end{bmatrix}, \tag{150}$$

$$\mathbf{H}_{3f} = -\begin{bmatrix} 12k_{11}/m & 0 \\ 0 & \frac{560}{213}k_{22}/m \end{bmatrix}, \qquad \mathbf{H}_{4f} = \begin{bmatrix} 0 & \frac{21}{5} \\ -\frac{196}{213} & 0 \end{bmatrix}. \tag{151}$$

It can be shown that if the effect of the axial displacement is neglected, the equation for the in-plane transverse deformation is given by

$$\ddot{q}_{f2} = (\dot{\theta}^2 - \beta^2)q_{f2}, \tag{152}$$

where $\beta = 1{\cdot}6214\sqrt{k_{22}/m}$. It is clear that this equation leads to an unstable solution if $\dot{\theta} > 1{\cdot}6214\sqrt{k_{22}/m}$. Comparing this stability result with stability result presented by equation (59), one can show that the reference translation increases the stability range.

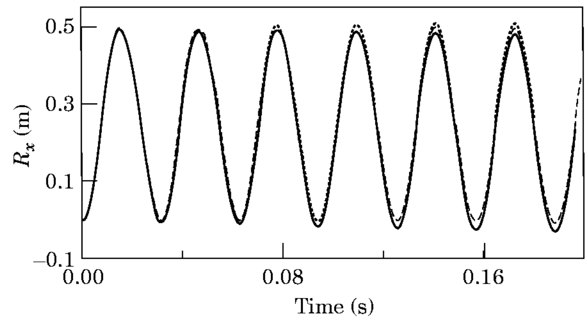

Figure 16. R_x displacement when the angular velocity $\dot{\theta} = 200$ rad/s: ——, LFM; · · · ·, NFM; –––, RBM.

15.1. NUMERICAL STUDY

In this section, the beam model and the initial conditions described in the first part of this paper are used to examine the effect of the longitudinal displacement u_g caused by the bending deformation on the coupling between the rigid body and the deformation modes. If the effect of the axial displacement is neglected, one obtains an unstable solution if the angular velocity $\dot{\theta} > 284{\cdot}4$ rad/s, as demonstrated by equation (152). In the numerical investigation presented in this section, three models are considered. The first model is the *rigid body model* (RBM) in which the rotating and translating beam is assumed to be rigid. The equations of motion of this model are presented in section 12 of this paper. The second model, referred to as the *linear flexible model* (LFM), does not include the effect of the longitudinal displacement caused by the bending deformation. In this model u_g is assumed to be equal to zero and a linear strain displacement relationship is used to formulate the constant stiffness matrix coefficients. In the third model, which is referred to as the *non-linear flexible model* (NFM), the effect of the longitudinal displacement u_g caused by bending is considered using the consistent complete model described in section 8.

Figures 16 and 17 show the results obtained for the reference motion using the three models (RBM, LFM, NFM) when the angular velocity of the beam is equal to 200 rad/s. It is clear from the results presented in these two figures that the elastic deformation does not have a significant effect on the rigid body motion of the beam at this low value of the angular velocity. Figures 18 and 19 show the results obtained using the NFM and RBM when the angular velocity is equal to 1000 rad/s. For this high value of the angular velocity ($\dot{\theta} > 284{\cdot}4$), the LFM has an unstable solution. The results presented in Figures 18 and 19 demonstrate that the reference motion of the beam is stable when the effect of the

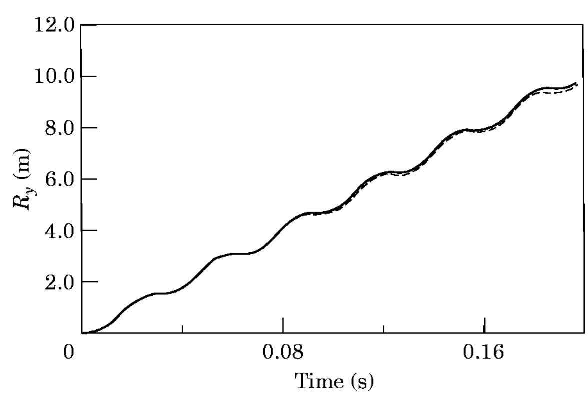

Figure 17. R_y displacement when the angular velocity $\dot{\theta} = 200$ rad/s; key as Figure 16.

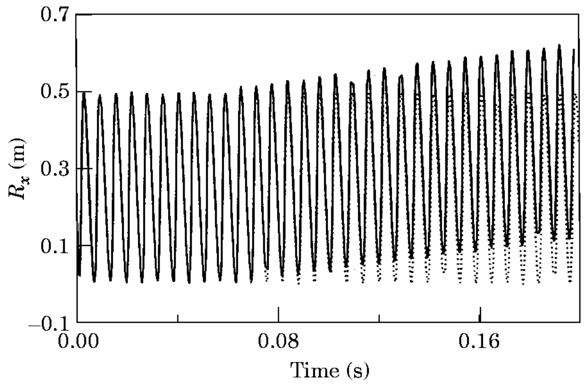

Figure 18. R_x displacement when the angular velocity $\dot{\theta} = 1000$ rad/s; ——, NFM; · · · ·, RBM.

longitudinal displacement due to bending is considered. As a consequence, the stability of the elastic modes leads to the stability of the rigid body modes of the simple rotating beam model used in this investigation.

In order to examine the effect of the reference motion on the elastic deformation of the beam, another model in which the reference motion of the beam is assumed to be equal to zero is considered. This model will be referred to as NRFM. Figures 20 and 21 show a comparison between the results obtained using the three different models (LFM, NFM, NRFM). In Figure 20, the axial displacement co-ordinate q_{f1} is plotted as a function of time when the angular velocity is equal to 200 rad/s. It is clear from the results presented in this figure that the reference motion and the non-linear terms do not have significant effect on the axial displacement at this low value of the angular velocity. In Figure 21, the in-plane transverse deformation co-ordinate is plotted as a function of time for the same value of the angular velocity. It is clear from this figure that the results obtained using the NFM lead to a lower oscillation amplitude as compared to the other models. Figures 22 and 23 show the numerical results obtained when the angular velocity is increased to 1000 rad/s.

As demonstrated by the numerical results presented in this section, the stability of the elastic modes resulting from the use of the effect of the longitudinal displacement caused by the bending deformation leads to the stability of the rigid body modes. It can be also demonstrated that the rigid body modes remain stable if the effect of the longitudinal displacement due to the bending deformation is consistently neglected when

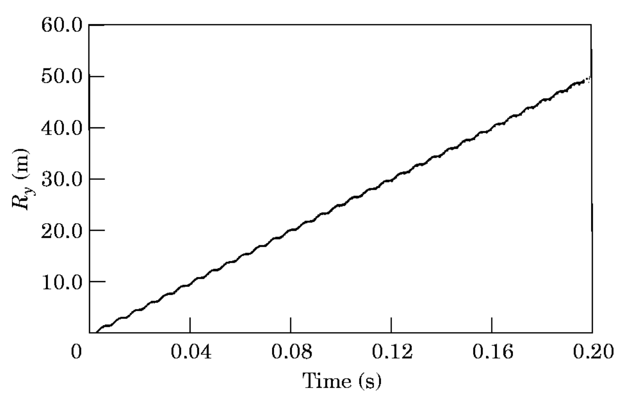

Figure 19. R_y displacement when the angular velocity $\dot{\theta} = 1000$ rad/s; key as Figure 18.

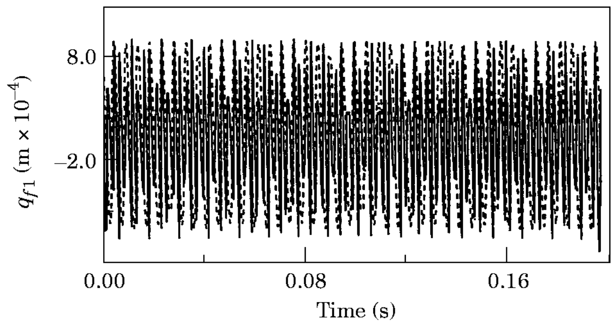

Figure 20. Axial deformation when the angular velocity $\dot{\theta} = 200$ rad/s; ——, LFM; · · · · , NFM; –––, NRFM.

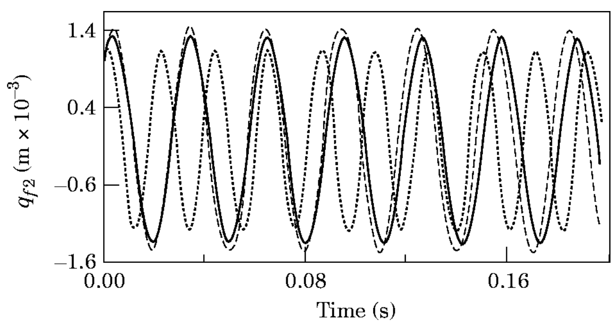

Figure 21. In-plane transverse deformation due to angular velocity $\dot{\theta} = 200$ rad/s; key as Figure 20.

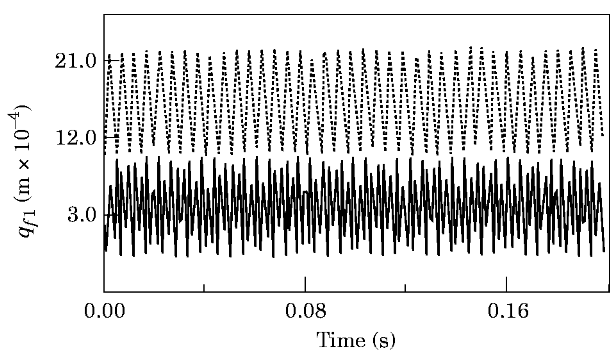

Figure 22. Axial deformation when the angular velocity $\dot{\theta} = 1000$ rad/s; ——, NFM; · · · · , NRFM.

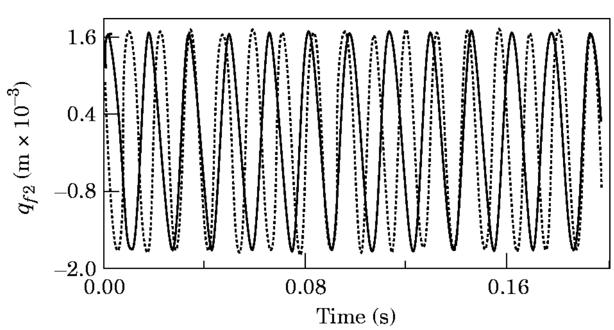

Figure 23. In-plane transverse deformation due to angular velocity $\dot{\theta} = 1000$ rad/s; key as Figure 22.

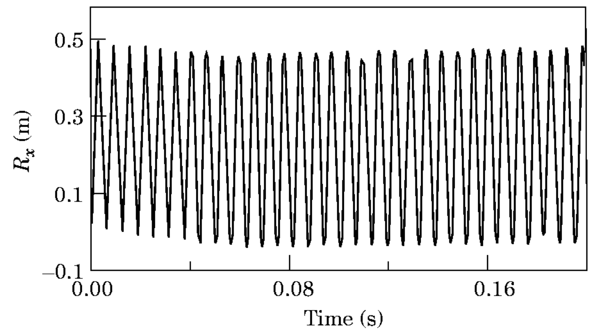

Figure 24. R_x displacement for CIM when the angular velocity $\dot{\theta} = 1000$ rad/s.

both the inertia and elastic forces are formulated. This is the case that corresponds to the consistent incomplete model discussed in section 9. Figures 24 and 25 show the reference displacements R_x and R_y as functions of time when the angular velocity of the beam is equal to 1000 rad/s. The results presented in these figures are obtained using the assumptions of the CIM discussed in section 10.

16. SUMMARY AND CONCLUSIONS

In many investigations on the rotating beam problem, it was pointed out that the use of geometric centrifugal stiffness in the inertia forces is necessary to obtain a stable solution for the beam at high values of angular velocity. It is clearly demonstrated in this paper that such an approach is not the only method which can be used to achieve beam stability in a high range of the angular velocities. This important result was demonstrated using simple beam models. A single-degree-of-freedom beam model was first used to examine the effect of higher order terms in the inertia forces on the dynamics and stability of motion of the beam. In this single-degree-of-freedom beam model, only the transverse deformation of the beam is considered. A two-degree-of-freedom model was also developed and used to examine the effect of the coupling between the axial and bending displacements. Three different non-linear models were developed in this investigation. The first model is the CCM in which the effect of the longitudinal displacement due to bending is considered in both the inertia and the elastic forces. The second model is the CIM in which the effect of the longitudinal displacement due to bending is neglected in formulating both the elastic and inertia forces. The third model is the SIM in which the

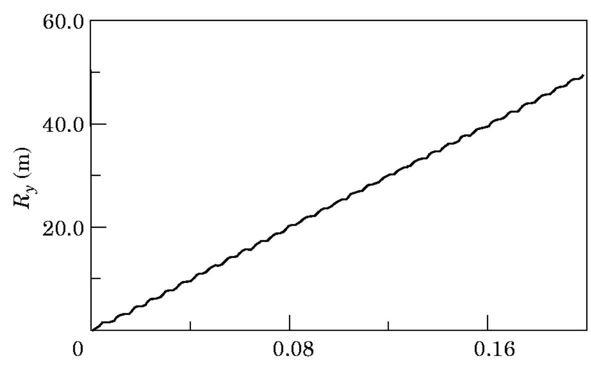

Figure 25. R_g displacement for CIM when the angular velocity $\dot{\theta} = 1000$ rad/s.

effect of the longitudinal displacement due to bending is considered in the inertia forces, but it is neglected in the elastic forces. The numerical results presented in this paper clearly demonstrate that all the three models (CCM, CIM, SIM) lead to a stable solution as the angular velocity of the beam increases. More significantly, these results show that it is not necessary to include the effect of the geometric centrifugal stiffening term in the inertia forces in order to obtain a stable solution for the rotating beam at higher values of angular velocity. If this effect is consistently neglected in both the inertia and elastic forces in a non-linear elastic model, the dynamic solution of the rotating beam equation remains stable at high values of the angular velocity.

The equations of motion that govern the dynamics of rotating and translating beams were also examined in this paper. The effect of longitudinal displacement caused by bending deformation on the dynamics and stability of the rotating and translating beams was discussed. It was shown that the reference translation of the beam increases the stability limit when the effect of axial displacement is neglected. Several models were developed and used to examine the effect of the coupling between the rigid body and deformation modes of the rotating and translating beams. It was demonstrated by the numerical results presented in this paper that the stability of the elastic modes leads to the stability of the rigid body modes at high values of the angular velocity of the beam reference. Stable solution of the elastic modes can be obtained by considering the effect of the longitudinal displacement due to bending or by using the consistent incomplete model. In both cases, the solution for the rigid body modes is stable regardless of the value of the angular velocity of the beam.

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