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FREE VIBRATION OF A RECTANGULAR PLATE CARRYING A CONCENTRATED MASS

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In a recent paper [1], the present author introduced a hybrid means to analyze the free vibration of dynamical systems which consist of a continuous structure combined with various lumped attachments. Using the assumed-modes method [2] with N component modes, the free vibration of such a combined dynamical system corresponds to the solution of a generalized eigenvalue problem of order $N \times N$, whose stiffness and mass matrices consist of diagonal matrices modified by the sum of R rank one matrices, where R correspond to the number of constraints or lumped attachments. Manipulating this generalized eigenvalue problem, the free vibration can be calculated instead by solving a much smaller characteristic determinant of order $R \times R$, leading to considerable computational advantages.

After the work was completed, reference [3] came to the attention of the author which treats the free vibration of a rectangular plate with lumped attachments. Specifically, the authors used the analytical and numerical combined method (ANCM) to determine the natural frequencies of a uniform rectangular flat plate carrying any number of point masses and grounded translational springs. Using the assumed-modes method, the eigenvalue equation for the aforementioned constrained plate is given by (see reference [3] for detailed derivation):

$$[\mathscr{K}]\bar{\boldsymbol{\eta}} = \omega^2[\mathscr{M}]\bar{\boldsymbol{\eta}}, \qquad [\mathscr{M}] = [\mathrm{I}] + \sum_{i=1}^{R} m_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}}, \qquad [\mathscr{K}] = [\Lambda] + \sum_{i=1}^{R} k_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}}. \tag{1–3}$$

In the above expressions, [I] denotes the identity matrix, [Λ] corresponds to a diagonal matrix whose ith element is given by the ith eigenvalue of the unconstrained plate, and

$$\boldsymbol{\phi}_i = [\phi_1(x_i, y_i), \ldots, \phi_j(x_i, y_i), \ldots, \phi_N(x_i, y_i)]^{\mathrm{T}}, \tag{4}$$

where $\phi_j(x_i, y_i)$ corresponds to the jth normalized eigenfunction or component mode of the unconstrained plate (without any lumped attachments) evaluated at (x_i, y_i), the location of the ith grounded translational spring–lumped mass attachment. The eigenfunctions are arranged in ascending order according to the magnitudes of the corresponding natural frequencies. Finally, note that both the stiffness and mass matrices consist of diagonal matrices modified by the sum of R rank one matrices.

The natural frequencies of the constrained plate must satisfy the following $N \times N$ characteristic determinant:

$$\det([\mathscr{K}] - \omega^2[\mathscr{M}]) = \det\left([\Lambda] + \sum_{i=1}^{R} k_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}} - \omega^2 \sum_{i=1}^{R} m_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}} - \omega^2[\mathrm{I}]\right)$$

$$= \det\left([\Lambda] + \sum_{i=1}^{R} \sigma_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}} - \omega^2[\mathrm{I}]\right) = 0, \tag{5}$$

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where $\sigma_i = k_i - m_i\omega^2$. By rearranging equation (5), one can show that the eigenvalues, ω^2, of the constrained plate are given by the zeros of the product of the following characteristic determinants of order $N \times N$:

$$\det([\Lambda] - \omega^2[\mathrm{I}]) \det\left([\mathrm{I}] + \sum_{i=1}^{R} \sigma_i([\Lambda] - \omega^2[\mathrm{I}])^{-1}\boldsymbol{\phi}_i\boldsymbol{\phi}_i^{\mathrm{T}}\right) = 0. \tag{6}$$

After some lengthy algebra, equation (6) can be shown to be identical to

$$\det([\Lambda] - \omega^2[\mathrm{I}]) \det[B] = \prod_{i=1}^{N} (\lambda_i - \omega^2) \det[B] = 0, \tag{7}$$

where the (i, j)th element of $[B]$, of size $R \times R$, is given by

$$b_{ij} = \sum_{r=1}^{N} \frac{\phi_r(x_i, y_i)\phi_r(x_j, y_j)}{\lambda_r - \omega^2} + \frac{1}{\sigma_i}\delta_i^j, \qquad i, j = 1, \ldots, R. \tag{8}$$

Note that each element of $[B]$ consists of a sum of N terms. In equations (8), $\phi_r(x_i, y_i)$ denotes the rth eigenfunction at (x_i, y_i) and δ_i^j is the Kronecker delta. For a uniform simply supported rectangular plate carrying any number of lumped masses and grounded translational springs, one can compute the natural frequencies by either solving a generalized eigenvalue problem of equation (1), of dimension $N \times N$, or by solving for the roots of the characteristic determinant of equation (7), of dimension $R \times R$. For $R \ll N$, equation (7) proves more computationally efficient to solve.

Consider a constrained rectangular plate with only a concentrated mass, m, at (x_c, y_c). Then equation (7) reduces to

$$\prod_{i=1}^{N} (\lambda_i - \omega^2)\left(1 - m\omega^2 \sum_{i=1}^{N} \frac{\phi_i^2(x_c, y_c)}{\lambda_i - \omega^2}\right) = 0. \tag{9}$$

If the constrained plate is simply supported, the unconstrained modes and the unconstrained eigenvalues are easily obtained from existing literature [4]. Then equation (9) becomes

$$\prod_{i=1}^{N} (\lambda_i - \omega^2)\left(1 - m\omega^2 \sum_{p=1}^{r} \sum_{q=1}^{s} \frac{4(\sin^2 p\pi x_c/a)(\sin^2 q\pi y_c/b)}{\rho ab(\pi^4[(p/a)^2) + (q/b)^2]D_E/\rho - \omega^2}\right) = 0, \tag{10}$$

where $N = r \times s$; a and b are the lengths of the plate in the x and y directions, respectively; ρ is the mass per unit area of the plate; and D_E denotes the flexural rigidity of the plate:

$$D_E = Eh^3/12(1 - v^2), \tag{11}$$

where E is the Young's modulus of the plate, h is the plate thickness and v denotes Poisson's ratio. When the lumped mass is not located at the node of any of the component modes, the eigenvalues of the constrained and unconstrained modes of the simply supported plate must be distinct; thus $\omega^2 \neq \lambda_i$, and equation (10) reduces to

$$1 - m\omega^2 \sum_{p=1}^{r} \sum_{q=1}^{s} \frac{4(\sin^2 p\pi x_c/a)(\sin^2 q\pi y_c/b)}{\rho ab(\pi^4[(p/a)^2 + (q/b)^2]D_E/\rho - \omega^2)} = 0, \tag{12}$$

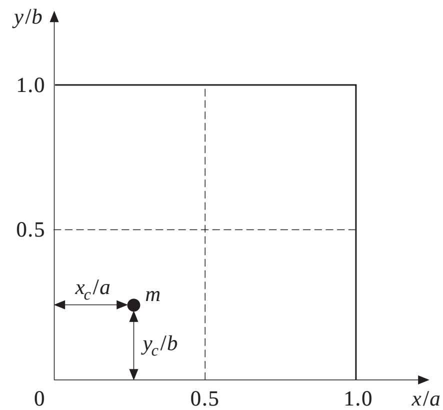

Figure 1. A uniform square simply supported plate carrying a concentrated mass $m = 50$ kg located at $x_c/a = 0{\cdot}25$ and $y_c/b = 0{\cdot}25$.

Rearranging equation (12) yields

$$1 = \frac{4m\omega^2}{abD_E} \sum_{p=1}^{r} \sum_{q=1}^{s} \frac{(\sin^2 p\pi x_c/a)(\sin^2 q\pi y_c/b)}{\pi^4[(p/a)^2 + (q/b)^2] - \omega^2\rho/D_E}. \tag{13}$$

In the limit as r and s approach infinity, the *exact* solution of a simply supported rectangular plate carrying a concentrated mass is recovered [5]. It should be emphasized again that the hybrid approach requires only the eigenfunctions and eigenvalues of a simply supported plate, which are readily obtainable. This approach leads to the exact solution, and is much simpler than solving the problem analytically (see reference [5] for detailed derivation).

To validate the present approach, the natural frequencies of a uniform simply supported square plate carrying a lumped mass as shown in Figure 1 (Figure 2 of reference [3]) are determined. The system parameters are identical to those used in reference [3] (see Table 2 of reference [3]), namely: $m = 50$ kg, x_c, $y_c = 0{\cdot}5$ m, $0{\cdot}5$ m, a, $b = 2$ m, 2 m, $h = 0{\cdot}005$ m, $E = 2{\cdot}051 \times 10^{11}$ N/m^2, $\rho = 39{\cdot}25$ kg/m^2, $\nu = 0{\cdot}3$. Since the lumped mass location lies on the nodal line of the square simply supported plate [4], equation (10) instead of (12) must be used to calculate the natural frequencies of the constrained plate (simply supported square plate with lumped mass) [1]. Table 1 lists the first five natural frequencies of

TABLE 1

The first five natural frequencies of Figure 1. *The system parameters are* $m = 50\,kg$, $x_c, y_c = 0{\cdot}5\,m$, $0{\cdot}5\,m$, a, $b = 2\,m$, $2\,m$, $h = 0{\cdot}005\,m$, $E = 2{\cdot}051 \times 10^{11}\,N/m^2$, $\rho = 39{\cdot}25\,kg/m^2$, $\nu = 0{\cdot}3$ (*identical to those of Table* 2 *of reference* [3]). *For the ANCM and the hybrid approaches*, 30 *modes are used*.

Natural frequency	ANCM [3]	Hybrid [1]	Exact [5]
1	31·81399	31·82478	31·82478
2	63·23190	63·31816	63·31816
3	95·41475	95·41495	95·41495
4	127·61601	127·74139	127·74139
5	180·59301	180·67666	180·67666

Figure 1, obtained by using ANCM [3], the hybrid approach [1] and the exact formulation [5]. Note that the third natural frequency of the constrained plate corresponds to ω_{12} of the unconstrained plate (the simply supported square plate without any lumped mass) [4]. For $r = 6$, $s = 5$, $N = r \times s = 30$, note that the hybrid approach leads to the exact solution, while the ANCM method leads to natural frequencies which are slightly lower than the exact results.

In brief, the hybrid approach described in reference [1] can also be used to analyze the free vibration of a simply supported rectangular plate carrying a concentrated mass. The approach can also be easily extended to study the free vibration of a rectangular plate with various boundary conditions (including clamped, free and simply supports) carrying any number of point masses and grounded translational springs.

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