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UNCONDITIONAL STABILITY DOMAINS OF STRUCTURAL CONTROL SYSTEMS USING DUAL ACTUATOR–SENSOR PAIRS

V. JAYACHANDRAN AND J. Q. SUN

Department of Mechanical Engineering, University of Delaware, Newark, DE 19716, *U.S.A.*

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1. INTRODUCTION

An earlier study in reference [1] has demonstrated the complications in placing physically dual as well as non-dual actuator–sensor pairs on one dimensional structures. In particular, it outlines the unconditional stability domains of an active damping SISO control system in the parameter space expanded by the sensor and actuator co-ordinates. It shows that colocated physically dual actuator–sensor pairs guarantee unconditional stability whereas physically non-dual pairs do not admit to the same generalization. In that paper, the unconditional stability domain was investigated by excluding non-minimum phase systems from the stability domain. However, minimum phase systems can be unstable for intermediate values of feedback gain. The present paper presents the unconditional stability domains when the system is either minimum phase or non-minimum phase and hence accounts for all possible cases of instability. The effect of damping on the stability domain is also studied in this paper. This study shows that a system can change its stability characteristics quite significantly as the structural damping is altered. While most results in the paper are numerical, analytical derivations for a two-mode case are presented. It should be noted that the present paper addresses one of the issues related to decentralized structural controls with colocated dual sensor–actuator pairs, viz., the stability robustness of the system with respect to the placement of sensors and actuators [2–7]. This control design has a potential to substantially reduce the complexity and cost of controllers for high dimensional systems [8–14]. It is well known that for dual sensor–actuator pairs, colocation guarantees the unconditional stability of the closed loop system. Such unconditionally stable systems allow large control gains without causing instability and hence, may achieve superior performance. However, in reality, one may not always have the luxury of having colocated dual sensors and actuators. From a practical point of view, it is very important to know the regions to place sensors and actuators that lead to unconditionally stable closed loop systems.

This paper uses the same simple structure as in reference [1] to illustrate the discussions. The remainder of the paper is outlined as follows. In section 2, the mathematical statement of the problem considered in the paper is presented, and the unconditional stability conditions for minimum and non-minimum phase systems are discussed. In section 3, numerical results of unconditional stability domains for an active damping SISO feedback control system are presented.

2. THE PROBLEM STATEMENT

Consider the deflection $w(x, t)$ of a beam in terms of a finite modal function expansion given by

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$$w(x, t) = \sum_{i=1}^{n} W_i(x)\eta_i(t), \tag{1}$$

where $W_i(x)$ are orthogonal eigenfunctions satisfying the necessary differential equation and boundary conditions [15, 16]. In this paper, one considers a point force given by

$$u(x, t) = u(t)\delta(x - x_i), \tag{2}$$

where x_i is the co-ordinate of the control force on the beam. The control law is a velocity feedback given by

$$u(t) = -K_D\dot{w}(x_0, t), \tag{3}$$

where x_0 is the sensor location, and K_D is the control gain. The stability of the closed loop system is determined by the roots of the characteristic equation

$$1 + K_D G(s) = 0, \tag{4}$$

where s is the Laplace transform variable and $G(s)$ is the open loop transfer function of the system given by

$$G(s) = s\sum_{k=1}^{n} W_k(x_i)W_k(x_0)\prod_{j=1, j\neq k}^{n}(s^2 + cs + \omega_j^2)\Bigg/\prod_{j=1}^{n}(s^2 + cs + \omega_j^2) \equiv N(s)/D(s). \tag{5}$$

Here, c is the damping coefficient for all the modes and ω_j is the natural frequency of the jth mode.

2.1. *Non-minimum phase systems*

When some of the open loop zeros, i.e., the roots of the equation $N(s) = 0$, lie on the right side of the complex s-plane, the system is non-minimum phase and is conditionally stable. In this case, there is an upper bound on K_D, above which the closed loop system becomes unstable. However, below this critical value of K_D, the system may not be well damped. In searching for unconditional stability domains, one first sees if the system is non-minimum phase. This is done be checking the sign of the coefficients of the polynomial $N(s)$ and solving for the roots of the equation $N(s) = 0$ as is done in reference [1]. This reference has covered this case in detail.

2.2. *Minimum-phase conditionally stable systems*

A minimum phase system (i.e., all the open loop zeros and poles are on the left side of the s-plane) may be conditionally stable. In this case, the closed loop system is unstable for some intermediate values of K_D. In this section, one seeks to exclude the sensor and actuator locations on the beam that render the closed loop system only conditionally stable when the open loop system is minimum phase.

Assume now that the system is minimum phase. One checks to see if the system is unstable for some intermediate values of K_D by checking if the root locus ever crosses the imaginary axis. Note that this test is conclusive for both minimum phase and non-minimum phase systems. However, this test is numerically more intensive. Hence, it is better to apply the test only to the minimum phase case in order to save computational effort. One rewrites the root locus equation in the form:

$$\frac{1}{K_D} = -\frac{N(s)}{D(s)} = -s\sum_{k=1}^{n}\frac{w_k(x_i)w_k(x_0)}{s^2 + cs + \omega_k^2}. \tag{6}$$

Let $s = \mathrm{j}\omega$. When the root locus intersects the imaginary axis, one gets the following expression for the gain K_D :

$$\frac{1}{K_D} = -\mathrm{j}\omega \left\{ \sum_{k=1}^{n} w_k\,(x_i) w_k\,(x_0) \frac{[(\omega_k^2 - \omega^2) - \mathrm{j}c\omega]}{(\omega_k^2 - \omega^2)^2 + c^2\omega^2} \right\}. \tag{7}$$

This implies that there exist real valued frequencies ω that satisfy the conditions

$$\sum_{k=1}^{n} w_k\,(x_i) w_k\,(x_0) \frac{(\omega_k^2 - \omega^2)}{(\omega_k^2 - \omega^2)^2 + c^2\omega^2} = 0, \qquad \sum_{k=1}^{n} w_k\,(x_i) w_k\,(x_0) \frac{c\omega^2}{(\omega_k^2 - \omega^2)^2 + c^2\omega^2} < 0. \tag{8}$$

The procedure to look for conditionally stable minimum phase systems is to find the positive real roots of the first equation for ω^2 and then check the inequality.

2.3. *Effect of damping*

Increasing the structural damping generally moves the open loop poles and zeros to the left of the s-plane. In case of certain non-minimum phase systems, a higher structural damping can cause the open loop zeros in the right half s-plane to move over to the left half and consequently make the system minimum phase. This crossover happens only in certain cases when the actuator–sensor locations are not too far away from the unconditional stability region. On the other hand, in case of minimum phase conditionally stable systems, all the poles and zeros are in the left half s-plane to begin with. Adding structural damping moves them further to the left causing the whole root locus to shift into the left half s-plane. Hence, there is always a critical value of damping, c_{min}, which will ensure unconditional stability of such systems. If the damping is increased beyond this critical value, equation (8) will have no real valued roots, implying that the system is unconditionally stable for all positive gains. The above discussions on the effect of damping have been verified in the numerical results presented in section 3. In the next section, a two-mode case is used to examine the sources of instability.

2.4. *Two mode case*

In order to get more physical insight into the system behavior as well as its instability, an analytical study is performed on a model with only two modes. The root locus of equation (4) is rewritten as

$$(s^2 + cs + \omega_1^2)\,(s^2 + cs + \omega_2^2) + K_D\, s[a_1\,(s^2 + cs + \omega_2^2) + a_2\,(s^2 + cs + \omega_1^2)] = 0, \tag{9}$$

where $a_1 = w_1\,(x_i) w_1\,(x_0)$ and $a_2 = w_2\,(x_i) w_2\,(x_0)$. For unconditional stability, the roots of this equation must always lie in the left half s-plane, i.e., they must have negative real parts for all positive values of K_D. Routh's array is used to specify stability conditions [17]:

$$\xi_1 = 2c + K_D\,(a_1 + a_2) > 0,$$

$$\xi_2 = 2c^3 + (\omega_1^2 + \omega_2^2)c + K_D\,[3(a_1 + a_2)c^2 + \omega_1^2\, a_1 + \omega_2^2\, a_2] + K_D^2\,(a_1 + a_2)^2 c > 0,$$

$$\xi_3 = \xi_2\,[(\omega_1^2 + \omega_2^2)c + K_D\,(\omega_2^2\, a_1 + \omega_1^2\, a_2)] - \xi_1^2\, \omega_1^2\, \omega_2^2 > 0, \tag{10}$$

where $K_D \geqslant 0$. The first inequality requires that $c > 0$, and $a_1 + a_2 > 0$. In order for the third inequality to hold for large gains, one must have $a_1\,\omega_2^2 + a_2\,\omega_1^2 > 0$. Furthermore, it can be shown that there exists a minimum damping c_{min} such that the second and third inequalities hold for all $K_D \geqslant 0$. This minimum damping has to be determined numerically. In summary, one has the following conditions for unconditional stability:

Condition 1: $c > c_{min} > 0$; condition 2: $a_1 + a_2 > 0$; condition 3: $a_1\,\omega_2^2 + a_2\,\omega_1^2 > 0$.

It is interesting to consider what happens when the unconditional stability breaks down for a minimum phase system. Let the two-mode case be used as an example to illustrate the discussion. When the actuator and sensor are on the same side of the node of the second normal mode shape, the open loop zero created by the first and second modes falls in-between the open loop poles associated with these modes. The departure angle at these poles as the gain increases from zero to positive infinite is always $-180°$ plus or minus a small angle [17]. Hence, the poles approach the zeros via a path completely in the left side of the s-plane (see Figure 1(a)). When the actuator and sensor pair are placed on either side of the node, one gets $\omega_1^2\,a_1 + \omega_2^2\,a_2 < 0$ and the zero moves to the region above the pole corresponding to the second mode. Now the departure angle at the second mode pole becomes $0°$ plus or minus a small angle. This pole then approaches the zero via a path swinging to the right of the pole (see Figure 1(b)). Depending on the level of damping c, the path may or may not cross the imaginary axis. When it does cross the imaginary axis, the minimum phase system becomes conditionally stable. One will see later that the damping in the system determines how far apart an actuator and sensor can be when they are placed on either side of a node in order to guarantee unconditional stability.

3. NUMERICAL RESULTS AND DISCUSSIONS

The same cantilever beam made of aluminum as in reference [1] is used in this study. The length of the beam is 1·0 m, the width is 0·1 m and the thickness is 0·01 m. Extensive computations have been done to delineate the unconditional stability domains in the (x_i, x_0) parameter space. Figure 2 shows the unconditional stability domain for the non-minimum phase case only, from reference [1]. Figure 3 shows the same results when both the minimum phase and non-minimum phase cases are included. The damping coefficient c is 1·0 Ns/m. As can be seen from the figures, the unconditional stability domains in Figure 3 are far smaller than in Figure 2, suggesting the frequent occurrence of instability in the intermediate ranges of feedback gain.

Similar remarks regarding the property of the unconditional stability domains and the effect of number of modes, i.e., the bandwidth of the system, to those in reference [1] can be made based on the numerical simulations: (1) As the number of modes increases, the

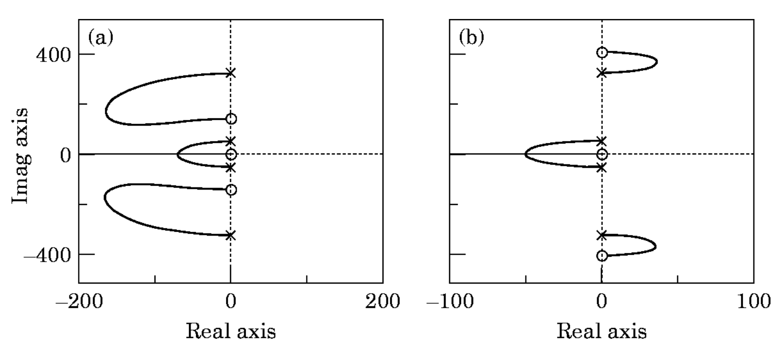

Figure 1. Root loci for the two mode system. (a) Unconditionally stable; $c = 1$ Ns/m; $(x_i, x_0) = (0{\cdot}3, 0{\cdot}6)$. (b) Conditionally stable; $c = 1$ Ns/m; $(x_i, x_0) = (0{\cdot}3, 0{\cdot}8)$.

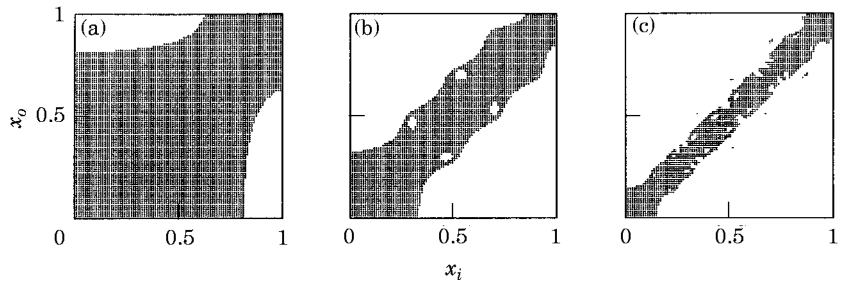

Figure 2. Unconditional stability domain for a point force-linear velocity sensor pair on a cantilever. Results from reference [1]; $c = 1$ Ns/m. The number of modes is (a) 2, (b) 5 and (c) 10; x_i is the actuator location and x_0 is the sensor location

unconditional stability domain in the (x_i, x_0) parameter space erodes away quickly. (2) The physically dual and colocated sensor-actuator pair always leads to a stable system.

An interesting observation can be made regarding the maximum possible separation between the actuator and sensor while unconditional stability is guaranteed. It is evident from the figures that the best place for non-colocated pairs is in the region close to the clamped end of the cantilever where there are fewer nodes of the mode shapes. However, this region offers low control authority. As seen in Figure 3, there are narrow regions or bottlenecks in the stability domain that separate relatively large regions of unconditional stability domains. These bottlenecks are located at the nodes of the different participating modes as shown in Figure 4, and become wider as the damping increases (see Figures 5 and 6). This observation implies that the placement of an actuator–sensor pair on two sides of a node will likely destabilize the system and that the structural damping increases the stability robustness with respect to the actuator-sensor placement. One also sees that certain non-minimum phase implementations that are observed to be unstable in Figure 2 become unconditionally stable as the damping is increased (the holes in Figure 2 disappear in Figures 5 and 6).

Figure 7 shows the root loci for the two mode system. The actuator–sensor pair co-ordinates are $(x_i, x_0) = (0{\cdot}3, 0{\cdot}8)$ and the loci are plotted for different damping values. The exact critical damping obtained numerically from equation (8) is 183·4 Ns/m. The effect of damping is clearly observable in the root loci.

The study of damping is extended to the system with more than two controlled modes. For the five-mode case, the actuator–sensor location $(x_i, x_0) = (0{\cdot}3, 0{\cdot}42)$ is found to be outside the unconditional stability domain, and the system is minimum phase. Using equation (8), one sweeps through the c-parameter space by increasing damping until the

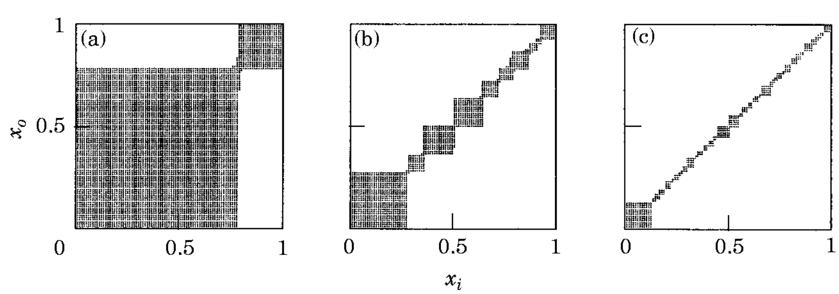

Figure 3. Unconditional stability domain for a point force-linear velocity sensor pair on a cantilever; $c = 1$ Ns/m. Results were obtained by using the new search algorithm. The number of modes is (a) 2, (b) 5 and (c) 10; x_i is the actuator location and x_0 is the sensor location.

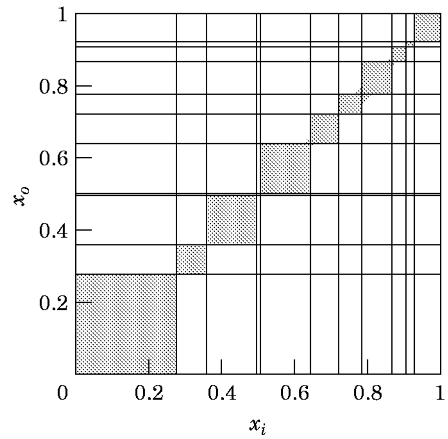

Figure 4. Unconditional stability domain for system with five modes superposed with nodal lines for all modes; x_i is the actuator location and x_0 is the sensor location.

system becomes unconditionally stable. For this example, the critical damping is found to be 722·4 Ns/m. The root loci of this system with $(x_i, x_0) = (0{\cdot}3, 0{\cdot}42)$ and different damping values are shown in Figure 8. The root loci indeed move to the left of the s-plane as damping increases, and the system becomes unconditionally stable once the critical damping is exceeded.

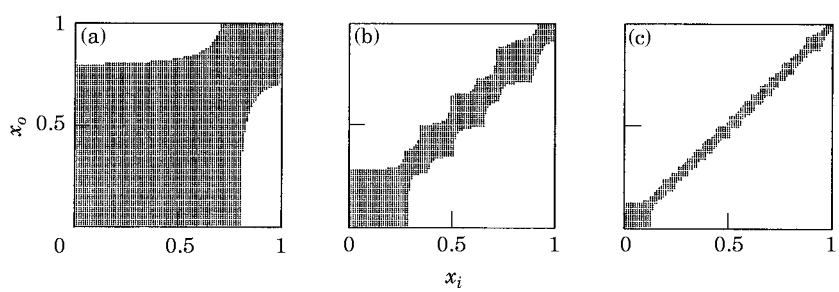

Figure 5. Unconditional stability domain for a point force-linear velocity sensor pair on a cantilever; $c = 100$ Ns/m. The number of modes is (a) 2, (b) 5 and (c) 10; x_i is the actuator location and x_0 is the sensor location.

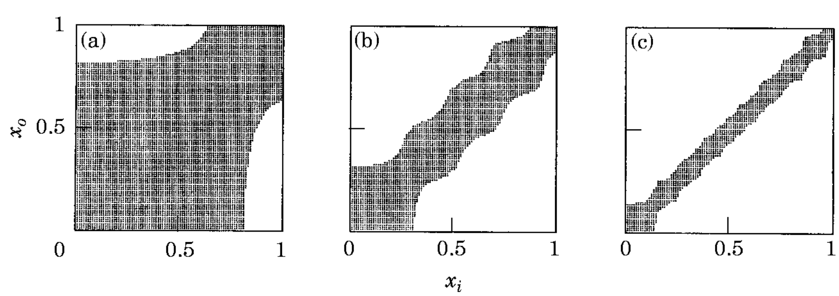

Figure 6. Unconditional stability domain for a point force-linear velocity sensor pair on a cantilever; $c = 1000$ Ns/m. The number of modes is (a) 2, (b) 5 and (c) 10; x_i is the actuator location and x_0 is the sensor location.

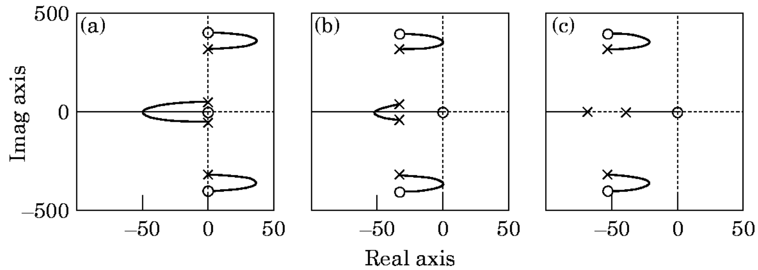

Figure 7. Root loci for the two mode system; $(x_i, x_0) = (0{\cdot}3, 0{\cdot}8)$. (a) Conditionally stable; $c = 1$ Ns/m. (b) Unconditionally stable at the critical damping $c_{min} = 183{\cdot}4$ Ns/m. (c) Unconditionally stable at a higher value of damping $c = 300$ Ns/m.

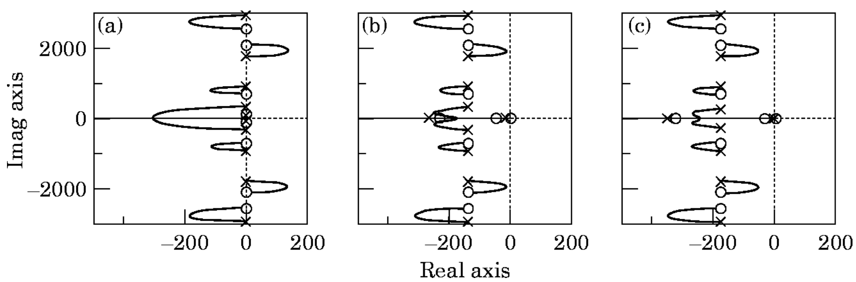

Figure 8. Root loci for the two mode system; $(x_i, x_0) = (0{\cdot}3, 0{\cdot}42)$. (a) Conditionally stable; $c = 1$ Ns/m. (b) Unconditionally stable at the critical damping $c_{min} = 772{\cdot}4$ Ns/m. (c) Unconditionally stable at a higher value of damping $c = 1000$ Ns/m.

4. CONCLUDING REMARKS

A revised set of numerical results for unconditional stability domains in the (x_i, x_0) parameter space of a SISO feedback control of a one dimensional structure has been presented. A physically dual actuator–sensor pair is considered in the study. It is found that instability can occur when the system is non-minimum phase as well as when it is minimum phase. In some non-minimum phase and all minimum-phase cases, there is a critical passive damping above which unconditional stability is guaranteed for all positive gains. It has also been found that placing an actuator-sensor pair on either side of a node of one of the participating modes will lead to a conditionally stable system when the system damping is sufficiently low. Passive damping provides stability robustness with respect to actuator–sensor placement. The study of this simple structure provides valuable insight into the behavior of more complex control systems.

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