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EXACT SOLUTIONS FOR THE LONGITUDINAL VIBRATION OF NON-UNIFORM RODS

B. M. Kumar and R. I. Sujith

Department of Aerospace Engineering, Indian Institute of Technology, Madras, 600036 *India*

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The objective of this paper is to present exact analytical solutions for the longitudinal vibration of rods with non-uniform cross-section. Using appropriate transformations, the equation of motion of axial vibration of a rod with varying cross-section is reduced to analytically solvable standard differential equations whose form depends upon the specific area variation. Solutions are obtained for a rod with a polynomial area variation and for a sinusoidal rod. The solutions are obtained in terms of special functions such as Bessel and Neumann as well as trignometric functions. Simple formulas to predict the natural frequencies of non-uniform rods with various end conditions are presented. The natural frequencies of non-uniform rods for these end conditions are calculated, and their dependence on taper is discussed. The governing equation for the problem is the same as that of wave propagation through ducts with non-uniform cross-sections. Therefore solutions presented here can be used to investigate such problems.

1. INTRODUCTION

The vibration of non-uniform beams and rods is a subject of considerable scientific and practical interest that has been studied extensively. A plethora of literature in this field exists and a comprehensive bibliographical account of previous works in this area will not be attempted here. The study of longitudinal vibration of non-uniform rods is important in the study of composite structures subjected to high velocity impact [1] and the study of foundations [2–4]. A study of vibration of tapered rods was undertaken by Eisenberger [5] who showed that the natural frequencies were only affected slightly by the taper. It was shown that the equation of motion of rods with conical cross-sections could be reduced to the form of a wave equation by a change of variable [6]. In a recent study, using a systematic approach, Abrate [7] seeks all the possible area variations for which exact solutions for the problem can be obtained. He obtained closed form solution for rods whose cross-section varies as $A(x) = A_0(1 + a[x/L])^2$. Exact analytical solutions also exist for exponential and catenoidal rods [8].

In the absence of exact solutions, the problem can be solved using approximate or numerical methods. While these approximate or numerical methods yield accurate answers, they often may not provide adequate insight into the physics of the problem. Moreover, since they cannot be implemented without the availability of a computer, it is often difficult to incorporate these solution approaches into practical design procedures. Therefore it is desirable, though often difficult to obtain exact solution to such problems. The availability of exact solutions will help in establishing the accuracy of the approximate or numerical solutions.

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2. THE EQUATION OF MOTION

The longitudinal motion of a rod with varying cross-section is governed by the differential equation [9]

$$(\partial/\partial x)[EA(x)\partial u/\partial x] = \rho A(x)\partial^2 u/\partial t^2. \tag{1}$$

Assuming a solution of the form $u(x, t) = U(x)\,\mathrm{e}^{\mathrm{i}\omega t}$, equation (1) reduces to the second order ordinary differential equation for the complex amplitude $U(x)$:

$$\mathrm{d}^2U/\mathrm{d}x^2 + (1/A)(\mathrm{d}A/\mathrm{d}x)(\mathrm{d}U/\mathrm{d}x) + \rho(\omega^2/E)U = 0. \tag{2}$$

Equation (2) has variable coefficients. Therefore, exact solutions of this equation for a general area variation $A(x)$ cannot be obtained. However, for certain specific area variations, exact solutions can be obtained. Exact solutions for uniform, conical, exponential and catenoidal rods are available in the literature [8]. In the following sections, using appropriate transformations, equation (2) will be reduced to analytically solvable differential equations for: (1) $A(x) = (ax + b)^n$ (polynomial variation) and (2) $A(x) = A_0 \sin^2(ax + b)$.

3. SOLUTION FOR POLYNOMIAL AREA VARIATIONS

In order to obtain an exact solution, equation (2) is rewritten with $A(x)$ as the independent variable [10, 11], yielding

$$(\mathrm{d}A/\mathrm{d}x)^2(\mathrm{d}^2U/\mathrm{d}A^2) + (1/A)(\mathrm{d}/\mathrm{d}x)[A\,\mathrm{d}A/\mathrm{d}x](\mathrm{d}U/\mathrm{d}A) + (\rho\omega^2/E)U = 0. \tag{3}$$

The above equation is solved for a rod with a cross-section area variation that is given by the following expression.

$$A = (ax + b)^n, \tag{4}$$

where n need not be an integer. Noting that

$$\mathrm{d}A/\mathrm{d}x = an(ax + b)^{n-1} \quad \text{and} \quad (\mathrm{d}A/\mathrm{d}x)(A[\mathrm{d}A/\mathrm{d}x]) = a^2 n(2n - 1)(ax + b)^{2n-2}, \tag{5, 6}$$

equation (3) can be re-written as

$$\mathrm{d}^2U/\mathrm{d}A^2 + ((2n - 1)n)(1/A)/(\mathrm{d}U/\mathrm{d}A) + (\rho\omega^2/Ea^2n^2)(1/A^{2-2/n})U = 0. \tag{7}$$

To simplify equation (7), the following variables w and z which replace U and A respectively are introduced:

$$U = wA^a, \qquad z = \lambda A^\sigma, \tag{8, 9}$$

where

$$\alpha = \tfrac{1}{2}(1/n - 1), \qquad \lambda = (\omega/a)\sqrt{\rho/E} \qquad \text{and} \qquad \sigma = 1/n. \tag{10}$$

Transforming equation (7) from the U–A space to the w–z space yields the ordinary differential equation

$$\mathrm{d}^2 w/\mathrm{d}z^2 + (1/z)(\mathrm{d}w/\mathrm{d}z) + (1 - v^2/z^2)w = 0, \tag{11}$$

where n is given by

$$v = (1-n)/2, \tag{12}$$

$$w = c_1 \mathrm{J}_v(z) + c_2 Y_v(z) \text{ when } v \text{ is an integer.} \tag{13a}$$

$$w = c_1 \mathrm{J}_v(z) + c_2 \mathrm{J}_{-v}(z) \text{ when } v \text{ is not an integer.} \tag{13b}$$

Therefore the axial displacement amplitude can be written as

$$U = A^{\alpha}[c_1 \mathrm{J}_v(\lambda A^{\sigma}) + c_2 Y_v(\lambda A^{\sigma})] \text{ when } v \text{ is an integer,} \tag{14a}$$

$$U = A^{\alpha}[c_1 \mathrm{J}_v(\lambda A^{\sigma}) + c_2 \mathrm{J}_{-v}(\lambda A^{\sigma})] \text{ when } v \text{ is not an integer.} \tag{14b}$$

Consider the longitudinal vibration of a rod with its cross-sectional area varying according to the equation $A(x) = ax + b$, equation (3) reduces to

$$\partial^2 U/\partial A^2 + (1/A)(\partial U/\partial A) + (\rho\omega^2/Ea^2)U = 0, \tag{15}$$

which is the Bessel's equation of the zeroth order, whose solution is given by [12, 13]

$$U = c_1 \mathrm{J}_0([\beta/a]A) + c_2 Y_0([\beta/a]A), \tag{16}$$

where $\beta = \omega\sqrt{\rho/E}$.

For the case of $A = (ax+b)^2$ the solution is

$$U = (1/A^{1/4})\,[c_1 \mathrm{J}_{1/2}([\beta/a]A^{1/2}) + c_2 \mathrm{J}_{-1/2}([\beta/a]A^{1/2})]. \tag{17}$$

This case was previously solved by Abrate [7]. Using the relation [13]

$$\mathrm{J}_{1/2}(z) = \sqrt{2/\pi z}\,\sin z, \tag{18}$$

it can easily be shown that for the case of $n = 2$, the solution given by equation (17) can be re-written in the form given by Abrate [7]:

$$U = (1/\sqrt{A})\,[c_1 \sin([\beta/a]A^{1/2}) + c_2 \cos([\beta/a]A^{1/2})] \tag{19}$$

4. NATURAL FREQUENCIES OF A NON-UNIFORM ROD—A NUMERICAL EXAMPLE

The natural frequencies of a rod with its cross-sectional area varying according to the equation

$$A = (ax+b)^4 \tag{20}$$

is discussed in this section. For a fixed–fixed rod, the boundary conditions are $u(0, t) = 0$ and $u(L, t) = 0$. This yields the set of two homogeneous algebraic equations:

$$(1/A_0^{3/8})\,[c_1 \mathrm{J}_{-3/8}([\beta/a]A_0^{1/4}) + c_2 \mathrm{J}_{3/8}([\beta/a]A_0^{1/4})] = 0, \tag{21}$$

$$(1/A_1^{3/8})\,[c_1 \mathrm{J}_{-3/8}([\beta/a]A_1^{1/4}) + c_2 \mathrm{J}_{3/8}([\beta/a]A_1^{1/4})] = 0, \tag{22}$$

where

$$A_1 = A(L) = (aL+b)^4 \tag{23}$$

TABLE 1

Non-dimensional natural frequencies of fixed–fixed rods with $A = (ax + b)^4$

	a		
Mode	0	1	2
1	3·141593	3·133487	2·386221
2	6·283185	6·278921	6·272251
3	9·424778	9·421905	9·417264
4	12·566371	12·564210	12·560670
5	15·707963	15·706230	15·703370
6	18·849556	18·848110	18·845270

Since these equations are homogeneous, they are solvable only when their determinant vanishes, which yields the relationship for the non-dimensional eigenvalue β:

$$\mathrm{J}_{-3/8}\left([\beta/a]A_0^{1/4}\right)\mathrm{J}_{3/8}\left([\beta/a]A_1^{1/4}\right) - \mathrm{J}_{3/8}\left([\beta/a]A_0^{1/4}\right)\mathrm{J}_{-3/8}\left([\beta/a]A_1^{1/4}\right) = 0 \tag{24}$$

Table 1 shows the eigenvalues for uniform rods ($a = 0$) and for tapered rods (fixed–fixed) with $a = 1, 2$, $b = 1$ and $L = 1$. The natural frequencies are presented in terms of β where $\beta = \omega\sqrt{\rho/E}$. For uniform rods, $\beta L = j\pi$, where j is an integer, the mode number. Table 1 indicates that the lowest natural frequencies are affected most by the taper. For higher modes, the natural frequencies are close to that of a uniform rod. The mode shape is given by

$$U = c_1\, A^{-3/8}\left[\mathrm{J}_{-3/8}\left(\frac{\beta}{a}A^{1/4}\right) - \left[\mathrm{J}_{-3/8}\left(\frac{\beta}{a}A_0^{1/4}\right)\Big/\mathrm{J}_{3/8}\left(\frac{\beta}{a}A_0^{1/4}\right)\right]\mathrm{J}_{3/8}\left(\beta A^{1/4}\right)\right] \tag{25}$$

The mode shape corresponding to the fourth natural frequency is plotted in Figure 1. One of the features that distinguish the mode shape of the tapered rod from that of a uniform rod is the evanescent behavior of the mode shape. The amplitude at the antinodes can be seen to be decreasing from one end to the other end.

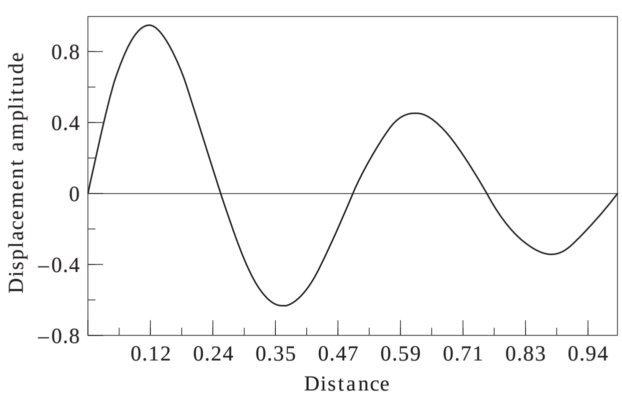

Figure 1. Mode shape corresponding to the fourth natural frequency of a fixed–fixed beam with cross-section area varying as $A(x) = (1 + x)^4$.

TABLE 2

Non-dimensional natural frequencies of fixed–free rods with $A = (ax + b)^4$

	a		
Mode	0	1	2
1	1·570796	–	–
2	4·712389	4·487482	4·404069
3	7·853982	7·721747	7·672932
4	10·995574	10·901630	10·866970
5	14·137167	14·064260	14·037360
6	17·278760	17·219170	17·197190

For fixed–free rods, the boundary conditions are $u(0, t) = 0$ and $(\partial u/\partial x)\,(L, t) = 0$ which results in the transcendental equation for the eigenfrequency.

$$
\begin{aligned}
&\mathrm{J}_{-3/8}\left(\beta \frac{A_0^{1/4}}{a}\right)\left[\frac{-3}{aL+b}\,\mathrm{J}_{3/8}\left(\beta \frac{A_1^{1/4}}{a}\right) - \beta \frac{A_0^{1/4}}{a}\,\mathrm{J}_{11/8}\left(\beta \frac{A_1^{1/4}}{a}\right) + \beta \frac{A_0^{1/4}}{a}\,\mathrm{J}_{-5/8}\left(\beta \frac{A_1^{1/4}}{a}\right)\right] \\
&\quad - \mathrm{J}_{3/8}\left(\beta \frac{A_0^{1/4}}{a}\right)\left[\frac{-3}{aL+b}\,\mathrm{J}_{-3/8}\left(\beta \frac{A_1^{1/4}}{a}\right) - \beta \frac{A_0^{1/4}}{a}\,\mathrm{J}_{5/8}\left(\beta \frac{A_1^{1/4}}{a}\right)\right. \\
&\quad \left. + \beta \frac{A_0^{/4}}{a}\,\mathrm{J}_{-11/8}\left(\beta \frac{A_1^{1/4}}{a}\right)\right] = 0. \qquad (26)
\end{aligned}
$$

Table 2 shows the eigenvalues for uniform rods ($a = 0$) and for tapered rods (fixed–free) with $a = 1$, 2 $b = 1$ and $L = 1$. For uniform rods, $\beta L = \frac{1}{2}(2j - 1)\pi$, where j is an integer, the mode number. Table 2 indicates that, for fixed–free rods, the lowest natural frequencies are affected most by the taper. For higher modes, the natural frequencies are close to that of a uniform rod. It is also interesting to note that taper reduces the natural frequency, and the first mode disappears. (The first mode is present until $a = 0{\cdot}97$).

For free–free rods, the boundary conditions are $(\partial u/\partial x)\,(0, t) = 0$ and $(\partial u/\partial x)\,(L, t) = 0$ which results in the transcendental equation for eigenfrequency

$$PS - QR = 0,$$

where

$$
\begin{aligned}
P &= -\frac{3}{b}\,\mathrm{J}_{-3/8}\left(\beta \frac{A_0^{1/4}}{a}\right) - \mathrm{J}_{5/8}\left(\beta \frac{A_0^{1/4}}{a}\right)\beta \frac{A_0^{1/4}}{a} + \mathrm{J}_{-11/8}\left(\beta \frac{A_0^{1/4}}{a}\right)\beta \frac{A_0^{1/4}}{a}, \\
Q &= -\frac{3}{aL+b}\,\mathrm{J}_{-3/8}\left(\beta \frac{A_1^{1/4}}{a}\right) - \mathrm{J}_{5/8}\left(\beta \frac{A_1^{1/4}}{a}\right)\beta \frac{A_0^{1/4}}{a} + \mathrm{J}_{-11/8}\left(\beta \frac{A_1^{1/4}}{a}\right)\beta \frac{A_0^{1/4}}{a}, \\
R &= -\frac{3}{b}\,\mathrm{J}_{3/8}\left(\beta \frac{A_0^{1/4}}{a}\right) - \mathrm{J}_{11/8}\left(\beta \frac{A_0^{1/4}}{a}\right)\beta \frac{A_0^{1/4}}{a} + \mathrm{J}_{-5/8}\left(\beta \frac{A_0^{1/4}}{a}\right)\beta \frac{A_0^{1/4}}{a}, \\
S &= -\frac{3}{aL+b}\,\mathrm{J}_{3/8}\left(\beta \frac{A_1^{1/4}}{a}\right) - \mathrm{J}_{11/8}\left(\beta \frac{A_1^{1/4}}{a}\right)\beta \frac{A_0^{1/4}}{a} + \mathrm{J}_{-5/8}\left(\beta \frac{A_1^{1/4}}{a}\right)\beta \frac{A_0^{1/4}}{a}. \qquad (27)
\end{aligned}
$$

Table 3 shows the eigenvalues for uniform rods ($a = 0$) and for tapered rods (free–free) with $a = 1, 2$, $b = 1$ and $L = 1$. For uniform rods, $\beta L = j\pi$, where j is an integer, the mode number. Table 3 indicates that, like the cases of fixed–fixed and fixed–free rods, the lowest natural frequencies are affected most by the taper. For higher modes, the natural frequencies are close to that of a uniform rod.

These observations about the natural frequencies are consistent with the observations of Abrate [7].

3. SOLUTION FOR AREA VARIATION OF THE FORM $A = A_0 \sin^2 (ax + b)$

In this section the exact solution for the longitudinal vibration of a rod with an area variation of the form

$$A = A_0 \sin^2 (ax + b) \tag{28}$$

is derived. To simplify equation (2) a new variable y is introduced:

$$y = U \sin (ax + b). \tag{29}$$

Transforming equation (2) from the U–x space to y–x space yields

$$\mathrm{d}^2 y/\mathrm{d}x^2 + [\rho\omega^2/E + a^2]y = 0 \tag{30}$$

whose solution is given by

$$y = c_1 \sin kx + c_2 \cos kx \tag{31}$$

where

$$k^2 = \rho\omega^2/E + a^2 \tag{32}$$

Therefore,

$$U = [1/\sin (ax + b)]\,[c_1 \sin kx + c_2 \cos kx] \tag{33}$$

It can also be seen that the above equation can be re-written using equation (18) as

$$U = [\sqrt{x}/\sin (ax + b)]\,[c_1\, \mathrm{J}_{1/2}(kx) + c_1\, \mathrm{J}_{-1/2}(kx)]. \tag{34}$$

For a fixed–fixed rod, the boundary conditions are $u(0, t) = 0$ and $u(L, t) = 0$. Therefore the natural frequencies of the rod are given by $kL = j\pi$, where k is given by equation (31), and m the mode number. Table 4 shows the non-dimensional eigenvalues ($\beta L = \omega\sqrt{\rho/E}$) for uniform rods ($a = 0$) and for tapered rods with $a = 1$, 2, $b = 1$ and $L = 1$.

TABLE 3

Non-dimensional natural frequencies of free–free rods with $A = (ax + b)^4$

	a		
Mode	0	1	2
1	3·141593	3·378458	3·286891
2	6·283185	6·425906	6·614998
3	9·424778	9·524152	9·671519
4	12·566371	12·642120	12·759890
5	15·707963	15·769030	15·866250
6	18·849556	18·900660	18·983120

TABLE 4

Non-dimensional natural frequencies of fixed–fixed rods with $A = A_0 \sin^2(ax + b)$

Mode	a = 0	1	2
1	3·141593	2·978189	2·422727
2	6·283185	6·203097	5·956376
3	9·424778	9·371576	9·210127
4	12·566371	12·526519	12·406195
5	15·707963	15·676100	15·580119
6	18·849556	18·823011	18·743152

In this case the mode shape is given by

$$\mathbf{U}_j(x) = c_1 \frac{\sin k_j x}{\sin(ax + b)}. \tag{35}$$

For fixed–free rods, the boundary conditions are $u(0, t) = 0$ and $(\partial u/\partial x)(L, t) = 0$ which give the following transcendental equation for the eigenfrequency.

$$[a/\tan(aL + b)] \tan kL = k. \tag{36}$$

Table 5 shows the eigenvalues for uniform rods ($a = 0$) and for tapered rods with $a = 1$, 2, $b = 1$ and $L = 1$.

For free–free rods, the boundary conditions are $(\partial u/\partial x)(0, t) = 0$ and $(\partial u/\partial x)(L, t) = 0$ which results in the transcendental equation for eigenfrequency:

$$\tan kl = [a - a \sin b/\sin(aL + b)]/[k \sin b + a^2/\sin(aL + b).] \tag{37}$$

Table 6 shows the eigenvalues for uniform rods ($a = 0$) and for tapered rods with $a = 1$, 2, $b = 1$ and $L = 1$.

In all the three cases, it can be seen that the lowest natural frequencies are affected most by taper. For higher modes, the natural frequencies are close to that of a uniform rod.

4. DISCUSSION

The authors have been able to obtain the solution to the problem for the case of a polynomial area variation, by transforming the differential equation such that the area A

TABLE 5

Non-dimensional natural frequencies of fixed–free rods with $A = A_0 \sin^2(ax + b)$

Mode	a = 0	1	2
1	1·570796	1·517638	2·148560
2	4·712389	4·702145	5·535762
3	7·853982	7·848311	8·632812
4	10·995574	10·991620	11·694640
5	14·137167	14·134120	14·757860
6	17·278760	17·276280	17·830600

TABLE 6

Non-dimensional natural frequencies of free–free rods with $A = A_0 \sin^2(ax + b)$

	a		
Mode	0	1	2
1	3·141593	3·0004297	1·5808147
2	6·283185	6·216901	5·113309
3	9·424778	9·380888	8·436760
4	12·566371	12·533530	11·721540
5	15·707963	15·681720	14·977670
6	18·849556	18·827700	18·210650

is the independent variable. The solutions are obtained as direct functions of area. It may be possible that other problems can be solved by using this approach.

One of the effects of taper on the vibration of non-uniform rods is that the amplitudes of the antinodes are not constant as it can easily be seen from Figure 1. This evanescent nature of the mode shape is reflected in the properties of Bessel functions [14]. The solutions presented in this paper are obtained in the form of Bessel functions. The solution developed earlier by Abrate [7] for a conical rod can also be expressed in the form of Bessel functions as is shown in equation (17). Therefore, it is speculated that for approximate techniques used for solving this problem, Bessel functions may be a better approximation for mode shapes than trigonometric functions.

The problem discussed in this paper is mathematically similar to the problem of sound propagation through ducts of varying cross-section, whose one-dimensional wave equation for simple harmonic time dependence is given by [15]

$$\partial^2 p/\partial x^2 + [1/A(x)]\,(\mathrm{d}A(x)/\mathrm{d}x)\,(\mathrm{d}p/\mathrm{d}x) + k^2 p = 0,$$

where p is the acoustic pressure amplitude, $A(x)$, the cross-section area, and k, the wave number. It can be seen that this equation is the same as equation (2). Therefore solutions developed for one problem can be used for the other.

5. CONCLUSIONS

Exact analytical solutions describing the longitudinal vibration of rods were obtained by transforming the equation of motion to standard differential equations which are analytically solvable in terms of special functions. Solutions are obtained for a rod with a polynomial area variation and for a sinusoidal rod. The solutions are obtained in terms of special functions such as Bessel and Neumann as well as trigonometric functions. Simple formulas to predict the natural frequencies of non-uniform beams with various end conditions are presented. It is shown that the lowest natural frequencies are affected most by the taper. For higher modes, the natural frequencies are close to that of a uniform rod. The mode shapes differ significantly from that of uniform rods. The mode shapes display an evanescent nature and this feature is easily captured by the Bessel functions. Therefore, it is speculated that for approximate techniques used for solving this problem, Bessel functions might be a better approximation for mode shapes than trigonometric functions.

The expressions obtained in this analysis are in terms of Bessel and trigonometric functions and are easy to evaluate. These closed form expressions presented herein can be used also as benchmarks for checking the results obtained from numerical or approximate methods. The governing equation for the problem is the same as that of wave propagation through ducts with non-uniform cross sections. Therefore solutions presented here can be used to investigate such problems.

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APPENDIX: NOMENCLATURE

a, b	constant in equation (4)
A	area
E	Young's Modulus
k	wave number in equation (32)
J_v	Bessel function of order v
j	mode number (integer)
L	length of the rod
n	constant in equation (4)
P, Q, R, S	variables in equation (27)
t	time
u	displacement
U	displacement amplitude
w	transformation variable in equation (8)
x	distance
y	transformation variable in equation (29)
Y_v	Neumann function of order v
z	transformation variable in equation (9)
α, λ, σ	transformation variables in equation (10)
β	non-dimensional frequency
ρ	mass per unit volume of the rod