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LETTERS TO THE EDITOR

ON THE EIGENVALUES OF VISCOUSLY DAMPED BEAMS, CARRYING HEAVY MASSES AND RESTRAINED BY LINEAR AND TORSIONAL SPRINGS

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1. INTRODUCTION

The problem of free vibrations of beams and rods with various boundary conditions and attached masses and springs has been investigated by many authors. A list of some of these studies can be found in references [1, 2]. Recently, in the context of a project study, it was necessary to derive the characteristic equation of a Bernoulli–Euler beam to which, in addition to springs and heavy masses, viscous dampers are also attached. During the efforts towards incorporation of the damping effect into the formulations, the present author observed that this effect was not taken into account in the studies cited above. Therefore, the aim of this note is to give a systematic formulation of the approximate characteristic equation of a beam carrying heavy masses restrained by linear and rotational springs and damped by linear viscous dampers, as shown in Figure 1.

2. THEORY

The partial differential equation of the free bending vibrations of a uniform beam according to Bernoulli–Euler theory, is the well known expression [3]

$$EI\, \partial^4 w(x, t)/\partial x^4 + m\, \partial^2 w(x, t)/\partial t^2 = F\,[w(x, t)]\, \delta(x - x_j) \tag{1}$$

where EI and m denote the bending rigidity and mass per unit length respectively. The operator $F[w(x, t)]$ on the right side represents any attachments like point and/or heavy masses, linear and/or rotational springs and dampers at the discrete points $x = x_j$ on the beam.

An approximate series solution of equation (1) can be taken in the form

$$w(x, t) \approx \sum_{r=1}^{n} w_r(x)\eta_r(t) \tag{2}$$

where the $w_r(x)$ are the orthogonal eigenfunctions of the beam without any appendages, normalized with respect to the mass density. The $\eta_r(t)$ are unknown and time dependent generalized co-ordinates.

After substitution of expression (2) into the differential equation (1) both sides of the equation are multiplied by the sth eigenfunction $w_s(x)$ and integrated over the beam length. By using the orthogonality property of the eigenfunctions, the system of modal equations, i.e., the system of differential equations for the $\eta_r(t)$, is obtained. In the previous works of the present author [1, 4] the attachments considered were point masses, linear, rotational springs and heavy masses. By combining the results of those works, the modal equations of the beam system in Figure 1 without the dampers can be given as

$$\ddot{\eta}_r(t) + \omega_r^2\, \eta_r(t) = N_r^*(t) + N_r^{**}(t) + N_r^{***}(t) + N_r^{****}(t), \qquad (r = 1, \ldots, n). \tag{3}$$

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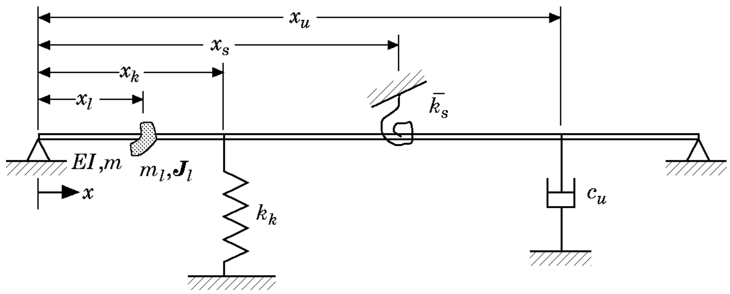

Figure 1. Viscously damped Bernoulli–Euler beam carrying heavy masses restrained by linear and torsional springs.

Here, ω_r denotes the rth eigenfrequency of the beam without any masses and springs, $N_r^*(t), \ldots, N_r^{****}(t)$ are the generalized forces corresponding to the point masses, linear or torsional springs and heavy masses, respectively:

$$N_r^*(t) = -\sum_{l=1}^{p} m_l\, w_r(x_l) \sum_{j=1}^{n} \ddot{\eta}_j(t) w_j(x_l),$$

$$N_r^{**}(t) = -\sum_{k=1}^{q} k_k\, w_r(x_k) \sum_{j=1}^{n} \eta_j(t) w_j(x_k),$$

$$N_r^{***}(t) = -\sum_{s=1}^{\upsilon} \bar{k}_s\, w_r'(x_s) \sum_{j=1}^{n} \eta_j(t) w_j'(x_s),$$

$$N_r^{****}(t) = -\sum_{l=1}^{p} J_l\, w_r'(x_l) \sum_{j=1}^{n} \ddot{\eta}_j(t) w_j'(x_l). \tag{4}$$

If it is assumed that the beam is additionally acted upon by v linear viscous dampers, it can be shown that the right side of equation (3) has to be ammended by the generalized force corresponding to the dampers, $N_r^{*****}(t)$, which has the form

$$N_r^{*****}(t) = -\sum_{u=1}^{v} c_u\, w_r(x_u) \sum_{j=1}^{n} \dot{\eta}_j(t) w_j(x_u). \tag{5}$$

As can be seen from Figure 1, c_u is the damping constant of the uth viscous damper and $w_r(x)$ denotes the rth eigenfunction of the beam without any masses, springs and dampers.

Hence, the modal equations of the combined system in Figure 1 can be formulated as

$$\ddot{\eta}_r(t) + \omega_r^2 \eta_r(t) + \sum_{l=1}^{p} \sum_{l=1}^{n} m_l\, w_r(x_l) w_j(x_l) \ddot{\eta}_j(t) + \sum_{k=1}^{q} \sum_{j=1}^{n} k_k\, w_r(x_k) w_j(x_k) \eta_j(t)$$

$$+ \sum_{s=1}^{\upsilon} \sum_{j=1}^{n} \bar{k}_s\, w_r'(x_s) w_j'(x_s) \eta_j(t) + \sum_{l=1}^{p} \sum_{j=1}^{n} J_l\, w_r'(x_l) w_j'(x_l) \ddot{\eta}_j(t)$$

$$+ \sum_{u=1}^{v} \sum_{j=1}^{n} c_u\, w_r(x_u) w_j(x_u) \dot{\eta}_j(t) = 0, \quad (r = 1, \ldots, n). \tag{6}$$

A solution of the form

$$\eta_r(t) = \bar{\eta}_r\, e^{\lambda t}, \quad (r = 1, \ldots, n), \tag{7}$$

where λ denotes an eigenvalue of the combined system, yields the following system of homogeneous equations:

$$\lambda^2\left(\bar{\eta}_r + \sum_{l=1}^{p}\sum_{j=1}^{n} m_l\, w_r(x_l) w_j(x_l)\bar{\eta}_j + \sum_{l=1}^{p}\sum_{j=1}^{n} J_l\, w_r'(x_l) w_j'(x_l)\bar{\eta}_j\right)$$

$$+ \lambda \sum_{u=1}^{v}\sum_{j=1}^{n} c_u\, w_r(x_u) w_j(x_u)\bar{\eta}_j$$

$$+ \omega_r^2\, \bar{\eta}_r + \sum_{k=1}^{q}\sum_{j=1}^{n} k_k\, w_r(x_k) w_j(x_k)\bar{\eta}_j + \sum_{s=1}^{v}\sum_{j=1}^{n} \bar{k}_s\, w_r'(x_s) w_j'(x_s)\bar{\eta}_j = 0,$$

$$(r = 1, \ldots, n). \tag{8}$$

In order to use the advantages of matrix notation, one can define

$$\bar{\eta} = [\bar{\eta}_1, \ldots, \bar{\eta}_n]^{\mathrm{T}}, \quad \mathbf{w}(x) = [w_1(x), \ldots, w_n(x)]^{\mathrm{T}}, \quad \mathbf{W}(x) = \mathbf{w}(x)\mathbf{w}^{\mathrm{T}}(x),$$

$$\mathbf{W}'(x) = \mathbf{w}'(x)\mathbf{w}'^{\mathrm{T}}(x), \quad \boldsymbol{\omega}^2 = \mathbf{diag}\,(\omega_i^2), \quad \mathbf{A}' = \sum_{k=1}^{q} k_k\, \mathbf{W}(x_k), \quad \mathbf{B}' = \sum_{l=1}^{p} m_l\, \mathbf{W}(x_l),$$

$$\mathbf{B}'' = \sum_{l=1}^{p} J_l\, \mathbf{W}'(x_l), \quad \mathbf{C}' = \sum_{s=1}^{v} \bar{\mathbf{k}}_s\, \mathbf{W}'(x_s), \quad \mathbf{D} = \sum_{u=1}^{v} \mathbf{c}_u'\, \mathbf{W}(x_u). \tag{9}$$

As can be seen from the definitions of $\mathbf{W}(x)$ and $\mathbf{W}'(x)$, $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{B}''$, $\mathbf{C}'$, and $\mathbf{D}$ are symmetric matrices.

Furthermore, by introducing the abbreviations

$$\mathbf{A} = \mathbf{A}' + \mathbf{C}' + \boldsymbol{\omega}^2, \qquad \mathbf{B} = \mathbf{B}' + \mathbf{B}'' + \mathbf{I} \tag{10}$$

with $\mathbf{I}$ being the n-dimensional unit matrix and then starting with the system of equations (8), the following general eigenvalue problem is obtained:

$$(\lambda^2\mathbf{B} + \lambda\mathbf{D} + \mathbf{A})\eta = \mathbf{0}. \tag{11}$$

This means that the characteristic values λ of the mechanical system are obtained as the roots of the characteristic equation

$$\det\,(\lambda^2\mathbf{B} + \lambda\mathbf{D} + \mathbf{A}) = 0. \tag{12}$$

It is known [5] that the equation of motion of a damped linear discrete system,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}, \tag{13}$$

can be formulated in the state space as

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}, \tag{14}$$

where the system matrix **A** is defined as

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}. \tag{15}$$

The characteristic values λ of the mechanical system (13) can either be determined as the roots of the characteristic equation

$$\det(\lambda^2\mathbf{M} + \lambda\mathbf{D} + \mathbf{K}) = 0 \tag{16}$$

or as the eigenvalues of the matrix **A**. Hence, making use of the analogy between (12) and (16), one is able to state that the characteristic values λ of the present beam system can also be obtained as the eigenvalues of the matrix $\mathbf{A}^*$, which is defined as

$$\mathbf{A}^* = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{B}^{-1}\mathbf{A} & -\mathbf{B}^{-1}\mathbf{D} \end{bmatrix}. \tag{17}$$

After having obtained the characteristic equation for a beam which can be subject to general boundary conditions, one now wishes to give the corresponding expressions for a clamped–free beam in dimensionless form. To this end, following definitions are introduced [6]:

$$\bar{x} = x/L, \qquad \mathbf{a}(\bar{x}) = [a_1(\bar{x}), \ldots, a_n(\bar{x})]^{\mathrm{T}},$$

$$a_k(\bar{x}) = \mathrm{ch}\,\overline{\beta}_k\bar{x} - \cos\overline{\beta}_k\bar{x} - \bar{\eta}_k\,\mathrm{sh}\,\overline{\beta}_k\bar{x} - \sin\overline{\beta}_k\bar{x}),$$

$$\bar{\eta}_k = (\mathrm{ch}\,\overline{\beta}_k + \cos\overline{\beta}_k)/(\mathrm{sh}\,\overline{\beta}_k + \sin\overline{\beta}_k), \quad \overline{\beta}_1 = 1{\cdot}875104, \quad \overline{\beta}_2 = 4{\cdot}694091,$$

$$\overline{\beta}_3 = 7{\cdot}854757, \ldots,$$

$$\overline{\mathbf{B}} = \mathbf{diag}\,(\overline{\beta}_k^4), \quad \omega_0^2 = EI/mL^4, \quad \bar{m}_l = m_l/mL, \quad \bar{J}_l = \bar{J}_l/mL^3, \quad \bar{c}_u = c_u/mL\omega_0^2,$$

$$\overline{k}_k = k_k/mL\omega_0^2, \quad \overline{\overline{k}}_s = \overline{k}_s/mL^3\omega_0^2, \qquad \lambda^* = \lambda/\omega_0, \qquad ()' = \mathrm{d}()/\mathrm{d}\bar{x}. \tag{18}$$

With these definitions, one notes that the eigenfrequencies of the beam without any attachments, i.e., the eigenfrequencies ω_k of the clamped–free beam can be represented in terms of the dimensionless frequency parameters $\overline{\beta}_k$ as [3]

$$\omega_k = \overline{\beta}_k^2\,\omega_0, \qquad (k = 1, 2, \ldots).$$

The characteristic equation (12) can be given as

$$\det(\lambda^{*2}\mathbf{M}^* + \lambda^*\mathbf{D}^* + \mathbf{K}^*) = 0, \tag{19}$$

with

$$\mathbf{M}^* = \mathbf{I} + \sum_{l=1}^{p} \bar{m}_l\, a(\bar{x}_l)a^{\mathrm{T}}(\bar{x}_l) + \sum_{l=1}^{p} \bar{J}_l\, a'(\bar{x}_l)a'^{\mathrm{T}}(\bar{x}_l), \qquad \mathbf{D}^* = \sum_{u=1}^{v} \bar{c}_u\, a(\bar{x}_u)a^{\mathrm{T}}(\bar{x}_u),$$

$$\mathbf{K}^* = \overline{\mathbf{B}} + \sum_{k=1}^{q} \bar{k}_k\, a(\bar{x}_k)a^{\mathrm{T}}(\bar{x}_k) + \sum_{s=1}^{v} \overline{\overline{k}}_s\, a'(\bar{x}_s)a'^{\mathrm{T}}(\bar{x}_s), \tag{20}$$

where $\bar{x} = x/L$.

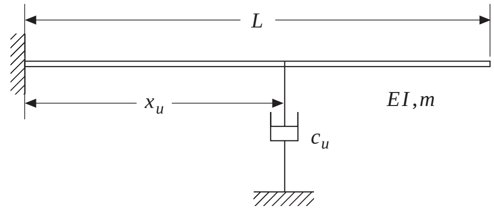

Figure 2. Viscously damped clamped–free Bernoulli–Euler beam.

The characteristic values λ^* of the system can also be determined as the eigenvalues of the matrix $\bar{\mathbf{A}}$, defined as

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{*-1}\mathbf{K}^* & -\mathbf{M}^{*-1}\mathbf{D}^* \end{bmatrix}. \tag{21}$$

3. APPLICATION TO A SIMPLE SYSTEM

One can take the system shown in Figure 2 as a simple application example. *EI* and *m* are the bending rigidity and mass per unit length of the beam. In reference [7], the sensitivity of the eigenvalues of this system with respect to small changes of the damping constant and the location of the damper was investigated. The following characteristic equation was obtained:

$$1 + c_u \lambda \sum_{k=1}^{n} \frac{w_k^2}{\lambda^2 + \omega_k^2} = 0. \tag{22}$$

Here w_k denotes $w_k(x_u)$. It can be shown that equation [22] can be rewritten in the notation of the present note as

$$1 + \bar{c}_u \lambda^* \sum_{i=1}^{n} \frac{a_i^2(\bar{x}_u)}{\lambda^{*2} + \bar{\beta}_i^4} = 0. \tag{23}$$

TABLE 1

Non-dimensional λ^* characteristic values; first column: eigenvalues of the matrix $\bar{\boldsymbol{A}}$ in equation (21); *second column: roots of equation* (23)

From equation (21)	From equation (23)
−0·0082235 ± 3·5160452i	−0·0082235 ± 3·5160452i
−0·1827152 ± 22·036269i	−0·1827152 ± 22·036269i
−0·7369632 ± 61·700122i	−0·7369632 ± 61·700122i
−1·1457409 ± 120·88565i	−1·1457409 ± 120·88565i
−0·8767038 ± 199·83633i	−0·8767038 ± 199·83633i
−0·2285508 ± 298·55027i	−0·2285508 ± 298·55027i
−0·0210505 ± 416·99053i	−0·0210505 ± 416·99053i
−0·4949405 ± 555·16234i	−0·4949405 ± 555·16234i
−0·9764712 ± 713·06949i	−0·9764712 ± 713·06949i
−0·7967546 ± 890·71920i	−0·7967546 ± 890·71924i

Consider now a system with the following physical data: $\bar{x}_u = 0{\cdot}2$, $\omega_0^2 = 54{\cdot}012345$ rad^2/s^2, $\bar{c}_u = 0{\cdot}137143$; these correspond to the mechanical system in reference [7].

The first column of Table 1 contains the non-dimensional λ^* values obtained as the eigenvalues of the $2n$ dimensional square matrix defined by equation (21). In the second column, those λ^* values are collected which are obtained as the roots of equation (23), which is essentially taken from reference [7]. In both cases $n = 10$ is taken. Inspection of the complex numbers in both columns indicates clearly that their agreement is excellent.

4. CONCLUSIONS

In this note, an approximate characteristic equation of a Bernoulli-Euler beam carrying heavy masses, restrained by linear and torsional springs and damped by linear viscous dampers is derived. The resulting expressions are applied to a simple system.

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