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PARAMETER ESTIMATION IN IMBALANCED NON-LINEAR ROTOR-BEARING SYSTEMS FROM RANDOM RESPONSE

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This study attempts to explore the possibility of estimation of linear and non-linear stiffness parameters of rolling element bearings in rotor systems experiencing harmonic excitation from unknown imbalance as well as random excitation from a variety of sources, primarily the progressive random deterioration of the bearing surfaces and subsurfaces. The random forces inflicted on the system are comparable to the harmonic imbalance forces, if the imbalance is not very significant. In earlier studies, the authors addressed the inverse problem of parameter estimation in non-linear rotor-bearing systems experiencing only random excitations, under the assumption that the rotor is perfectly balanced. The problem of parameter estimation, in a non-linear rotor-bearing system experiencing small residual imbalance forces along with random forces, is transformed into one of slowly varying parameters through the stochastic averaging procedure. The resulting equations are modelled as an approximate Markovian process and a Fokker–Planck equation is derived to describe it. The Fokker–Planck equation is solved and processed further, to obtain the bearing stiffness parameters. The procedure has the advantage that it does not require an estimate of the excitation forces (harmonic and random) and works directly on the measured response signals of the system. The algorithm is illustrated for a laboratory rotor rig and the results are compared with those obtained through an existing analytical model. Estimates of the unknown imbalance of the rotor, its angular location and the damping ratio are also obtained, as by-products of the procedure developed.

1. INTRODUCTION

Expressions for linear stiffness of isolated rolling element bearings are derived by establishing a relationship between the load carried by the bearing and its deformation, commonly through the classical solution for the local stress and deformation of two elastic bodies apparently contacting at a single point [1]. The early studies [2, 3] on bearings concern vibrations caused due to geometric imperfections of contact surfaces. Procedures [4, 5] are available for estimation of bearing stiffness under static loading conditions. Rolling element bearings are known to possess highly non-linear stiffness characteristics. The problem of identifying the non-linear stiffness characteristics was approached by Kononenko and Plakhtienko [6] through the Krylov–Bogoliubov–Mitropolsky procedure. Kraus *et al*. [7] presented a method for the extraction of rolling element bearing stiffness and damping under operating conditions. The method is based on experimental modal analysis combined with a mathematical model of the rotor-bearing support system. The method is applied for investigation of the effect of speed, preload and free outer race bearings on stiffness and damping. Muszynska [8] has developed a perturbation technique for estimation of these parameters. The technique involves a controlled input excitation to be given to the bearings. Goodwin [9] reviewed the experimental approaches to rotor

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support impedance measurement. Non-linear stochastic contact vibrations and friction at a Hertzian contact have been studied by Hess *et al*. [10]. The experimentation involves excitation of the bearings either externally by a white Gaussian random normal load or within the contact region by a rough surface input and the analytical approach is based on the solution of the Fokker–Planck equation.

A technique for estimation of non-linear stiffness in a rotor-bearing system, based on analysis of its random response, has been developed by Tiwari and Vyas [11]. The governing non-linear equation with a random excitation force, resulting from random imperfections of the bearing surfaces and assembly is modelled through the Fokker–Planck equation. The solution of the Fokker–Planck equation is further processed for linear and non-linear bearing stiffness parameters. The analysis involved a rotor with a rigid shaft carrying a single disc at its midspan. The study was extended by Tiwari and Vyas [12] for the more involved problem of rotors with flexible shafts carrying more than one disc. The procedures had the advantage that they do not require an estimate of the excitation forces and work directly on the response signals from the bearing caps.

The above studies concerned single-degree-of-freedom and multi-degree-of-freedom systems, where the rotor was assumed to be balanced. The present study explores the possibility of estimation of linear and non-linear stiffness parameters of the bearing, from the bearing cap signals, when the rotor is not balanced, and the imbalance is unknown. The problem is formulated as a single-degree-of-freedom system, treating the rotor as rigid rotor and bearings as non-linear flexible supports. The excitation to the system consists of harmonic forces due to the imbalance and random forces due to arbitrary deviations, of bearing contact surfaces and subsurfaces, from their ideal design and their progressive deterioration during operation. These random forces can also be contributed from random sources such as inaccuracies in the rotor-bearing housing assemblies, etc., and are comparable to the harmonic excitation forces, if the imbalance is not significantly large. The interaction effects between harmonic and random forces are ignored in the present study. The parameter estimation procedure is based on the averaging technique of Bogoliubov and Mitropolsky [13] for deterministic non-linear systems, extended by Stratonovich [14] for stochastic differential equations. The governing equation of motion is transformed from the rapidly varying variables, namely displacement and velocity, to variables, amplitude and phase, varying slowly with time. Stochastic averaging is carried out to take into account the effect of the random excitation multiplied by a correlated term, so as to model the slowly varying amplitude as an approximate Markovian process. A second order stochastic approximation is carried out and a one-dimensional Fokker–Planck equation is derived to describe the Markovian amplitude process. The response to the Fokker–Planck equation is derived and processed further for parameter estimation. The procedure developed is illustrated for estimation of stiffness parameters of the rolling element bearings of a laboratory rotor rig. The experimental results are compared with those obtained through the analytical guidelines of Harris [5] and Ragulskis *et al*. [4]. Estimates of the unknown imbalance of the rotor, its angular location and the damping ratio are also obtained, as by-products of the procedure developed.

2. RANDOM RESPONSE OF IMBALANCED ROTORS

The study is restricted to rotors with the disc mounted on a rigid shaft and the non-linear spring force being contributed by the bearings (Figure 1). The damping to the system is taken to be linear and the disc carries an imbalance F_0. The governing equation for the system is written as

$$m\ddot{x} + c\dot{x} + k_L x + k_{NL} G(x) = F_0 \cos(\omega t + \theta) + \psi(t). \tag{1}$$

In the above k_L and k_{NL} are the linear and non-linear stiffness parameters of the bearings and $G(x)$ can be a polynomial in x. The rotor mass is m and ω is its rotational speed. The angular location of the imbalance with respect to a reference point on the shaft is measured by θ. The random excitation $\psi(t)$ is contributed by the bearing surface imperfections, caused by the random deviations from their standard theoretical design and progressive surface and subsurface deterioration. In addition it can be contributed by inaccuracies in the rotor-bearing housing assembly and other such sources.

The spring force non-linearity in rolling element bearings is generally taken to be cubic in nature [4], i.e.,

$$G(x) = x^3. \tag{2}$$

The concept of averaging principle, developed by Bogoliubov and Mitropolsky [13] for deterministic non-linear vibration transforms the equation, involving vibrations which are rapidly varying with time, to a set of simple equations for slowly varying response co-ordinates. This principle, extended by Stratonovich [14] for stochastic differential equations, has been employed to analyse the rotor-bearing system governed by equation (1).

Defining

$$\lambda = k_{NL}/k_L \tag{3}$$

and since $(1/\lambda)$ is a small quantity (rolling element bearings being highly non-linear), equation (1) can be rewritten in terms of the small parameter $\varepsilon = (1/\lambda)$ as

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, \zeta(t))$$

where

$$f(x, \dot{x}, \zeta(t)) = [(F_0 \lambda/m)\cos(\omega t + \theta) + \zeta(t)\lambda - 2\omega_n \lambda \xi \dot{x} - (\omega_n^2 - \omega^2)\lambda x - \omega_n^2 \lambda^2 x^3], \tag{4}$$

$$\omega_n^2 = k_L/m, \qquad \xi = c/2m\omega_n, \qquad \zeta(t) = \psi(t)/m.$$

Because ε is small, the response can be taken to be harmonic in time with frequency ω and with slowly varying amplitude, $A(t)$ and phase $\varphi(t)$, i.e.,

$$x(t) \approx A(t)\cos[\omega t + \varphi(t)], \qquad \dot{x}(t) \approx -\omega A(t)\sin[\omega t + \varphi(t)]. \tag{5}$$

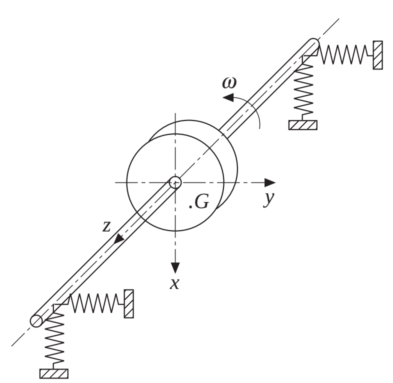

Figure 1. Rotor-bearing model.

Equation (4) can now be expressed as a set of standard form equations in terms of the slowly varying parameters $A(t)$ and $\varphi(t)$ as

$$A = [x^2 + (\dot{x}^2/\omega^2)]^{1/2}, \qquad \varphi = -\arctan[\dot{x}/\omega x] - \omega t, \tag{6}$$

or

$$\dot{A} = \varepsilon G[A, \varphi, \zeta(t)], \qquad \dot{\varphi} = \varepsilon H[A, \varphi, \zeta(t)], \tag{7}$$

where

$$G[A, \varphi, \zeta(t)] = G_{av}(A, \varphi) - \{\zeta(t)\lambda/\omega\} \sin(\omega t + \varphi),$$

$$H[A, \varphi, \zeta(t)] = H_{av}(A, \varphi) - \{\zeta(t)\lambda/\omega A\} \cos(\omega t + \varphi), \tag{8}$$

with

$$\begin{aligned} G_{av}(A, \varphi) = {} & \{-(\omega\lambda A/2) + (\omega_n^2 \lambda A/2\omega) + (\omega_n^2 \lambda^2 A^3/4\omega)\} \sin 2(\omega t + \varphi) \\ & + \{\omega_n \lambda \xi A\} \cos 2(\omega t + \Phi) - (F_0 \lambda/2m\omega) \sin(2\omega t + \varphi + \theta) \\ & + (\omega_n^2 \lambda^2 A^3/8\omega) \sin 4(\omega t + \varphi) - \{(F_0 \lambda/2m\omega) \sin(\varphi - \theta) + \omega_n \lambda \xi A\}, \\ H_{av}(A, \varphi) = {} & -(\omega_n \lambda \xi) \sin 2(\omega t + \varphi) + \{-(\omega\lambda/2) + (\omega_n^2 \lambda/2\omega) \\ & + (\omega_n^2 \lambda^2 A^2/2\omega)\} \cos 2(\omega t + \varphi) - (F_0 \lambda/2m\omega A) \cos(2\omega t + \varphi + \theta) \\ & + (\omega_n^2 \lambda^2 A^2/8\omega) \cos 4(\omega t + \varphi) + \{-\omega\lambda/2) - (F_0 \lambda/2m\omega A) \cos(\varphi - \theta) \\ & + (\omega_n^2 \lambda/2\omega) + (3\omega_n^2 \lambda^2 A^2/8\omega)\}. \end{aligned} \tag{9}$$

3. STOCHASTIC AVERAGING

It can be seen that the right side of equations (7) contain (employing the terminology of Stratonovich [14]) "oscillatory" terms, i.e., harmonic functions of ωt, along with randomly "fluctuating" terms, i.e., $-\{\zeta(t)\lambda/\omega\} \sin(\omega t + \varphi)$ and $-\{\zeta(t)\lambda/\omega A\} \cos(\omega t + \varphi)$, which contain the random force term $\zeta(t)$. However, due to the presence of ε in equation (4), the parameters A and φ vary slowly with time and can be assumed to remain constant over a cycle of oscillation. The averaging process for A and φ can be carried out in two stages—by stochastic averaging and elimination of the randomly fluctuating terms involving random force term $\zeta(t)$ and then averaging over a cycle of oscillation for the removal of the oscillating terms involving harmonic functions of ωt.

The approach to obtaining the response of the system, including stochastic averaging, can be simplified by providing arguments similar to those in the previous sections and treating the random excitation to the system as ideal white noise with zero mean and Gaussian distribution. For a zero mean random excitation $\zeta(t)$, the expressions (7) for $\dot{A}$ and $\dot{\varphi}$ can be stochastically averaged to write the non-fluctuating amplitude term, $\dot{A}_{nf}$ and the non-fluctuating phase term, $\dot{\varphi}_{nf}$ as

$$\dot{A}_{nf} = \varepsilon G_{av}(A, \varphi), \qquad \dot{\varphi}_{nf} = \varepsilon H_{av}(A, \varphi). \tag{10}$$

Continuing the averaging process over a cycle, one gets the non-fluctuating, non-oscillatory amplitude and phase terms as

$$\dot{A}_{nf\text{-}no} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \varepsilon G_{av}[A, \varphi]\,\mathrm{d}t = -[(F_0/2m\omega)\sin(\varphi-\theta) + (\omega_n \xi A)],$$

$$\dot{\varphi}_{nf-no} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \varepsilon H_{av}[A, \varphi]\,\mathrm{d}t$$

$$= [-(\omega/2) - (F_0/2m\omega A)\cos(\varphi-\theta) + (\omega_n^2/2\omega) + (3\omega_n^2\lambda A^2/8\omega)]. \qquad (11)$$

Considering equations (7), (8) and (9) and putting the fluctuating terms back into the above expressions, the non-oscillating approximations for the amplitude and phase are

$$\dot{A}_{no} = -[(F_0/2m\omega)\sin(\varphi-\theta) + (\omega_n\xi A)] - \{\zeta(t)/\omega\}\sin(\omega t + \varphi),$$

$$\dot{\varphi}_{no} = [-(\omega/2) - (F_0/2m\omega A)\cos(\varphi-\theta) + (\omega_n^2/2\omega) + (3\omega_n^2\lambda A^2/8\omega)]$$
$$- \{\zeta(t)/\omega A\}\cos(\omega t + \varphi). \qquad (12)$$

The equations (12) can be condensed as

$$\dot{A}_{no} = \varepsilon G_1[A, \varphi] - \{\zeta(t)/\omega\}\sin(\omega t + \varphi), \qquad \dot{\varphi}_{no} = \varepsilon H_1[A, \varphi] - \{\zeta(t)/\omega A\}\cos(\omega t + \varphi),$$

with

$$G_1[A, \varphi] = -[(F_0\lambda/2m\omega)\sin(\varphi-\theta) + \omega_n\lambda\xi A],$$

$$H_1[A, \varphi] = [-(\omega\lambda/2) - (F_0\lambda/2m\omega A)\cos(\varphi-\theta) + (\omega_n^2\lambda/2\omega) + (3\omega_n^2\lambda^2 A^2/8\omega)]. \qquad (13)$$

The trucated equations (11), giving $\dot{A}_{nf\text{-}no}$ and $\dot{\varphi}_{nf\text{-}no}$ or the truncated equations (12), giving $\dot{A}_{no}$ and $\dot{\varphi}_{no}$ can be taken as approximations of $\dot{A}$ and $\dot{\varphi}$. However, either approximation does not, adequately, reflect the influence of non-linearity in the system, for while the expression for $\dot{\varphi}_{nf\text{-}no}$ (or $\dot{\varphi}_{no}$) does involve the non-linearity parameter λ, the one for $\dot{A}_{nf\text{-}no}$ (or $\dot{A}_{no}$) does not. A higher order of approximation for $\dot{A}$ and $\dot{\varphi}$ is, therefore, essential to represent adequately the effects of the non-linearity on the statistical characteristics of the response.

4. SECOND ORDER AVERAGING

For a higher order approximation, instead of obtaining $\dot{A}_{no}$ and $\dot{\varphi}_{no}$ as in equation (12), the procedure of asymptotic method developed by Bogoliubov [14] can be employed, whereby the non-fluctuating approximations $\dot{A}_{nf}$ and $\dot{\varphi}_{nf}$, of equation (10) are taken to have the form

$$\dot{A}_{nf} = \dot{A}^* + \varepsilon\dot{u}(A^*, \varphi^*), \qquad \dot{\varphi}_{nf} = \dot{\varphi}^* + \varepsilon\dot{v}(A^*, \varphi^*), \qquad (14)$$

where $\dot{A}^*$ and $\dot{\varphi}^*$ are expressed as

$$\dot{A}^* = \varepsilon G_{av}^*(A^*, \varphi^*) = \varepsilon G_1^*(A^*, \varphi^*) + \varepsilon^2 G_2^*(A^*, \varphi^*) + \dots,$$

$$\dot{\varphi}^* = \varepsilon H_{av}^*(A^*, \varphi^*) = \varepsilon H_1^*(A^*, \varphi^*) + \varepsilon^2 H_2^*(A^*, \varphi^*) + \dots. \qquad (15)$$

Similarly the variations $\dot{u}$ and $\dot{v}$ are expressed in series form as

$$\dot{u}(A^*, \varphi^*) = \dot{u}_1(A^*, \varphi^*) + \varepsilon\dot{u}_2(A^*, \varphi^*) + \dots,$$

$$\dot{v}(A^*, \varphi^*) = \dot{v}_1(A^*, \varphi^*) + \varepsilon\dot{v}_2(A^*, \varphi^*) + \dots. \qquad (16)$$

Equations (14), along with the series expansions of equations (15) and (16) are substituted into the stochastically averaged equations (10). Noting that

$$\dot{u} = (\partial u/\partial A^*)\dot{A}^* + (\partial u/\partial \varphi^*)(\omega + \dot{\varphi}^*), \qquad \dot{v} = (\partial v/\partial A^*)\dot{A}^* + (\partial v/\partial \varphi^*)(\omega + \dot{\varphi}^*), \quad (17)$$

the terms with identical power of ε are equated to obtain the equations governing the successive approximations. The equation governing the terms involving the first approximation (of order ε^1) is, thus, obtained as

$$G_1^*(A^*, \varphi^*) + \omega(\partial u_1/\partial \varphi^*) = G_{av}^*(A^*, \varphi^*),$$
$$H_1^*(A^*, \varphi^*) + \omega(\partial v_1/\partial \varphi^*) = H_{av}^*(A^*, \varphi^*). \quad (18)$$

In the above equation, the term $G_{av}(A, \varphi)$, of equation (10) has been transformed to $G_{av}^*(A^*, \varphi^*)$.

The right side of equations (18) involve G_{av}^* and H_{av}^*, which contain both the oscillatory and the non-oscillatory terms. The functions $u(A^*, \varphi^*)$ and $v(A^*, \varphi^*)$ are now chosen in such a way that $G_1^*(A^*, \varphi^*)$ and $H_1^*(A^*, \varphi^*)$ contain no oscillatory terms. Thus, terms involving $(\partial u/\partial \varphi^*)$ and $(\partial v/\partial \varphi^*)$ are equated to the oscillatory parts of G_{av}^* and H_{av}^*, and $G_1^*(A^*, \varphi^*)$ and $H_1^*(A^*, \varphi^*)$ are equated to the non-oscillatory parts, to obtain

$$G_1^*(A^*, \varphi^*) = -[(F_0\lambda/2m\omega)\sin(\varphi^* - \theta) + \omega_n\lambda\xi A^*],$$
$$H_1^*(A^*, \varphi^*) = [-(\omega\lambda/2) - (F_0\lambda/2m\omega A^*)\cos(\varphi^* - \theta) + (\omega_n^2\lambda/2\omega) + (3\omega_n^2\lambda 2A^{*2}/8\omega],$$
$$(19)$$

and

$$\omega(\partial u_1(A^*, \varphi^*)/\partial \varphi^*) = \{-(\omega\lambda A^*/2) + (\omega_n^2\lambda A^*/2\omega) + (\omega_n^2\lambda^2 A^{*3}/4\omega)\}\sin 2(\omega + \varphi^*)$$
$$+ \{\omega_n\lambda\xi A^*\}\cos 2(\omega t + \varphi^*) - (F_0\lambda/2m\omega)\sin(2\omega t + \varphi^* + \theta)$$
$$+ (\omega_n^2\lambda^2 A^3/8\omega)\sin 4(\omega t + \varphi),$$
$$\omega(\partial v_1(A^*, \varphi^*)/\partial \varphi^*) = -(\omega_n\lambda\xi)\sin 2(\omega t + \varphi^*) + \{-(\omega\lambda/2) + (\omega_n^2\lambda/2\omega)$$
$$+ (\omega_n^2\lambda^2 A^{*2}/2\omega)\}\cos 2(\omega t + \varphi^*)$$
$$- (F_0\lambda/2m\omega A^*)\cos(2\omega t + \varphi^* + \theta)$$
$$+ (\omega_n^2\lambda^2 A^{*2}/8\omega)\cos 4(\omega t + \varphi^*). \quad (20)$$

Comparison of equations (19) and (13) reveals

$$G_1^*(A^*, \varphi^*) = G_1(A, \varphi), \qquad H_1^*(A^*, \varphi^*) = H_1(A, \varphi). \quad (21)$$

Equations (20) give

$$u_1(A^*, \varphi^*) = \{(\lambda A^*/4) - (\omega_n^2\lambda' A^*/4\omega^2) - (\omega_n^2\lambda^2 A^{*3}/8\omega^2)\}\cos 2(\omega t + \phi^*)$$
$$+ (\omega_n\lambda\xi A^*/2\omega)\sin 2(\omega t + \varphi^*) + (F_0\lambda/2m\omega^2)\cos(2\omega t + \varphi^* + \theta)$$
$$+(\omega_n^2\lambda^2 A^{*3}/32\omega^2)\cos 4(\omega t + \varphi^*), \quad (22)$$
$$v_1(A^*, \varphi^*) = (\omega_n\lambda\xi/2\omega)\cos 2(\omega t + \varphi^*) + \{-(\lambda/4) + (\omega_n^2\lambda/4\omega^2)$$
$$+ (\omega_n^2\lambda^2 A^{*2}/4\omega^2)\}\sin 2(\omega t + \varphi^*) - (F_0\lambda/2m\omega^2 A^*)\sin(2\omega t + \varphi^* + \theta)$$
$$+ (\omega_n^2\lambda^2 A^{*2}/32\omega^2)\sin 4(\omega t + \varphi^*). \quad (23)$$

For second approximation (of order ε^2), the governing equations are, similarly, obtained as

$$G_2^* + \omega(\partial u_2/\partial\varphi^*) = (\partial G_{av}^*/\partial A^*)u_1 + (\partial G_{av}^*/\partial\varphi^*)v_1 - (\partial u_1/\partial A^*)G_1^* - (\partial u_1/\partial\varphi^*)H_1^*,$$

$$H_2^* + \omega(\partial v_2/\partial\varphi^*) = (\partial H_{av}^*/\partial A^*)u_1 + (\partial H_{av}^*/\partial\varphi^*)v_1 - (\partial v_1/\partial A^*)G_1^* - (\partial v_1/\partial\varphi^*)H_1^*, \quad (24)$$

from which the non-oscillatory $G_2^*(A^*, \varphi^*)$ and $H_2^*(A^*, \varphi^*)$ are obtained as

$$G_2^*(A^*, \varphi^*) = (F_0\lambda/4m\omega^2)\{-(5\omega\lambda/4) + (5\omega_n^2\lambda/4\omega) + (A^{*2}\omega_n^2\lambda^2/\omega)\}\sin(\varphi^* - \theta) + (5\omega_n\lambda\xi F_0/8m\omega^2)\cos(\varphi^* - \theta),$$

$$H_2^*(A^*, \varphi^*) = \{(3A^{*2}\omega_n^2\lambda^3/8\omega) - (3A^{*2}\omega_n^4\lambda^3/8\omega^3) - (51A^{*4}\omega_n^4\lambda^4/256\omega^3) - (\omega\lambda^2/8) - (\omega_n^4\lambda^2/8\omega^3)\} - \{(F_0\lambda^2/4m\omega A^*) - (F_0\omega_n^2\lambda^2/4m\omega^3A^*) - (17A^*F_0\omega_n^2\lambda^3/32m\omega^3)\}\cos(\varphi^* - \theta) - (F_0\omega_n\lambda^2\xi/2m\omega^2A^*)\sin(\varphi^* - \theta). \quad (25)$$

Substitution from equations (19) and (25) into equation (15) gives

$$\dot{A}^* = -\varepsilon[(F_0\lambda/2m\omega)\sin(\varphi^* - \theta) + \omega_n\lambda\xi A^*] + \varepsilon^2[(F_0\lambda/4m\omega^2)\{-(5\omega\lambda/4) + (5\omega_n^2\lambda/4\omega) + (A^{*2}\omega_n^2\lambda^2/\omega)\}\sin(\varphi^* - \theta) + (5\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^* - \theta), \quad (26)$$

$$\dot{\varphi}^* = \varepsilon[-(\omega\lambda/2) - (F_0\lambda/2m\omega A^*)\cos(\varphi^* - \theta) + (\omega_n^2\lambda/2\omega) + (3\omega_n^2\lambda^2A^{*2}/8\omega)] + \varepsilon^2[\{(3A^{*2}\omega_n^2\lambda^3/8\omega) - (3A^{*2}\omega_n^4\lambda^3/8\omega^3) - (51A^{*4}\omega_n^4\lambda^4/256\omega^3) - (\omega\lambda^2/8) - (\omega_n^4\lambda^2/8\omega^3)\} - \{(F_0\lambda^2/4m\omega A^*) - (F_0\omega_n^2\lambda^2/4m\omega^3A^*) - (17A^*F_0\omega_n^2\lambda^3/32m\omega^3)\}\cos(\varphi^* - \theta) - (F_0\omega_n\lambda^2\xi/2m\omega^2A^*)\sin(\varphi^* - \theta)]. \quad (27)$$

Noting equation (14) and that the oscillatory terms are confined to the variables u and v, the non-fluctuating, non-oscillatory amplitude term $A_{nf\text{-}no}$ becomes

$$\dot{A}_{nf\text{-}no} = \dot{A}^*, \quad (28)$$

where $\dot{A}^*$ is given by equation (26).

Consideration of equations (28), (7) and (8) enables writing the non-oscillatory amplitude term, A_{no}, as

$$\dot{A}_{no} = \dot{A}_{nf\text{-}no} - \varepsilon[(\zeta(t)\lambda/\omega)\sin(\omega t + \varphi^*)] - \varepsilon[(F_0\lambda/2m\omega)\sin(\varphi^* - \theta) + \omega_n\lambda\xi A^*] + \varepsilon^2[(F_0\lambda/4m\omega^2)\{-(5\omega\lambda/4) + (5\omega_n^2\lambda/4\omega) + (A^{*2}\omega_n^2\lambda^2/\omega)\}\sin(\varphi^* - \theta) + (5\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^* - \theta)] - \varepsilon[(\zeta(t)\lambda/\omega)\sin(\omega t + \varphi^*)]. \quad (29)$$

Similarly

$$\dot{\varphi}_{no} = \varepsilon[-(\omega\lambda/2) - (F_0\lambda/2m\omega A^*)\cos(\varphi^* - \theta) + (\omega_n^2\lambda/2\omega) + (3\omega_n^2\lambda^2A^{*2}/8\omega)] + \varepsilon^2[\{(3A^{*2}\omega_n^2\lambda^3/8\omega) - (3A^{*2}\omega_n^4\lambda^3/8\omega^3) - (51A^{*4}\omega_n^4\lambda^4/256\omega^3) - (\omega\lambda^2/8) - (\omega_n^4\lambda^2/8\omega^3)\} - \{(F_0\lambda^2/4m\omega A^*) - (F_0\omega_n^2\lambda^2/4m\omega^3A^*) - (17A^*F_0\omega_n^2\lambda^3/32m\omega^3)\}\cos(\varphi^* - \theta) - (F_0\omega_n\lambda^2\xi/2m\omega^2A^*)\sin(\varphi^* - \theta)] - \varepsilon[\{\zeta(t)\lambda/\omega A^*\}\cos(\omega t + \varphi^*)]. \quad (30)$$

The expressions in equations (29) and (30) are taken as approximations of the amplitude and phase terms, $\dot{A}$ and $\dot{\varphi}$, i.e.,

$$\dot{A}^* \approx \dot{A}_{no}, \qquad \dot{\varphi}^* \approx \dot{\varphi}_{no}. \tag{31}$$

The amplitude term, A^* and phase term, φ^* approximations are correlated with random excitation force $\zeta(t)$. However, since $\zeta(t)$ is assumed to be a broad band random process, its correlation time is much smaller than the time constant characterising the rate of change of amplitude A^* and phase φ^*, which are slowly varying functions of time. It can be assumed that the values of $\zeta(t)$ are statistically independent of the values of A^*, i.e., amplitude A^* can be approximated as a Markov process [14, 16–18]. In addition, amplitude changes much more rapidly than the phase, and hence the amplitude manages to establish an equilibrium amplitude distribution $p(A^* \mid \varphi^*)$ for every value of phase φ^*.

The Fokker–Planck equation for amplitude A^*, from equation (31), can be formulated as

$$\begin{aligned} &-(\partial/\partial A^*)\,[\{-\varepsilon(F_0\lambda/2m\omega)\sin(\varphi^*-\theta)-\varepsilon(\omega_n\lambda\xi A^*)+\varepsilon^2(F_0\lambda/4m\omega^2)\{-(5\omega\lambda/4)\\ &\quad+(5\omega_n^2\lambda/4\omega)+(A^{*2}\omega_n^2\lambda^2/\omega)\}\sin(\varphi^*-\theta)+\varepsilon^2(5\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^*-\theta)\\ &\quad+\varepsilon^2\{S(\zeta;\omega)\lambda^2/8\omega^2A^*\}\}p]+\varepsilon^2\{S(\zeta;\omega)\lambda^2/8\omega^2\}\,(\partial^2p/\partial A^{*2})=\partial p(A^*)/\partial t, \end{aligned} \tag{32}$$

where $S(\zeta;\omega)$ is spectral density of the random excitation $\zeta(t)$ at the frequency ω. For a stationary case equation (31) reduces to

$$\begin{aligned} &-(\partial/\partial A^*)\,[\{-\varepsilon(F_0\lambda/2m\omega)\sin(\varphi^*-\theta)-\varepsilon(\omega_n\lambda\xi A^*)+\varepsilon^2(F_0\lambda/4m\omega^2)\{-(5\omega\lambda/4)\\ &\quad+(5\omega_n^2\lambda/4\omega)+(A^{*2}\omega_n^2\lambda^2/\omega)\}\sin(\varphi^*-\theta)+\varepsilon^2(5\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^*-\theta)\\ &\quad+\varepsilon^2\{S(\zeta;\omega)\lambda^2/8\omega^2A^*\}\}p]+\varepsilon^2\{S(\zeta;\omega)\lambda^2/8\omega^2\}\,(\partial^2p/\partial A^{*2})=0. \end{aligned} \tag{33}$$

The solution to the stationary Fokker–Planck equation, (33), is

$$\begin{aligned} p(A^*)&=cA^*\exp[-\{8\omega^2/\varepsilon^2\lambda^2S(\zeta,\omega)\}\{\varepsilon(A^*F_0\lambda/2m\omega)\sin(\varphi^*-\theta)\\ &\quad+\varepsilon(A^{*2}\omega_n\lambda\xi/2)-\varepsilon^2(F_0\lambda/4m\omega^2)\{-(5A^*\omega\lambda/4)+(5A^*\omega_n^2\lambda/4\omega)\\ &\quad+(A^{*3}\omega_n^2\lambda^2/3\omega)\}\sin(\varphi^*-\theta)-\varepsilon^2(5A^*\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^*-\theta)\}]. \end{aligned} \tag{34}$$

5. EXTRACTION OF ROTOR-BEARING PARAMETERS

The probability density function for any two values A_i^* and A_{i+1}^*, of the amplitude (with $A_{i+1}^* > A_i^*$), can be written from equation (34), as

$$\begin{aligned} p(A_i^*)&=cA_i^*\exp[-\{8\omega^2/\varepsilon^2\lambda^2S(\zeta,\omega)\}\{\varepsilon(A_i^*F_0\lambda/2m\omega)\sin(\varphi^*-\theta)\\ &\quad+\varepsilon(A_i^{*2}\omega_n\lambda\xi/2)-\varepsilon^2(F_0\lambda/4m\omega^2)\{-(5A_i^*\omega\lambda/4)+(5A_i^*\omega_n^2\lambda/4\omega)\\ &\quad+(A_i^{*3}\omega_n^2\lambda^2/3\omega)\}\sin(\varphi^*-\theta)-\varepsilon^2(5A_i^*\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^*-\theta)\}], \end{aligned} \tag{35}$$

$$\begin{aligned} p(A_{i+1}^*)&=cA_{i+1}^*\exp[-\{8\omega^2/\varepsilon^2\lambda^2S(\zeta,\omega)\}\{\varepsilon(A_{i+1}^*F_0\lambda/2m\omega)\sin(\varphi^*-\theta)\\ &\quad+\varepsilon(A_{i+1}^{*2}\omega_n\lambda\xi/2)-\varepsilon^2(F_0\lambda/4m\omega^2)\{-(5A_{i+1}^*\omega\lambda/4)+(5A_{i+1}^*\omega_n^2\lambda/4\omega)\\ &\quad+(A_{i+1}^{*3}\omega_n^2\lambda^2/3\omega)\}\sin(\varphi^*-\theta)-\varepsilon^2(5A_{i+1}^*\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^*-\theta)\}]. \end{aligned} \tag{36}$$

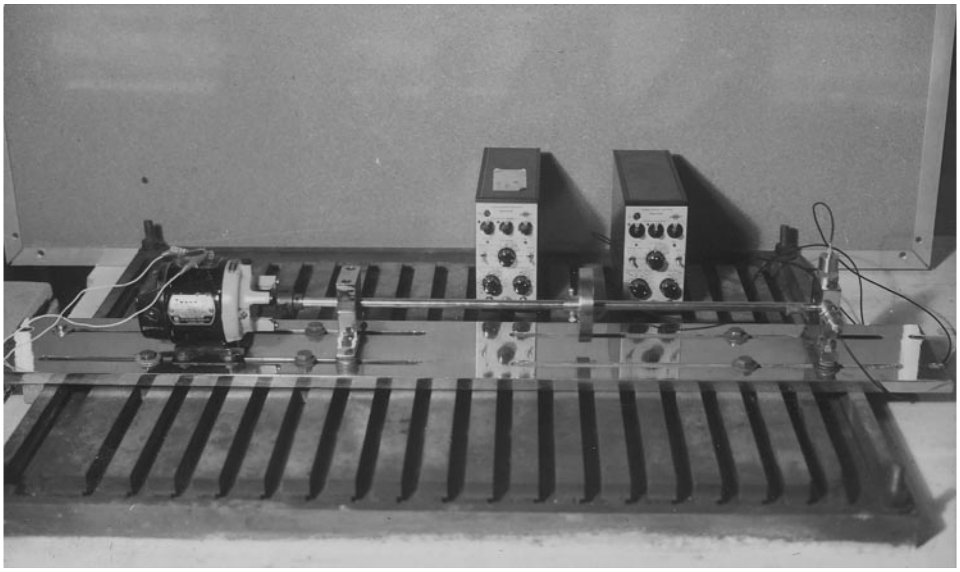

Figure 2. Rotor-bearing set-up.

Defining, $\Delta A_i^* = (A_{i+1}^* - A_i^*)$, for small ΔA_i^*, one can write, from equations (35) and (36)

$$[p(A_{i+1}^*)/p(A_i^*)] = (A_{i+1}^*/A_i^*) \exp[-\{8\omega^2/\varepsilon^2\lambda^2 S(\zeta,\omega)\}\{\varepsilon(\Delta A_i^* F_0\lambda/2m\omega)\sin(\varphi^* - \theta) + \varepsilon(A_i^*\Delta A_i^*\omega_n\lambda\xi) - \varepsilon^2(F_0\lambda/4m\omega^2)\{-(5\Delta A_i^*\omega\lambda/4) + (5\Delta A_i^*\omega_n^2\lambda/4\omega) + (A_i^{*2}\Delta A_i^*\omega_n^2\lambda^2/\omega)\}\sin(\varphi^* - \theta) - \varepsilon^2(5\Delta A_i^*\omega_n\lambda^2\xi F_0/8m\omega^2)\cos(\varphi^* - \theta)\}]. \quad (37)$$

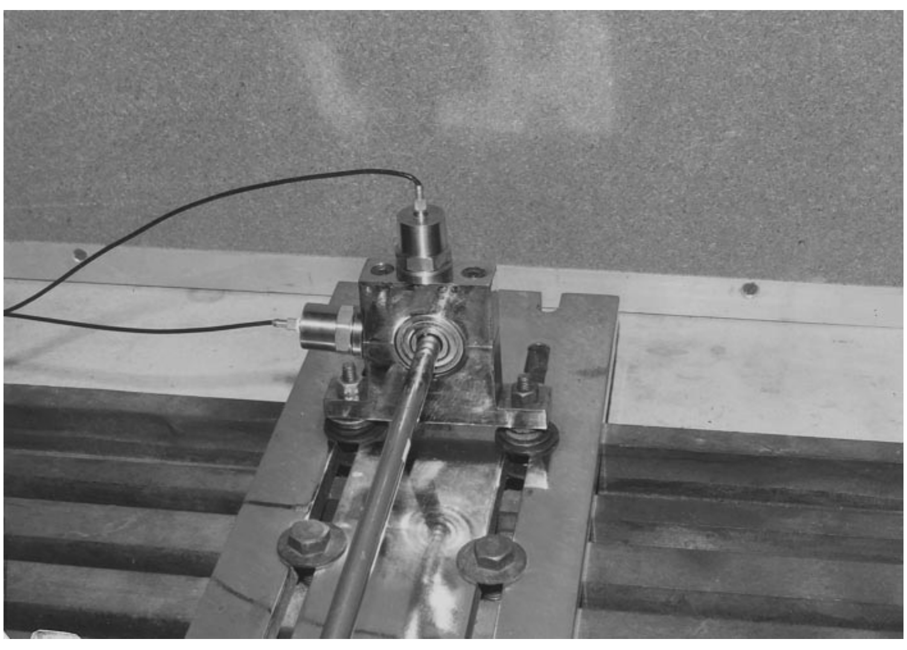

Figure 3. Accelerometer mountings.

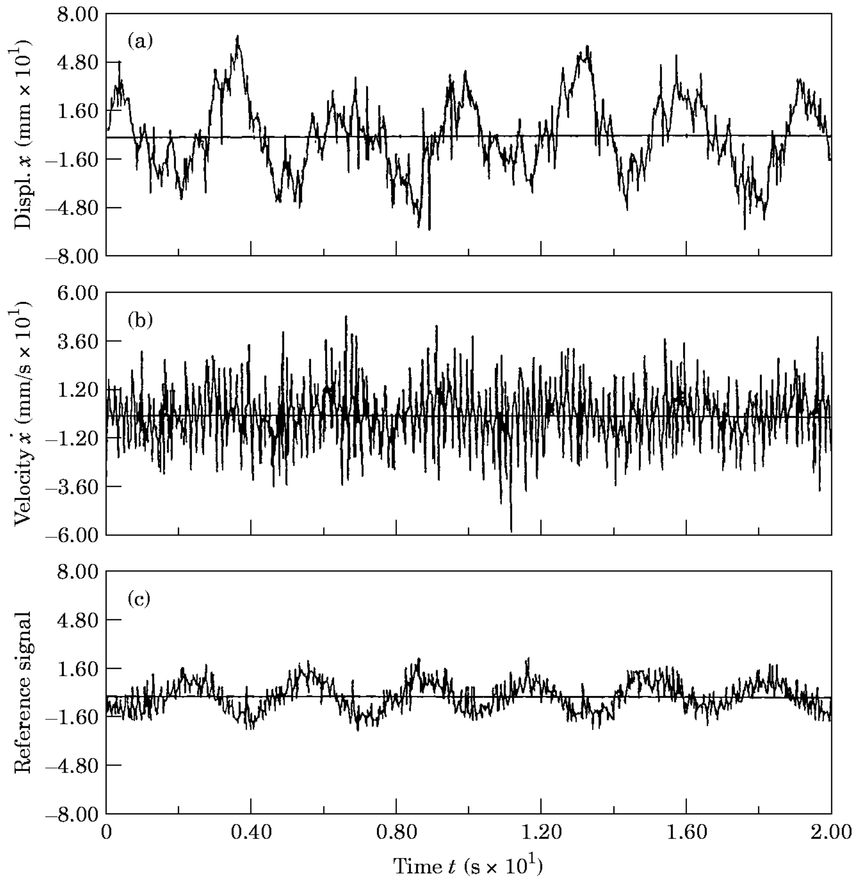

Figure 4. Displacement, velocity and reference signals at 1800 r.p.m.

For N amplitude values, $A_1^*, A_2^*, \ldots, A_N^*$, equation (37) is expressed as set of $(N-1)$ linear simultaneous algebraic equations,

$$
\begin{aligned}
&[(1/8\Delta A_i^*\omega^2)\ln\{A_{i+1}^*p(A_i^*)/A_i^*p(A_{i+1}^*)\}]\{S(\zeta,\omega)/\omega_n\xi F_0\cos\theta\} \\
&\quad+[13/16m\omega]\{\sin(\varphi-\theta)/\omega_n\xi\cos\theta\}-[5/16m\omega^3]\{\omega_n\sin(\varphi^*-\theta)/\xi\cos\theta\} \\
&\quad-[A_i^{*2}/4m\omega^3]\{\omega_n\lambda\sin(\varphi^*-\theta)/\xi\cos\theta\}+[A_i^*]\{1/F_0\cos\theta\} \\
&\quad+[5\sin\varphi^*/8m\omega^2]\{\tan\theta\} \\
&=-[5\cos\varphi(/8m\omega^2] \qquad i=1,2,\ldots,(N-1). \qquad (38)
\end{aligned}
$$

Equations (38) are used to estimate the parameters ω_n, λ, F_0, ξ, $S(\zeta,\omega)$ and θ using the Least Squares procedure.

The amplitude displacement and velocity data (x and $\dot{x}$) is obtained experimentally, and using equation (6), the amplitude A and phase φ are computed along with the probability function, $p(A)$, to be fed into equation (38) for parameter estimation. However, equation (38) involves A^*, φ^* and $p(A^*)$ and as an initial approximation the experimentally obtained A, φ and $p(A)$ are taken as A^*, φ^* and $p(A^*)$ respectively.

The proposed method is illustrated for a laboratory rotor shown in Figures 2 and 3. The rig consists of a disc of mass $m = 0{\cdot}41$ kg, centrally mounted on the shaft supported in two identical bearings. The shaft is driven through a flexible coupling by a motor. The

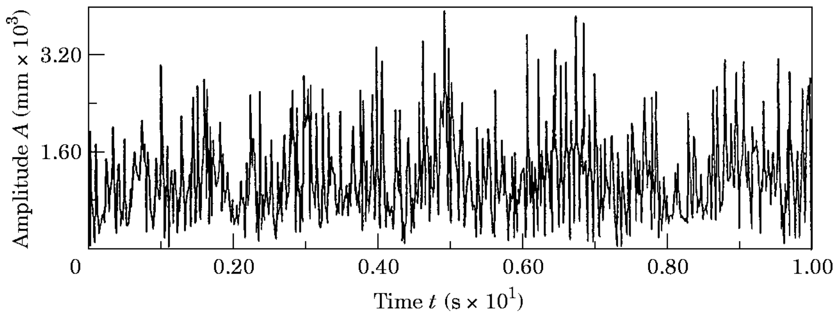

Figure 5. Amplitude variation of the measured response.

accelerometers are mounted on one of the bearing caps to pick up the vibration signals. A reference signal can be picked up from the shaft by a non-contact eddy current proximity probe. To reduce the noise level in the measured response outside the interest of the frequency range, bandpass filter has been used along with the charge amplifier. The rotor is dynamically balanced and then a known imbalance mass is attached at the disc. The shaft is rotated at a particular speed and the signals are picked up. The experiment is repeated for different set of known imbalance masses and for different speeds.

Experimentally obtained displacement and velocity signals along with the corresponding reference signal are shown, for a rotor speed of 1800 r.p.m. in Figure 4. The amplitude A and phase φ signals computed from these measured data are shown in Figures 5 and 6. The probability density function, $p(A)$, of the amplitude is shown in Figure 7. The bearing parameters estimated for initial approximation of using experimentally obtained $A(t)$ as $A^*(t)$ and $\varphi(t)$ as $\varphi^*(t)$ in equations (38) are given in Table 1.

For a more accurate estimation of estimated parameters an iterative scheme is used. The parameters estimated from the above initial approximation are substituted in equations (22, 23), along with the initial assumption of taking the experimental A and φ to be A^* and φ^* respectively, to compute the variations u and v. These values of u and v and the experimental A and φ are employed in equation (14) to get new approximations for A^* and φ^*. The new approximations, A^* and φ^*, are now employed in equation (38) for a fresh parameter estimation and thus, the iterative cycle can be continued. The final set of parameters estimated after such iteration is given in Table 2.

The closeness, between the known imbalance introduced in the rotor and its angular location and the corresponding experimentally estimated values, is a measure of the validity of the procedure developed. The estimated damping ratios also appear physically reasonable.

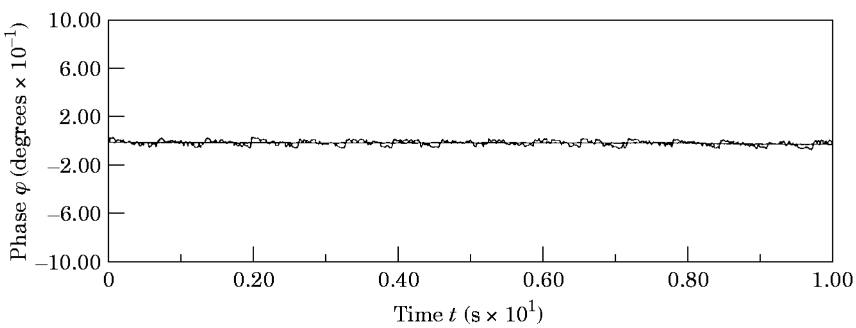

Figure 6. Phase variation of the measured response.

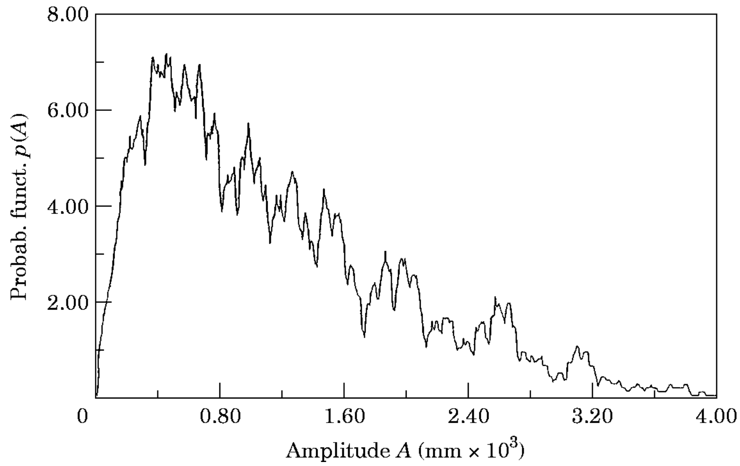

Figure 7. Probability density distribution of the response amplitude.

6. VALIDATION OF ESTIMATED STIFFNESS

The analytical formulations of Harris [5] and Ragulskis *et al*. [4], which are based on Hertzian contact theory, are employed for comparison of the bearing stiffness parameters k_L and k_{NL}, obtained by the procedure developed. The total elastic force at the points of contact of the ith ball with the inner and outer races is expressed as [4]

$$F_i = K_n(g + x\cos\eta_i + y\sin\eta_i)^{3/2}, \tag{39}$$

and its projection along the line of action of the applied force is

$$F_i = K_n(g + x\cos\eta_i + y\sin\eta_i)^{3/2}\cos\eta_i, \tag{40}$$

where g is the radial preload or pre-clearance between the ball and the races and x and y are the displacements of the moving ring in the direction of the radial load and perpendicular to the direction of the radial load respectively. η_i is the angle between the lines of action of the radial load (direction of displacement of the moving ring) and the radius passing through the center of the ith ball. K_n is a coefficient of proportionality depending on the geometric and material properties of the bearing. The specifications of the test bearing are: Ball bearing type SKF 6200, Number of balls 6, Ball diameter 6 mm, Bore diameter 10 mm, Outer diameter 30 mm, Pitch diameter 20 mm, Inner groove radius 3·09 mm, Outer groove radius 3·09 mm, Allowable pre-load 0–2 microns. The value of K_n, for the test bearing with the above specifications, is estimated by the method suggested by Harris [5] as $2{\cdot}82 \times 10^5$ N/mm$^{1{\cdot}5}$. The total elastic force in the direction of the applied force is

$$F = \sum_{i=1}^{n} F_i, \tag{41}$$

where n is the total number of balls in the bearing. Using the condition of zero elastic force in the direction perpendicular to the elastic load, the deformation, y, perpendicular to the radial force line is expressed as

$$y = \sum_{i=1}^{n} [g + x\cos(\eta_i)]^{3/2}\sin(\eta_i) \Bigg/ \sum_{i=1}^{n} [g + x\cos(\eta_i)]^{1/2}\sin^2(\eta_i). \tag{42}$$

TABLE 1

Experimentally estimated parameters (after initial approximation)

Speed ω (r.p.m.)	Imbalance me (gm cm) (at $\theta = 0°$)	Estimated parameters				
		φ (°)	θ (°)	me (gm cm)	ξ	$k(x)$ (N/mm)
1800	10·5	0·7	0·1	16·80	0·062	$1·21 \times 10^4–0·58 \times 10^{10}x^2$
–	17·5	1·4	0·2	22·73	0·071	$1·23 \times 10^4–0·83 \times 10^{10}x^2$
–	24·5	6·9	0·1	30·64	0·022	$1·39 \times 10^4–1·06 \times 10^{10}x^2$
1400	10·5	3·0	0·3	17·24	0·083	$0·86 \times 10^4–0·84 \times 10^{10}x^2$
–	17·5	3·3	0·5	23·45	0·082	$0·79 \times 10^4–0·70 \times 10^{10}x^2$
–	24·5	4·6	0·2	16·56	0·042	$1·12 \times 10^4–0·63 \times 10^{10}x^2$

Equations (40) and (42) are used in equation (41) and the bearing stiffness is determined as a function of the deformation x as

$$k(x) = \partial F/\partial x. \tag{43}$$

It can be seen that the bearing stiffness is critically dependent on the preloading, g, of the balls. While the manufacturer may, at times, provide the preload range, the exact value of the preloading of the bearing balls in the shaft-casing assembly, especially during operations which have involved wear and tear, would be difficult to determine. The stiffness of the test bearing is listed in Table 3 as a function of the radial deformation, x, for various allowable preload values, g. The bearing stiffnesses obtained experimentally, using the procedure developed, Table 2, shows good resemblance to theoretically possible values. It is to be noted that the theoretical stiffness calculations are based on formulations which analyse the bearing in isolation of the shaft.

TABLE 2

Experimentally estimated parameters (after iteration)

Speed ω (r.p.m.)	Imbalance me (gm cm) (at $\theta = 0°$)	Estimated parameters				
		φ (°)	θ (°)	me (gm cm)	ξ	$k(x)$ (N/mm)
1800	10·5	2·9	0·3	14·23	0·023	$1·31 \times 10^4–0·92 \times 10^{10}x^2$
–	17·5	1·6	0·1	20·32	0·036	$1·41 \times 10^4–1·10 \times 10^{10}x^2$
–	24·5	3·2	0·2	27·32	0·032	$1·60 \times 10^4–1·28 \times 10^{10}x^2$
1400	10·5	1·7	0·2	15·43	0·063	$0·94 \times 10^4–0·94 \times 10^{10}x^2$
–	17·5	3·2	0·4	21·40	0·063	$0·93 \times 10^4–1·01 \times 10^{10}x^2$
–	24·5	1·8	0·1	23·56	0·037	$1·37 \times 10^4–0·72 \times 10^{10}x^2$

TABLE 3

Experimental and theoretical [4, 5] *bearing stiffness parameters*

Preload (mm)	Theroetical stiffness (Radial (N/mm))
0·0002	$1·20 \times 10^4–4·01 \times 10^{10}x^2$
0·0003	$1·47 \times 10^4–2·18 \times 10^{10}x^2$
0·0004	$1·69 \times 10^4–1·42 \times 10^{10}x^2$
0·0005	$1·89 \times 10^4–1·02 \times 10^{10}x^2$
0·0006	$2·08 \times 10^4–6·09 \times 10^{10}x^2$

7. CONCLUSION

A procedure for the estimation of linear and non-linear stiffness parameters of rolling element bearings supporting a rigid rotor having random excitation in addition to an unknown imbalance, makes certain engineering approximations, including idealisation of the excitations from bearing surface and assembly imperfections as white noise sources and works directly on the response signals, which can be conveniently picked up at the rotor-bearing caps. The procedure has the advantage that it does not require a knowledge of the excitation forces. The procedure, in addition to the stiffness parameters, also provides estimates of the magnitude and angular location of the imbalance and the damping ratio.

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