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A RATIONAL HARMONIC BALANCE APPROXIMATION FOR THE DUFFING EQUATION OF MIXED PARITY

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Interesting analysis has been reported by Mickens and Semwogerere [1] recently recommending a rational function,

$$x(t) = A\cos\omega t/(1 + B\cos 2\omega t), \tag{1}$$

for the non-linear one-dimensional oscillator differential equation,

$$\ddot{x} + f(x) = 0, \tag{2}$$

where $f(x)$ is an analytic function of x at $x = 0$ and

$$x(0) = x_0 \not\equiv 0, \qquad \dot{x}(0) = 0. \tag{3}$$

Here ω is the angular frequency, x_0 is the maximum amplitude and overdots denote differentiation with respect to time, t. They have examined a particular case of the function, $f(x) = x^3$ to conclude that the form of the solution (1) provides an excellent approximation to the actual solution of the equation (2) which is, however, true for $f(x) = x^3$ or odd functions. When $f(x)$ is not an odd function, the approximate periodic solution (1) for the equation of motion (2) needs a modification.

Most of the one-dimensional oscillators that occur in practical applications have functions $f(x)$ that are polynomial [2] and hence they are analytic at $x = 0$. In order to demonstrate the necessity to modify the expression (1), the well known Duffing equation will be considered in which the restoring force function, $f(x)$, is of the form [3–9]:

$$f(x) = \alpha x + \beta x^2 + \gamma x^3 + \delta \tag{4}$$

The approximate periodic solution (1) for the equation of motion (2) having $f(x)$ in the form (4) gets modified to

$$x(t) = (C + A\cos\omega t)/(1 + B\cos 2\omega t). \tag{5}$$

After the use of trigonometric identities and application of the method of harmonic balance to retain only constant terms and terms involving $\cos\omega t$, $\cos 2\omega t$ and $\cos 3\omega t$, four equations are obtained. From these equations, one obtains

$$\omega^2 = [(1 + B + \tfrac{1}{2}B^2)\alpha + 2C(2 + B)\beta + 3(\tfrac{1}{4}A^2 + C^2)\gamma]/(1 + B - \tfrac{11}{2}B^2), \tag{6}$$

$$C(4 - 10B^2)\alpha + [C^2(4 - 6B^2) + A^2(2 - 2B - 3B^2)]\beta + C[4C^2 + (6 - 9B)A^2]\gamma + (4 - 12B^2 - \tfrac{9}{2}B^4)\delta = 0 \tag{7}$$

0022–460X/97/440597 + 03 $25.00/0/sv971178

$$3B(4-3B)\,[2(2+2B+B^2)\alpha+4C(2+B)\beta+3(A^2+4C^2)\gamma]$$
$$+\,2(2+2B-11B^2)\,[B(4+B)\alpha+4BC\beta+A^2\gamma]=0 \qquad (8)$$

The fourth equation for the four unknowns ω, A, B and C is provided by using the initial conditions (3) in (5) as

$$A-(1+B)x_0+C=0 \qquad (9)$$

The equations (7)–(9) are solved numerically using the Newton–Raphson's iterative procedure with the initial guess values $A = x_0$; $B = \gamma A^2/(16\alpha + 10\gamma A^2)$; and $C = [2\beta A^2(B-1) - 4\delta]/(4\alpha + 6\gamma A^3)$.

To verify the adequacy of the proposed approximate periodic solution (5) for the Duffing equation of mixed parity, the following three cases:

(1) $\alpha = 0$, $\beta = 0$, $\gamma = 1$, $\delta = 0$; (Mickens [4]).
(2) $\alpha = 1$, $\beta = -0{\cdot}2$, $\gamma = 0$, $\delta = -1$; (Mickens [5], Rao and Rao [6])
(3) $\alpha = 1$, $\beta = -2{\cdot}2518$, $\gamma = 2{\cdot}54328$, $\delta = 0$; (Rao [9])

have been examined. The results are presented in Table 1.

It can be seen from Table 1 that in case (1) the solution (5) with $C = 0$ is good, that is, Micken's result. In case (2) the solution (5) with $B = 0$ is also good, whereas the solution (1) proposed by Mickens and Semwogerere [1] is 87% higher than the actual value of ω. In case (3), the solution (1) gives 12·5% higher than the actual value of ω whereas the solution (5), proposed herein, gives 0·5% lower than the actual value of ω. The approximate solution (5) proposed herein for the Duffing equation of mixed-parity in which the restoring force function, $f(x)$, is of the form (4), is found to be very close to the exact solutions as noticeable in all the above cases.

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Table 1

Comparison of angular frequency, ω for the amplitude, $x_0 = 3$

	Method of Ref. [1] equation (1)			Present study equation (5)				Exact solution
Case	A	B	ω	A	B	C	ω	ω
1	2·7298	−0·0901	2·5414	2·7298	−0·0901	0·0	2·5414	2·5414
2	3·0	0·0	1·0	1·2153	0.0	1·7847	0·5349	0·5349
3	2·7463	−0·0846	4·1797	2·4623	−0·0876	0·2748	3·6940	3·7138

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