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TRANSIENT ANALYSIS OF A COUPLED NON-LINEAR SLOW-VARIANT SYSTEM

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1. INTRODUCTION

Many physical and engineering problems have features which may be qualitatively described by coupled systems of non-linear oscillators. The natural frequencies of these systems may be combined through non-linear interactions so as to produce internal resonances. In most of the work done on the subject, the natural frequencies are assumed to be time independent, with the resonant conditions satisfied for all the time. These stationary oscillator systems have been dealt with by ordinary multi-scale methods in many articles and books, such as in references [1, 2] etc. But many problems of physical interest are governed by systems with slowly varying coefficients. The ordinary perturbation theory cannot handle this kind of problem because of mathematical difficulties. In paper [3], a one-dimensional oscillator with slowly varying frequency was discussed and an improved multi-scale method was proposed. In another expository paper, Kevorkian [4] has summarized the perturbation techniques and results for a general system of first order equations that model various weakly non-linear oscillatory motions with slowly varying parameters. Recently, Bosley [5] used the canonical averaging techniques to deal with the slowly varying oscillatory systems in Hamiltonian standard forms to very high orders and study the adiabatic invariance. Kevorkian and Bosley [6, 7] have discussed a model problem of two oscillators with weakly non-linear coupling and with either constant or slowly varying frequencies to survey perturbation techniques based on the improved multi-scale method applied to resonance problems.

In this paper, the following quadratically coupled non-linear oscillator system is discussed:

$$\begin{aligned}\ddot{x} + a^2(\varepsilon t)x &= \varepsilon y^2;\\ \ddot{y} + b^2(\varepsilon t)y &= \varepsilon x^2,\end{aligned} \tag{1}$$

where ε is a small positive quantity and $a(\varepsilon t), b(\varepsilon t) > 0$. The asymptotic solutions of this system will be derived when $a(\varepsilon t)$ and $b(\varepsilon t)$ vary slowly with time.

2. ASYMPTOTIC SOLUTION OF SYSTEM (1)

It is not difficult to see that as the ordinary multi-scale method fails in dealing with this case an improved method must be introduced instead.

2.1. *Outer expansion*

Firstly, the outer expansions of system (1) are discussed. A slow time scale $\hat{t} = \varepsilon t$ is introduced with the following two fast time scales:

$$\mu = (1/\varepsilon)\int_0^{\hat{t}} a(s)\,\mathrm{d}s, \qquad \nu = (1/\varepsilon)\int_0^{\hat{t}} b(s)\,\mathrm{d}s. \tag{2}$$

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It is assumed that the solution can be expanded in the form

$$\begin{cases} x = x_0(\mu, \hat{t}) + \varepsilon x_1(\mu, \hat{t}) + \varepsilon^2 x_2(\mu, \hat{t}) + \mathcal{O}(\varepsilon^3); \\ y = y_0(v, \hat{t}) + \varepsilon y_1(v, \hat{t}) + \varepsilon^2 y_2(v, \hat{t}) + \mathcal{O}(\varepsilon^3), \end{cases} \tag{3}$$

where the x_i's only depend on μ, $\hat{t}$ and the y_i's only depend on v, $\hat{t}$ (see [3, 6]). Substituting (2) and (3) into (1) and letting the coefficients of the same order be equal to zero, one gets

$$\begin{cases} x_0 = \hat{\alpha}_0(\hat{t}) \cos[\mu + \hat{\phi}_0]; \\ y_0 = \hat{\beta}_0(\hat{t}) \cos[\mu + \hat{\psi}_0], \end{cases} \tag{4}$$

and

$$\begin{cases} x_1 = \hat{\alpha}_1(\hat{t}) \cos[\mu + \hat{\phi}_1(\hat{t})] + \hat{\beta}_0^2(\hat{t})/(2a^2) + \{\hat{\beta}_0^2(\hat{t})/[(2(a^2 - 4b^2)]\} \cos 2(v + \hat{\psi}_0); \\ y_1 = \hat{\beta}_1(\hat{t}) \cos[v + \hat{\psi}_1(\hat{t})] + \hat{\alpha}_0^2(\hat{t})/(2b^2) + \{\hat{\alpha}_0^2(\hat{t})/[(2(b^2 - 4a^2)]\} \cos 2(\mu + \hat{\phi}_0), \end{cases} \tag{5}$$

where $\hat{\alpha}_0(\hat{t}) = \hat{\alpha}_0(0)\sqrt{a(0)/a(\hat{t})}$, $\hat{\beta}_0(\hat{t}) = \hat{\beta}_0(0)\sqrt{b(0)/b(\hat{t})}$, $\hat{\phi}_0$, $\hat{\psi}_0$, $\hat{\alpha}_0(0)$ and $\hat{\beta}_0(0)$ are constants, $\hat{\alpha}_1(\hat{t})$, $\hat{\beta}_1(\hat{t})$, $\hat{\phi}_1(\hat{t})$ and $\hat{\psi}_1(\hat{t})$ are to be determined by higher order terms.

2.2. *Inner expansion*

In what follows one considers the specific slowly varying parameters:

$$a(\hat{t}) = 2b_0 + a_1(\hat{t} - t_0), \qquad b(\hat{t}) = b_0 + b_1(\hat{t} - t_0). \tag{6}$$

It is easy to see that a 2 : 1 internal resonance (i.e., $a(\hat{t}) \approx 2b(\hat{t})$) will take place when $\hat{t} \approx t_0$. To solve this problem, the following slow time scale is introduced:

$$\overline{t} = \varepsilon^{-1/2}(\hat{t} - t_0) \tag{7}$$

The fast time scale is taken as t. Moreover, suppose that

$$\begin{cases} x = \bar{x}_0(t, \bar{t}) + \varepsilon^{1/2}\bar{x}_1(t, \overline{t}) + \varepsilon\bar{x}_2(t, \overline{t}) + \mathcal{O}(\varepsilon^{3/2}); \\ y = \bar{y}_0(t, \bar{t}) + \varepsilon^{1/2}\bar{y}_1(t, \overline{t}) + \varepsilon\bar{y}_2(t, \overline{t}) + \mathcal{O}(\varepsilon^{3/2}), \end{cases} \tag{8}$$

Substituting (8) into (1) in the same way as the previous discussion, one has the following asymptotic solutions for $|\overline{t}| \to \infty$:

$$\begin{aligned} x = {} & \bar{\alpha}_0(0) \cos[\rho + \overline{\phi}_0(0)] + \varepsilon^{1/2}\{[A_1(0) + \operatorname{sgn}(\overline{t})(r_1 I_1 - r_2 I_2)] \cos\rho \\ & + [B_1(0) + \operatorname{sgn}(\overline{t})(r_2 I_1 + r_1 I_2)] \sin\rho + p_1 \overline{t} \cos\rho + p_2 \overline{t} \sin\rho \\ & - (r_1 \sin 2\theta - r_2 \cos 2\theta)\}/[(a_1 - 2b_1)\overline{t}]\} + \mathcal{O}(\varepsilon) + \mathcal{O}(\overline{t}^{-3}); \\ y = {} & \overline{\beta}_0(0) \cos[\theta + \overline{\psi}_0(0)] + \varepsilon^{1/2}[C_1(0) \cos\theta \\ & + D_1(0) \sin\theta + c_1 \overline{t} \cos\theta + c_2 \overline{t} \sin\theta] + \mathcal{O}(\varepsilon) + \mathcal{O}(\overline{t}^{-3}), \end{aligned} \tag{9}$$

where

$$\rho = 2b_0 t + a_1 \overline{t}^2/2, \qquad \theta = b_0 t + b_1 \overline{t}^2/2 \tag{10}$$

and $\bar{\alpha}_0(0)$, $\overline{\beta}_0(0)$, $\overline{\phi}_0(0)$, $\overline{\psi}_0(0)$, $A_1(0)$, $B_1(0)$, $C_1(0)$, $D_1(0)$ are constants;

$$p = -[\bar{\alpha}_0(0)a_1/(4b_0)]\, e^{i\overline{\phi}_0(0)} \equiv p_1 + ip_2;$$

$$r = [i\overline{\beta}_0^2(0)/(8b_0)]\, e^{-2i\overline{\psi}_0(0)} \equiv r_1 + ir_2;$$

$$c = -[\overline{\beta}_0(0)b_1/(2b_0)]\,\mathrm{e}^{-\mathrm{i}\overline{\psi}_0(0)} \equiv c_1 + \mathrm{i}c_2;$$
$$I = \sqrt{\pi/|a_1 - 2b_1|}\,[1 + \mathrm{i}\,\mathrm{sgn}\,(a_1 - 2b_1)]/2 \equiv I_1 + \mathrm{i}I_2. \tag{11}$$

2.3. *Matching in overlapping domain*

To obtain the uniformly valid expansions of the system (1) for all the time, we must match the results of sections 2.1 and 2.2 in their overlapping domain. Introduce a new time variable:

$$t_\eta = (\hat{t} - t_0)/\varepsilon^\eta, \tag{12}$$

where $0 \leqslant \eta_1 < \eta < \eta_2 \leqslant 1/2$, η is to be determined by the following process. Substituting (12) into (2) and (10), then

$$\begin{cases} \mu = \tau_0/\varepsilon + \varepsilon^{\eta-1}2b_0\,t_\eta + \varepsilon^{2\eta-1}a_1\,t_\eta^2/2; \\ \nu = k_0/\varepsilon + \varepsilon^{\eta-1}b_0\,t_\eta + \varepsilon^{2\eta-1}b_1\,t_\eta^2/2, \end{cases} \tag{13}$$

$$\begin{cases} \rho = 2b_0\,t_0/\varepsilon + \varepsilon^{\eta-1}2b_0\,t_\eta + a_1\,\varepsilon^{2\eta-1}t_\eta^2/2; \\ \theta = b_0\,t_0/\varepsilon + \varepsilon^{\eta-1}2b_0\,t_\eta + b_1\,\varepsilon^{2\eta-1}t_\eta^2/2, \end{cases} \tag{14}$$

where

$$\tau_0 = \int_0^{t_0} a(s)\,\mathrm{d}s, \qquad k_0 = \int_0^{t_0} b(s)\,\mathrm{d}s. \tag{15}$$

Comparing (13) with (14), one has

$$\begin{cases} \mu = (\tau_0 - 2b_0\,t_0)/\varepsilon + \rho; \\ \nu = (k_0 - b_0\,t_0)/\varepsilon + \theta; \\ \hat{t} = t_0 + \varepsilon^\eta t_\eta. \end{cases} \tag{16}$$

Thus, the outer and inner expansions for x are given respectively as:

$$\begin{aligned} x^o(t_\eta, \varepsilon) = {} & \hat{\alpha}_0(0)\sqrt{a(0)/(2b_0)}\,\{\cos[\rho + (\tau_0 - 2b_0\,t_0)/\varepsilon + \hat{\phi}_0] \\ & - \varepsilon^\eta(a_1\,t_\eta/(4b_0))\cos[\rho + \tau_0/\varepsilon + \hat{\phi}_0]\} \\ & + \varepsilon^{1-\eta}\{\hat{\beta}_0^2(0)b_0/[8b_0(a_1 - 2b_1)t_\eta]\}\cos 2[(k_0 - b_0\,t_0)/\varepsilon + \hat{\psi}_0 + \theta] \\ & + \mathcal{O}(\varepsilon^{4\eta-1}) + \mathcal{O}(\varepsilon); \end{aligned} \tag{17}$$

$$\begin{aligned} x^i(t_\eta, \varepsilon) = {} & \bar{\alpha}_0(0)\cos[\rho + \overline{\phi}_0(0)] + \varepsilon^{1/2}\{[A_1(0) + r_2 I_2 - r_1 I_1]\cos\rho \\ & + [B_1(0) - r_2 I_1 - r_1 I_2]\sin\rho\} + \varepsilon^\eta p_1\,t_\eta\cos\rho + \varepsilon^\eta p_2\,t_\eta\sin\rho \\ & - \varepsilon^{1-\eta}(r_1\sin 2\theta - r_2\cos 2\theta)/[(a_1 - 2b_1)t_\eta] + \mathcal{O}(\varepsilon^{2-3\eta}) \end{aligned}$$

The singular term of x_1 in equation (5) contributes the last term in $x^o(t_\eta, \varepsilon)$ and the contribution of the rest of x_1 is $\mathcal{O}(\varepsilon)$. The remainder of order $\varepsilon^{2-3\eta}$ in $x^i(t_\eta, \varepsilon)$ represents the terms of order $\overline{t}^{-3}$ which are neglected in the inner expansion. By letting $2 - 3\eta = 4\eta - 1$, one has $\eta = 3/7$. In order to match the results to $\mathcal{O}(\varepsilon^{1/2})$, one must require

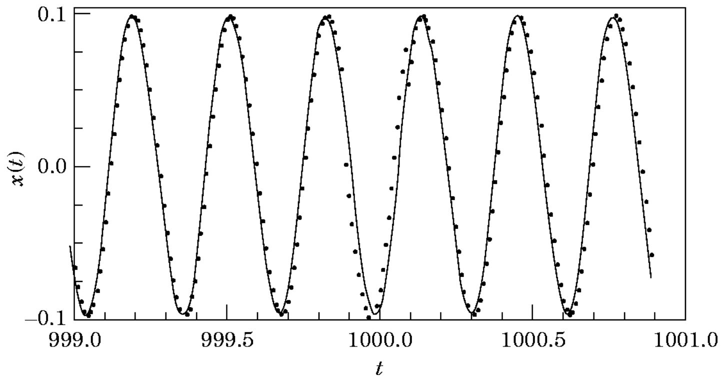

Figure 1. Variation of $x(t)$ with time ($\varepsilon = 0{\cdot}001$): ——, numerical; · · · ·, theoretical.

that $[x^o(t_\eta, \varepsilon) - x^i(t_\eta, \varepsilon)]/\varepsilon^{1/2} \to 0$ (for $\varepsilon \to 0$ and t_η fixed), so some coefficients in these expansions can be obtained as follows:

$$\begin{cases} \bar{\alpha}_0(0) = \hat{\alpha}_0(0)\sqrt{a(0)/(2b_0)}\,; \\ \overline{\phi}_0(0) = (\tau_0 - 2b_0 t_0)/\varepsilon + \hat{\phi}_0. \end{cases} \tag{18}$$

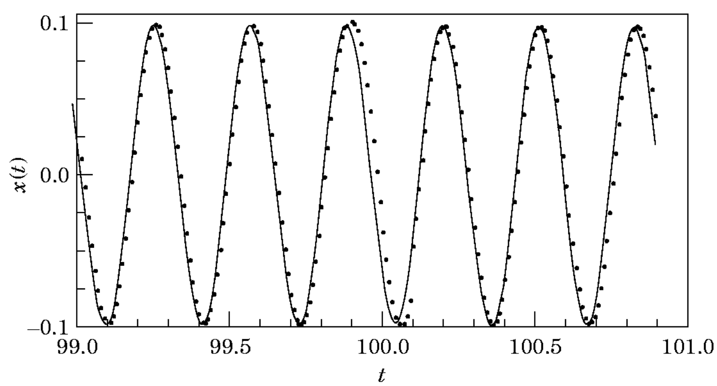

Figure 2. Variation of $x(t)$ with time ($\varepsilon = 0{\cdot}01$): key as Figure 1.

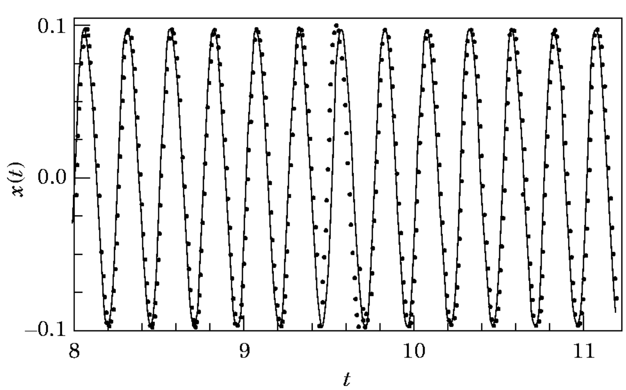

Figure 3. Variation of $x(t)$ with time ($\varepsilon = 0{\cdot}1$): key as Figure 1.

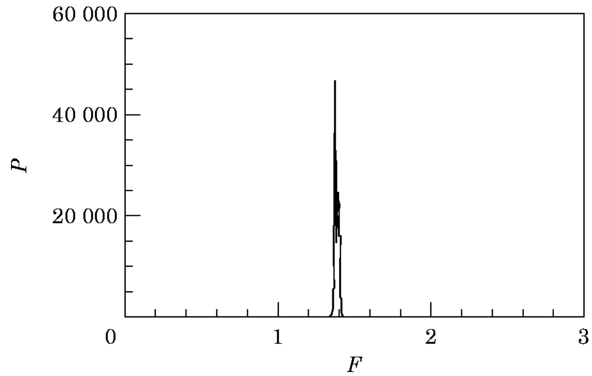

Figure 4. Power spectrum ($\varepsilon = 0{\cdot}001$).

As the outer expansion does not include the terms of $\mathcal{O}(\varepsilon^{1/2})$, one knows that

$$\begin{cases} A_1(0) = r_1 I_1 - r_2 I_2; \\ B_1(0) = r_2 I_1 + r_1 I_2. \end{cases} \tag{19}$$

Similarly, one obtains

$$\begin{aligned} &\bar{\beta}_0(0) = \hat{\beta}_0(0)\sqrt{b(0)/b_0}; \\ &\bar{\psi}_0(0) = (k_0 - b_0 t_0)/\varepsilon + \hat{\psi}_0; \\ &C_1(0) = 0; \\ &D_1(0) = 0. \end{aligned} \tag{20}$$

From the definitions of r_i and p_i ($i = 1, 2$) in (11) one knows that the inner and outer expansions are matched with each other.

3. NUMERICAL RESULTS

In order to verify the present theory, some numerical results are given in Figures 1–3 (solid lines represent numerical results, dots represent theoretical results, and $b_0 = 10$, $a_1 = 1$, $b_1 = 1{\cdot}5$, $t_0 = 1$). The internal resonance will occur when $\varepsilon t \approx 1$. Here one lets all x take the form of the inner expansion when $|t - t_0/\varepsilon| \leqslant 0{\cdot}1$ and the outer expansion when $|t - t_0/\varepsilon| > 0{\cdot}1$, it is easy to see that the errors become small following the decrease of the

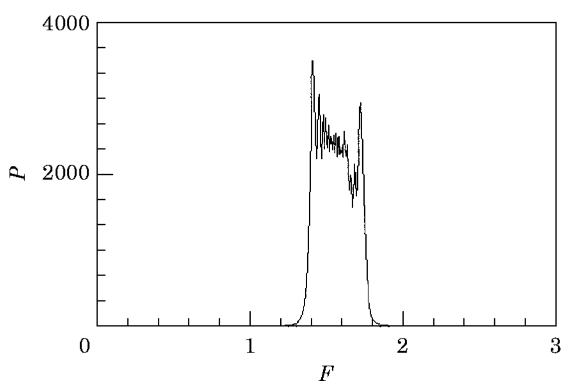

Figure 5. Power spectrum ($\varepsilon = 0{\cdot}01$).

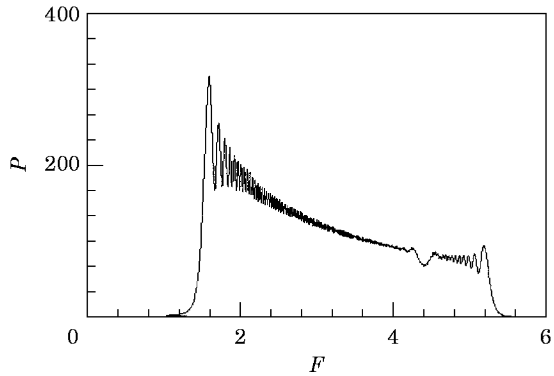

Figure 6. Power spectrum ($\varepsilon = 0{\cdot}1$).

small parameter ε. Moreover, similar results can be obtained for other values of the parameters. It shows that the improved multi-scale method is valid in the analysis of slowly varying oscillatory systems.

Moreover, through a power spectrum analysis, one finds that the solution for $\varepsilon = 0{\cdot}001$ has only a single frequency (see Figure 4), but the power spectrum grows increasingly wider when ε increases gradually (see Figure 5), and becomes continuous over greater frequency intervals so that the system (1) appears to be chaotic (See Figure 6). It means that the chaotic behavior of the systems is greatly influenced by the non-stationary variations.

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